MATH. SCAND. 28 (1971), 182–188

THE CONTRACTIBILITY OF THE HOMEOMORPHISM GROUP OF SOME PRODUCT SPACES BY WONG'S METHOD

PETER L. RENZ

Let X be a Hausdorff topological space. Let ω be the set of non-negative integers, and let $Y = X^\omega$. Let H be the homeomorphism group of Y provided with the compact open topology. Let the map p of Y onto Y be defined by $p(x_0, x_1, x_2, x_3, \ldots) = (x_1, x_0, x_2, x_3, \ldots)$ and let e be the identity element in H. If Z is a topological space, an *isotopy* of Z will be a continuous map F from $Z \times [0,1]$ to Z such that $F(\cdot, t)$ is a homeomorphism of Z onto Z for each fixed $t \in [0,1]$. Two maps f and g are *isotopic* if there is an isotopy F such that $f(\cdot) = F(\cdot, 0)$ and $g(\cdot) = F(\cdot, 1)$.

THEOREM 1. *With p, e and H as above, if p is isotopic to e, then H is contractible.*

Raymond Y. T. Wong [3] has shown that the isotopy condition is satisfied when $X = [0,1]$ (the case of the Hilbert cube) and $X = (0,1)$ and that it implies the homeomorphism groups of X^ω are arcwise connected in these cases. Our work uses his methods to prove contractibility without Wong's condition that X be separable and metrizable. A further remark at the end of this paper indicates that the same theorems are valid for arbitrary infinite powers of X.

Theorem 1 is not, however, the most direct generalization of Wong's result. The following result is more directly related to Wong's theorem. This formulation (Theorem 2) is partly suggested by correspondence with David W. Henderson and conversations with Richard Schori.

Let A be a topological space. The *cone over A*, which we denote by $\mathrm{C}(A)$, is the topological space $A \times [0,1]$ with A and $A \times \{0\}$ identified in the natural way and $A \times \{1\}$ identified to a point. The point $A \times \{1\}$ is called the *summit* of $\mathrm{C}(A)$. We define a *generalized isotopy of Y over A*

Received February 2, 1969; in revised form October 15, 1969.

to be a continuous map $F\colon A \times Y \to Y$ such that $F(a, \cdot)$ is a homeomorphism of Y for each $a \in A$.

THEOREM 2. *Let Y, p, and e be as above, let p be isotopic to e, and let A be a topological space. Then any generalized isotopy of Y over A may be extended to a generalized isotopy of Y over* $\mathrm{C}(A)$.

The construction for both of these results is identical but neither result implies the other except in some special cases.

Definitions of the maps.

Subscripts denote coordinates and our maps are defined by specifying the coordinates of the images as follows: For $n \in \omega$ define $\pi^{(n)}$ by

$$\bigl(\pi^{(n)}(s)\bigr)_m = \begin{cases} s_m & \text{if } m < n\,, \\ s_{m+1} & \text{if } m \geqq n \end{cases}$$

for all $s \in Y$. Define a map $\alpha^{(n)}\colon C(Y;Y) \to C(Y;Y)$ by

$$\bigl(\alpha^{(n)}f(s)\bigr)_m = \begin{cases} \bigl(f(s)\bigr)_m & \text{if } m < n\,, \\ s_m & \text{if } m = n\,, \\ \bigl(f(s)\bigr)_{m-1} & \text{if } m > n \end{cases}$$

for all $s \in Y$.

The map $\pi^{(n)}$ on Y is easily seen to be continuous, since its coordinate projections are continuous. If Z and W are topological spaces, we use the notation $C(Z, W)$ for the continuous functions from Z to W. The same argument shows that $f \in C(Y;Y)$ implies $\alpha^{(n)}f \in C(Y;Y)$. Furthermore, it is easily shown that the map $h \to h\circ\pi^{(n)}$ is a continuous map of H into $C(Y;Y)$ in the compact open topology. We show that $\alpha^{(n)}$ is a continuous map. Sets of the form $\{h \in H \mid h(K) \subset U\}$, where K ranges over the compact subsets of Y and U ranges over the basic open subsets of Y, form a sub-basis for the compact open topology of $C(Y;Y)$, so it suffices to show that if $\alpha^{(n)}h(K) \subset U$ with K and U as above, then there is some neighborhood W of h such that $h' \in W$ implies $\alpha^{(n)}h'(K) \subset U$.

Since U is a basic open set in a product space, $U = \times\,\{U^m \mid m \in \omega\}$, where $\times$ denotes the Cartesian product of a family and where the U^m for $m \in \omega$ are open subsets of X almost all of which are equal to X. Define an open set U' as follows:

$$U' = \times\,\{U'^m \mid m \in \omega\} \quad \text{where} \quad U'^m = \begin{cases} U^m & \text{if } m < n\,, \\ U^{m+1} & \text{if } m \geqq n\,. \end{cases}$$

Then by direct computation one has that

$$h' \in V = \{h' \in H \mid h'(K) \subset U'\}$$

implies

$$\alpha^{(n)} h'(K) \subset U .$$

Define $h(n)$ as follows: $h^{(n)} = \alpha^{(n)}(h \circ \pi^{(n)})$. The above shows that $h^{(n)}$ is continuous on Y and that $h \to h^{(n)}$ is continuous in the compact open topology. If $h \in H$, then $(h^{(n)})^{-1}$ may be easily seen to be $(h^{-1})^{(n)}$. Thus $h^{(n)}$ is one to one and $(h^{-1})^{(n)}$ is onto and since $(h^{-1})^{-1} = h$, it follows that $h^{(n)}$ is onto and thus a homeomorphism. That is, $h \to h^{(n)}$ is a map of H into H.

The first shrinking of *H*.

We show that the identity map on H is homotopic to the map $h \to h^{(0)}$ of H into $H^{(0)} = \{h \in H \mid h(s)_0 = s_0\}$. To do this we define a family of isotopies $\varphi^{(n)}$, for $n \in \omega$, of Y with the following properties:

(a) $\varphi^{(n)} : Y \times [n, n+1] \to Y$,
(b) $\varphi^{(n)}(s, n) = s$,
(c) $\varphi^{(n)}(s, n+1) = (s_0, s_1, \ldots, s_{n-1}, s_{n+1}, s_n, s_{n+2}, \ldots)$,
(d) $\varphi^{(n)}$ leaves the first n coordinates pointwise fixed.

Let φ' be the isotopy connecting e with p which exists by hypothesis. For $s \in Y$ define $T_k(s)$ to be the sequence $(s_k, s_{k+1}, s_{k+2}, \ldots, s_{k+j}, \ldots)$. We may define the $\varphi^{(n)}$ by specifying the coordinates of $\varphi^{(n)}(s,t)$ as follows:

$$\left(\varphi^{(n)}(s,t)\right)_m = \begin{cases} s_m & \text{if } m < n , \\ \left(\varphi'(T_n(s), t-n)\right)_{m-n} & \text{if } m \geqq n . \end{cases}$$

Define $\Phi \colon H \times [0, \infty) \to H$ as follows:

$$\Phi(h,t) = \varphi^{(n)}(\cdot, t) \circ h^{(n)} \circ \varphi^{(n)}(\cdot, t) \quad \text{for all } h \in H$$

if $n \leqq t \leqq n+1$. Notice that

$$\begin{aligned} \varphi^{(n)}(\cdot, n+1) \circ h^{(n)} \circ \varphi^{(n)}(\cdot, n+1) &= h^{n+1}(\cdot) \\ &= \varphi^{(n+1)}(\cdot, n+1) \circ h^{(n+1)} \circ \varphi^{(n+1)}(\cdot, n+1) . \end{aligned}$$

This resolves the ambiguities in the definition of Φ. Since $\varphi^{(n)}(\cdot, t)$ is a homeomorphism, $\Phi(h,t) \in H$ for all $h \in H$ and all $t \in [0, \infty)$. We extend Φ to $H \times [0, \infty]$ by defining $\Phi(h, \infty) = h$. In order to show Φ is a homotopy connecting $h \to h^{(0)}$ with the identity map on H, it is sufficient to show Φ is jointly continuous. To do this it suffices to show $(h^{(n)}, t) \to \Phi(h,t)$ is continuous, since $(h,t) \to (h^{(n)}, t)$ is continuous. Thus for $t \neq \infty$ it is useful to show the following:

LEMMA. *If $\psi\colon S\times[0,1]$ is a homotopy of a topological space S and h is a homeomorphism of S onto itself, then $(h,t)\to\psi(\cdot,t)\circ h$ and $(h,t)\to h\circ\psi(\cdot,t)$ are both homotopies of $C(S;S)$ in the compact open topology.*

PROOF. First we show $(h,t)\to h\circ\psi(\cdot,t)$ is continuous.

Given K, compact, and U, open in S, such that $h(\psi(K,t))\subset U$; notice that $\psi(K,t)\subset h^{-1}(U)$. Since ψ is jointly continuous, there exist neighborhoods $V_k\times I_k$ of (k,t) such that $\psi(V_k,I_k)\subset h^{-1}(U)$. But since K is compact, there is a finite set $K_0\subset K$ such that $K\subset\bigcup\{V_k\mid k\in K_0\}$. Let $I=\bigcap\{I_k\mid k\in K_0\}$. Then let K' be $\psi(K,I)$. Notice that $h(K')\subset U$. Let h' be K',U-close to h, that is, $h'(K')\subset U$. Then if $t'\in I$ and h' as above, $h'(\psi(K,I))\subset U$. Hence $(h,t)\to h\circ\psi(\cdot,t)$ is continuous.

Second we show $(g,t)\to\psi(\cdot,t)\circ g$ is continuous. Let K be compact and U open in S. Suppose $\psi(g(K),t)\subset U$. By the joint continuity of ψ and the compactness of $g(K)$ there exists a finite family (V_k,I_k) of neighborhoods of $(g(k),t)$ indexed by $K_0\subset K$ such that $g(K)\subset\bigcap\{V_k\mid k\in K_0\}$ and $\psi(V_k,I_k)\subset U$. Let $I=\bigcap\{I_k\mid k\in K_0\}$ and let $U'=\bigcup\{V_k\mid k\in K_0\}$. Then U' is open in S and $g(K)\subset U'$ so that if g' is K,U'-close to g and $t'\in I$, $\psi(g'(K),t')\subset U$. Hence $(g,t)\to\psi(\cdot,t)\circ g$ is continuous.

The proof of the Lemma follows from the first and second assertions above.

From the lemma and the definition of Φ we see that Φ consists of a sequence of homotopies, $\varphi^{(n)}(\cdot,t)\circ g\circ\varphi^{(n)}(\cdot,t)$, which carry $h^{(n)}$ to $h^{(n+1)}$ as t goes from n to $n+1$. Thus $\Phi\colon H\times[0,\infty)\to H$ is continuous. It remains to check that Φ is continuous at ∞. We use the notation $[t]=$greatest integer less than t, for $t\in R$. Let $\tau^{(t)}=\pi^{([t])}\circ\varphi^{([t])}(\cdot,t)$ for $0\leqq t<\infty$ and $\tau^{(\infty)}=$identity map of Y. Then $\tau^{(t)}\colon Y\times[0,\infty]\to Y$ may easily be seen to be a homotopy. Thus

$$(h,t)\to h\circ\tau^{(t)} = h\circ\pi^{([t])}\circ\varphi^{([t])}(\cdot,t)$$

is jointly continuous. But

$$\Phi(h,t) = \varphi^{[t]}(\cdot,t)\circ\big(\alpha^{([t])}(h\circ\tau^{(t)})\big)\,.$$

The joint continuity of Φ at ∞ follows from the joint continuity of $h\circ\tau^{(t)}$ and the fact that $\varphi^{[t]}(\cdot,t)$ and $\alpha^{([t])}$ eventually leave every finite set of coordinates fixed and therefore every open set of Y invariant. Thus if $h'\circ\tau^{(t')}$ lies within some neighborhood, K,U-close to $h=\Phi(h,\infty)$, for sufficiently large t', then $\varphi^{[t']}(\cdot,t)\alpha^{([t'])}$ leaves $h'\circ\tau^{(t')}$ within the same K,U-neighborhood of h. Thus Φ may be extended to a map Φ' so that $\Phi'\colon H\times[0,\infty]$ is a homotopy between the identity map on H and the map $h\to h^{(0)}$.

Shrinking H to $\{e\}$.

Let $H^{(-1)} = H$ and define

$$H^{(n)} = \{h \in H \mid h(s)_m = s \text{ for all } m \leqq m \text{ and all } s \in Y\} .$$

We have just shown that the identity map on $H^{(-1)}$ is homotopic to a map of $H^{(-1)}$ into $H^{(0)}$. In an exactly similar manner the identity map on $H^{(n)}$ is homotopic over $H^{(n)}$ to a map of $H^{(n)}$ into $H^{(n+1)}$. Let such homotopies

$$\Phi^{(n)}: H^{(n)} \times [n, n+1] \to H^{(n)}$$

be given. Now for $h \in H^{(-1)} = H$ and $n = -1, 0, 1, 2, \ldots$ define by induction a jointly continuous map $\psi: H \times [-1, \infty) \to H$ and $h^{((n))}$ as follows:

$$\begin{aligned} h^{((-1))} &= h , \\ \psi(h,t) &= \Phi^{(n)}(h^{((n))}, t), \quad t \in [n, n+1] , \\ h^{((n+1))} &= \Phi^{(n)}(h^{((n))}, n+1) . \end{aligned}$$

Then for $t > n$, we have $\psi(H,t) \subset H^{(n)}$. As a consequence for t sufficiently large $\psi(H,t)$ falls within any given neighborhood of e. Thus we may continuously extend $\psi: H \times [-1, \infty) \to H$ to a map $\psi': H \times [-1, \infty] \to H$ by defining $\psi'(h, \infty) = e$ for all $h \in H$, then ψ' is the desired contraction of H to $\{e\}$. For convenience we will assume that the above construction has been normalized to yield a contraction ψ^* such that $\psi^*: H \times [0,1] \to H$.

The generalized isotopy theorem.

With ψ^* as constructed above and F a generalized isotopy as in Theorem 2, we define a map $F^*: A \times [0,1] \times Y \to Y$ as follows:

$$F^*(a,t,y) = \psi^*(F(a, \cdot), t)(y) .$$

We know that for each $a \in A$ and $t \in [0,1]$, $F^*(a,t,\cdot)$ is a homeomorphism of Y onto itself. Clearly $F^*(a,0,\cdot) = F(a,\cdot)$ and $F^*(a,1,\cdot) = e(\cdot)$. Since the cone over A is $A \times [0,1]$ with the base, $A \times \{0\}$, identified with A and the top, $A \times \{1\}$, identified with the summit of the cone, and since F^* restricted to $A \times \{1\}$ is constantly the identity map on Y, we know that F^* defines an isotopy extending F to $\mathrm{C}(A)$ if F^* is continuous.

The construction of ψ involved steps of several types. The first type was forming $\Phi: H \times [0, \infty) \to H$ by joining together overlapping arcs of the form $\varphi(\cdot, t) \circ h \circ \varphi(\cdot, t)$ where φ is an isotopy. Applying this construction to the generalized isotopy F we have $\varphi\big(F(a, \varphi(y,t)), t\big)$ which is a composition of continuous functions and thus a continuous function of a, t, and y. Thus we see that the first step in constructing Φ leads to a generalized isotopy on $(A \times [0,\infty)) \times Y$.

The second step involves extending Φ to $H \times [0,\infty)$. This can be done continuously in the compact open topology because $\Phi(\cdot,t)$ eventually leaves every coordinate fixed as $t \to \infty$. This limiting behavior of Φ also insures that the map given by $\Phi'(f(a,\cdot),t)(y)$ of $(A \times [0,\infty]) \times Y$ to Y is continuous.

The third step involves joining together overlapping arcs of maps of the type constructed in the first two steps. This leads to a continuous map given by $\psi'(F(a,\cdot),t)(y)$ of $(A \times [0,\infty]) \times Y$ to Y. This extension is again continuous because $\psi(F(a,\cdot),t)$ eventually agrees coordinatewise with the identity map on Y as t tends to infinity. (In fact, this agreement is uniform on Y as t tends to infinity.) A normalization to the unit interval will give ψ^* without affecting continuity. Thus Theorem 2 is proved.

The notion of an *invertible isotopy* has been introduced and found useful. A *generalized invertible isotopy* is an isotopy F of $A \times Z$ into Z such that the map F' of $A \times Z$ into Z is also an isotopy where $F'(a,z) = F(a,\cdot)^{-1}(z)$. That is, an isotopy is *invertible* if its inverse is also an isotopy. It is easily seen that if p is invertibly isotopic to e then the construction of Φ may be modified by taking

$$\Phi(h,t) = \varphi^{(n)}(\cdot,t)^{-1} \circ h \circ \varphi^{(n)}(\cdot,t)\,,$$

where the $\varphi^{(n)}$ are now invertible isotopies. This will insure that the maps Φ, Φ', ψ, ψ' and ψ^* constructed as above are group homomorphisms of H. This construction will also insure that the corresponding isotopies constructed in the proof of Theorem 2 are invertible isotopies. These observations lead to the following corollaries.

COROLLARY 1. *If p is invertibly isotopic to e then H may be contracted to $\{e\}$ over itself by a map $\psi^*: H \times [0,1] \to H$ which is for every $t \in [0,1]$ a group homomorphism.*

COROLLARY 2. *If p is invertibly isotopic to e, then every invertible generalized isotopy of Y over A may be extended to an invertible generalized isotopy of Y over* $\mathrm{C}(A)$.

David W. Henderson has suggested the following observations which he has found useful. If X is a topological linear space and all of our isotopies leave the origin of $Y = X^\omega$ fixed, then the isotopy extension obtained in Corollary 2 will also have this property. The same holds for the case of invertible isotopies. He has applied results of this sort to the theory of microbundles on infinite dimensional manifolds. The general construction given here should be of use in constructing isotopies with particular properties to solve specific problems.

Isotopy and arcs in *H*.

It is well known that H need not be a topological group (see for example Bourbaki [1, Chapter X, Section 4 and associated exercises]). The operation of composition on $H \times H$ to H is in general continuous only on compacta. A topological space A is a k-space (see Kelley [2, pages 230 and 231]) if continuity on A is equivalent to continuity on compacta in A. In particular locally compact or first countable spaces are k-spaces. If Y and H are both k-spaces, then the natural mappings of $H \times H$ onto H and $H \times Y$ onto Y which are ordinarily only continuous on compacta are continuous. In this case the notions of an arc in H and an isotopy of $[0,1] \times Y$ onto Y agree. In this situation the two theorems proved are equivalent. This equivalence may be proved by using the fact that the valuation map from $H \times Y$ to Y is continuous on compacta for one direction. The other direction may be proved by using the converse of Theorem 3 of Bourbaki [1, Chapter X, Section 4, page 302] plus the fact that one is dealing with k-spaces.

It should be noted that if X satisfies the hypotheses of our theorems, then X^S does as well, where S is any non-empty set. Thus all of the theorems could be restated for arbitrary infinite powers $S \times \omega$ of X.

BIBLIOGRAPHY

1. N. Bourbaki, *General topology*, Part 2, of *Elements of mathematics*, Addison-Wesley, Reading, Massachusetts, 1966.
2. J. L. Kelley, *General topology*, Van Nostrand, New York, 1955.
3. R. Y. T. Wong, *On homeomorphisms of certain infinite dimensional spaces*, Trans. Amer. Math. Soc. 128 (1967), 148–154.

UNIVERSITY OF WASHINGTON, SEATTLE, U.S.A.
AND
REED COLLEGE, PORTLAND, OREGON, U.S.A.