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RADIAL NTH DERIVATIVES OF BLASCHKE PRODUCTS

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1.

In this paper, we consider the boundary behavior of Blaschke products

$$B(z) = \prod_{k=1}^{\infty} \frac{\bar{a}_k}{|a_k|} \frac{a_k - z}{1 - \bar{a}_k z} \tag{1.0}$$

in the unit disk $|z| < 1$. Here $|a_k| < 1$, $\sum (1 - |a_k|) < \infty$ and $a_k = 0$ is permitted with the understanding that then $\bar{a}_k / |a_k| \equiv 1$. The bounded, analytic function (1.0) converges almost everywhere to a limit of modulus 1 as z tends non-tangentially to the boundary $|z| = 1$. Thus $B(z)$ has a natural reflection (which we again denote by $B(z)$), analytic at those points of $|z| > 1$ which are not in the closure of the set $\{\bar{a}_k^{-1}\}$. The extended function $B(z)$ is represented by the product (1.0) in $|z| < 1$ and $|z| > 1$ and satisfies $B(z)\bar{B}(\bar{z}^{-1}) \equiv 1$ for all such z.

Some time ago, Frostman [6] proved the following theorem on the radial limits of B and its derivative.

THEOREM 1. (i) *Necessary and sufficient that*

$$\lim_{r \to 1-0} f(re^{ix}) = L$$

exist and $|L| = 1$ *for* $f = B$, *and every subproduct of* B, *is that*

$$\sum_{k=1}^{\infty} \frac{1 - |a_k|}{|e^{ix} - a_k|} < \infty .$$

(ii) *Necessary and sufficient that*

$$\lim_{r \to 1-0} B(re^{ix}) = L \quad \textit{and} \quad \lim_{r \to 1-0} B'(re^{ix}) = M$$

exist and $|L| = 1$ *is that*

$$\sum_{k=1}^{\infty} \frac{1 - |a_k|}{|e^{ix} - a_k|^2} < \infty .$$

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The derivatives of Blaschke products were considered again by Cargo in [4], where there was proved

THEOREM 2. *Let* $N > 1$ *and*

$$\sum_{k=1}^{\infty} \frac{1-|a_k|}{|e^{ix}-a_k|^{2N}} < \infty . \tag{1.1}$$

Then, for $f = B$, *and every subproduct of* B,

$$\lim_{r\to 1-0} f^{(n)}(re^{ix})$$

exists for $0 \leqq n \leqq N$.

Furthermore, Cargo [4, p. 347] conjectured that the converse of Theorem 2 is correct. (Actually Cargo's conjecture as well as his theorem were more general than Theorem 2 and involved tangential limits of derivatives of $B(z)$ and its subproducts. Through the work of Linden and Somadasa [8], however, the converse of the more general theorem of Cargo [4, Theorem 5] is seen to be false. Thus, all that remains of Cargo's conjecture is the converse of Theorem 2.)

In our recent paper [1], we strengthened Theorem 2 to the point where only

$$\sum_{k=1}^{\infty} \frac{1-|a_k|}{|e^{ix}-a_k|^{N+1}} < \infty \tag{1.2}$$

(instead of (1.1)) is assumed. This made it seem reasonable that Theorem 2 with (1.1) replaced by (1.2) should have a converse.

The purpose of this paper is to prove (in Section 3) the following generalization of Theorem 1.

THEOREM 3. (i) *Let* $N > 0$ *be even. Necessary and sufficient that* $f^{(N)}(re^{ix})$ *be bounded as* $r \to 1-0$, *for* $f = B$, *and every subproduct of* B, *is that* (1.2) *hold.*

(ii) *Let* N *be odd. Necessary and sufficient that*

$$\lim_{r\to 1-0} B^{(j)}(re^{ix}) = L_j$$

exist for $0 \leqq j \leqq N-1$, *that* $B^{(N)}(re^{ix})$ *be bounded as* $r \to 1-0$, *and that*

$$L_j = \lim_{R\to 1+0} B^{(j)}(Re^{ix}) \tag{1.3}$$

for $0 \leqq j \leqq N-1$, *is that* (1.2) *hold.*

The analogy between (1.3) and the condition $|L| = 1$ of Theorem 1 (ii) is clear. That some condition like (1.3) is unavoidable will be seen at the end of Section 4 below. A simple consequence of Theorem 3 is the converse of Theorem 2 (with (1.1) replaced by (1.2)). Actually, slightly more is contained in

COROLLARY 1. *Let* $N > 0$. *Necessary and sufficient that* $B^{(N)}(re^{ix})$ *be bounded and that*

$$\lim_{r\to 1-0} f^{(2[\frac{1}{2}N])}(re^{ix}) = M \tag{1.4}$$

exists for $f = B$, *and any subproduct of* B, *is that* (1.2) *hold.*

At first glance, Theorem 3(i) appears stronger than Theorem 1 (i) inasmuch as no analogue of the condition $|L| = 1$ is needed. In the case of Theorem 1 (i), however, Cargo [3] has shown that the condition $|L| = 1$ may be removed. Our methods easily yield the following extension of this (rather difficult) result of Cargo's.

COROLLARY 2. *If* $B^{(N)}(re^{ix})$ *is bounded and if* (1.4) *holds for any subproduct of* B, *then*

$$\lim_{r\to 1-0} f^{(N)}(re^{ix}) \quad \text{and} \quad \lim_{R\to 1+0} f^{(N)}(Re^{ix})$$

exist and are equal for any subproduct f *of* B.

Section 2 below contains two lemmas, one of which includes Theorem 2 above (with (1.1) replaced by (1.2)). In Section 3 we adapt the methods of [1] to prove Theorem 3 and its corollaries. In Section 4, we outline analogous results for general bounded analytic functions.

2.

The following lemma is essentially the same as our extension [1, Lemma 3.1] of (Cargo's) Theorem 2 above. Since the statement we need here differs from these earlier versions, we include the proof.

LEMMA 1. *Let* B *be a Blaschke product whose zeros satisfy* (1.2) *for some non-negative integer* N. *Then*

$$\lim_{r\to 1-0} B^{(N)}(re^{ix}) \quad \text{and} \quad \lim_{R\to 1+0} B^{(N)}(Re^{ix})$$

exist and are equal.

PROOF. For convenience, we shall assume $x = 0$. We use the equation

$$B'(z) = B(z) \sum_{k=1}^{\infty} (1-|a_k|^2)[(z-a_k)(1-\bar{a}_k z)]^{-1}. \tag{2.1}$$

The series on the right converges uniformly on compact subsets of the plane which are at a positive distance from the zeros and poles of B. We rewrite (2.1) as

$$B'(z) = \sum B_k(z)(1-|a_k|^2)/(1-\bar{a}_k z)^2 , \tag{2.2}$$

where $B_n(z)=B(z)(1-\bar{a}_n z)/(z-a_n)$.

The case $N=0$ of this lemma is from Frostman [6] (Theorem 1 (i) above). Proceeding by induction, we suppose the lemma true for integers less than N. Differentiating (2.2) $N-1$ times by Leibnitz' rule, we obtain

$$B^{(N)}(z) = \sum_{j=0}^{N-1} \binom{N-1}{j} \sum_{k=1}^{\infty} B_k^{(N-1-j)}(z) \frac{(j+1)!\, \bar{a}_k^{\,j}(1-|a_k|^2)}{(1-\bar{a}_k z)^{j+2}} .$$

By the induction hypothesis and (2.1) it follows that the series on the right in this last equation converges for $1-\varepsilon \leqq r \leqq 1+\varepsilon$, where ε is chosen so that B has no zeros on $1-\varepsilon \leqq r \leqq 1$. We claim the convergence is uniform there. Again by the induction hypothesis, it is enough to show

$$\sum_{k=1}^{\infty} B_k(r)\bar{a}_k^{\,N-1}(1-|a_k|^2)/(1-\bar{a}_k r)^{N+1} \tag{2.3}$$

converges uniformly for $1-\varepsilon \leqq r \leqq 1+\varepsilon$. For $r \leqq 1$, the absolute value of the nth term in (2.3) is at most

$$(1-|a_n|^2)|1-\bar{a}_n r|^{-(N+1)} \leqq 2^{N+1}(1-|a_n|^2)|1-\bar{a}_n|^{-(N+1)}$$

since $|1-\bar{a}_n r| > \frac{1}{2}|1-\bar{a}_n|$. For $r>1$, that term is dominated by

$$(1-|a_n|^2)(r\,|r^{-1}-a_n|)^{-(N+1)} \leqq (1-|a_n|^2)\,|r^{-1}-a_n|^{-(N+1)} .$$

Now, it is easy to see that (1.2) implies that there can be at most a finite number of the a_k such that

$$|\operatorname{Im} a_k| \leqq 1 - \operatorname{Re} a_k .$$

If we eliminate these, we have

$$|r^{-1}-a_n| \geqq 2^{\frac{1}{2}}\,|1-a_n| \quad \text{for } r \geqq 1 .$$

Hence we obtain, for the nth term in (2.3)

$$(1-|a_n|^2)\,|1-\bar{a}_n r|^{-(N+1)} \leqq 2^{\frac{1}{2}(N+1)}(1-|a_n|^2)\,|1-a_n|^{-(N+1)}$$

and the uniform convergence is established.

We also include in this section a technical lemma.

LEMMA 2. *Suppose $g(s)$ has $N+M$ continuous derivatives on some open* *interval I, and suppose $g^{(N+M)}$ is differentiable except at one point $s_0 \in I$ and that $g^{(N+M+1)}$ is continuous and bounded on $I-\{s_0\}$. Suppose also that there is a point $a \in I$ such that $g(a)=\ldots=g^{(N-1)}(a)=0$. Let $h(s)=g(s)(s-a)^{-N}$ for s in I. Then h is $M+1$ times differentiable on I and*

$$h^{(M+1)}(s) = \int_0^1 \ldots \int_0^1 g^{(M+N+1)}\left(a+t_1\ldots t_N(s-a)\right) v(t)\, dt_1\ldots dt_N$$

where $v(t)$ is a monomial in $t_1,\ldots,t_N$.

Proof. Since $g(a)=0$,

$$\begin{aligned} g(s) &= \int_0^1 \frac{d}{dt} g\left(a+t(s-a)\right) dt = (s-a)\int_0^1 g'\left(a+t(s-a)\right) dt \\ &= (s-a)\int_0^1\int_0^1 \frac{d}{du}\, g'\left(a+ut(s-a)\right) du\,dt \\ &= (s-a)^2\int_0^1\int_0^1 g''\left(a+ut(s-a)\right) t\, du\,dt\,. \end{aligned}$$

Continuing this procedure, we get

$$h(s) = \int_0^1 \ldots \int_0^1 g^{(N)}(a+t_1\ldots t_N(s-a))m(t_1,\ldots,t_N)\,dt_1\ldots dt_N$$

where m is a monomial. Now it is not hard to see that the hypotheses on g permit us to differentiate under the integral sign $M+1$ times to obtain the desired form for $h^{(M+1)}(s)$. This completes the proof of the lemma.

3.

If B is a Blaschke product, we shall consider the invariant subspace BH^2 of the Hilbert space H^2; one may consult [7] for a discussion of H^2 and its invariant subspaces. It was shown in [5] that for each $f \in (BH^2)^\perp = H^2 \ominus BH^2$ there is a unique function $F(z)$ of bounded characteristic, defined for $|z|>1$, such that

$$\lim_{r\to 1-0} f(re^{ix}) = \lim_{R\to 1+0} F(Re^{ix})$$

for almost all x. Moreover, F is holomorphic, except possibly at the poles of B.

Now let $|w| < 1$, and let n be a fixed non-negative integer, and define

$$(3.1) \qquad k_w(z) = \frac{n!\, z^n - B(z) \sum_{j=0}^{n} \binom{n}{j} \bar{B}^{(j)}(w)\, (n-j)!\, z^{n-j}\, (1-\bar{w}z)^j}{(1-\bar{w}z)^{n+1}}.$$

It is not hard to verify that $k_w \in (\mathrm{BH}^2)^{\perp}$ and in fact that k_w is the unique function (Cauchy kernel) in $(\mathrm{BH}^2)^{\perp}$ with the property that

$$f^{(n)}(w) = (f, k_w) \quad \text{for all } f \in (\mathrm{BH}^2)^{\perp}.$$

Notice that the right-hand side of (3.1) defines a function of bounded characteristic for $|z| > 1$ which has radial limits almost everywhere equal to those of $k_w(z)$. Thus it follows by [5, Theorem 1] that the right side of (3.1) is holomorphic away from the poles of B. In particular, if $B(w) \neq 0$, then the numerator of k_w must vanish to order $n+1$ at $\bar{w}^{-1}$.

Now we shall prove Theorem 3. Actually, we shall only prove (ii) and then prove Corollary 1, part of whose assertion is Theorem 3 (i).

Proof of Theorem 3 (ii). The sufficiency proof is contained in Lemma 1, so we shall confine ourselves to the proof of necessity. As before, we assume $x = 0$ for convenience.

Note first that, by assumption, the function $B(r)$ has $N-1$ continuous derivatives on $(0,2)$, and $B^{(N)}(r)$ exists on $(0,2)$ except possibly at $r = 1$. By hypothesis, $B^{(N)}(r)$ is bounded for $0 \leqq r < 1$. It follows from differentiating the functional equation $B(z)\bar{B}(\bar{z}^{-1}) = 1$ N times that $B^{(N)}(R)$ is bounded on $1 < R \leqq 2$ (assuming, as we may, that no a_k is real). Now take $0 < p < 1$ and $N = 2n+1$ and let k_p be defined by (3.1). We have $k_p(z) = g_p(z)(1-pz)^{-(n+1)}$, where g_p is the numerator in (3.1). We apply Lemma 2, to obtain

$$k_p^{(n)}(r) = (-p)^{-(n+1)} \int_0^1 \cdots \int_0^1 g_p^{(N)}(p^{-1} + t_1 \cdots t_{n+1}(r - p^{-1}))\, v(t)\, dt_1 \cdots dt_{n+1}$$

and thus

$$|k_p^{(n)}(p)| \leqq p^{-(n+1)} \int \cdots \int |g_p^{(N)}(p^{-1} + t_1 \cdots t_{n+1}(p - p^{-1}))|\, dt_1 \cdots dt_{n+1}$$

is bounded as $p \to 1$. But $k_p^{(n)}(p) = (k_p, k_p) = \|k_p\|^2$, and, as we showed in [1], the uniform boundedness of $\|k_p\|$ as $p \to 1-0$ is equivalent to (1.2), and this proves (ii).

Proof of Corollary 1. Once again, we need prove only necessity. We proceed by induction. For $N = 1$, the assumption (1.5) and the theorem of Cargo [3, Corollary 2] imply that (1.3) holds and the case $N=1$ follows from Theorem 3 (ii). For larger N, we consider separately the cases N odd and N even.

First, assume N is even. By induction, we may assume

$$\sum (1-|a_k|)/|1-a_k|^N < \infty . \tag{3.2}$$

Thus Lemma 1 implies that $B(r)$ has $N-1$ continuous derivatives on $I=(0,2)$ and the hypothesis implies $B^{(N)}$ is continuous and bounded on $(0,1)\cup(1,2)$. One may now apply exactly the same reasoning as in the proof of Theorem 3 (ii) to show that, if $N=2n$, then $k_p{}^{(n-1)}(p)$ is uniformly bounded as $p \to 1-0$. Now we must show that this is sufficient to imply (1.2).

From (3.2), it follows that at most a finite number of the a_n satisfy both $\mathrm{Re}\,a_m \geqq 0$ and $|\mathrm{Im}\,a_m| < 1-\mathrm{Re}\,a_m$. We shall consider four subproducts of B. Let B_1 denote the product whose zeros satisfy $\mathrm{Re}\,a_m \geqq 0$ and $\mathrm{Im}\,a_m > 1-\mathrm{Re}\,a_m$. Let the zeros of B_2 be the a_m satisfying $\mathrm{Re}\,a_m \geqq 0$ and $\mathrm{Im}\,a_m < \mathrm{Re}\,a_m - 1$. Let B_3 be the product whose zeros are those with negative real part and let B_4 contain the remaining zeros of B. We shall prove that (1.2) holds for each of the products B_1, B_2, B_3, B_4. This is obvious for B_3 and B_4 since the zeros of B_3 are bounded away from 1, and B_4 is finite. As for B_1 and B_2, we can conclude from the original hypothesis, together with what has gone on already, that $k_p{}^{(n-1)}(p)$ is uniformly bounded, where k_p comes from (3.1) and the zeros of B_1 (resp. B_2). Thus the corollary will follow if we can infer from this that (1.2) holds if, in addition to hypothesis, it is assumed that $\mathrm{Im}\,a_m < 1-\mathrm{Re}\,a_m$ (resp. $>$) and $\mathrm{Re}\,a_m \geqq 0$.

To do this, denote

$$B_m(z) = \prod_{k=1}^{m-1} \frac{z-a_k}{1-\bar{a}_k z}$$

and

$$h_m(z) = B_m(z)(1-|a_m|^2)^{\frac{1}{2}}/(1-\bar{a}_m z) .$$

The functions $h_1, h_2, \ldots$ form an orthonormal basis for $(\mathrm{BH}^2)^{\perp}$ and it is easy to see that the expansion of k_p is given by

$$k_p(z) = \sum_{k=1}^{\infty} \bar{h}_k{}^{(n)}(p)\, h_k(z)$$

so that

$$k_p{}^{(n-1)}(p) = \sum_{k=1}^{\infty} \bar{h}_k{}^{(n)}(p)\, h_k{}^{(n-1)}(p) . \tag{3.3}$$

Expanding the derivatives of the h_k by Leibnitz' rule and noting that, by Lemma 1, $|B_m^{(j)}(p)|$ is uniformly bounded for all m, p and $0 \leqq j \leqq n$, we see that the boundedness of $k_p^{(n-1)}(p)$ as $p \to 1$ implies the uniform boundedness of

$$\sum_{k=1}^{\infty} \frac{\bar{B}_k(p) a_k^n}{(1-a_k p)^{n+1}} \frac{B_k(p) \bar{a}_k^{n-1}(1-|a_k|^2)}{(1-\bar{a}_k p)^n}$$

as $p \to 1$. This implies that

$$\sum_{k=1}^{\infty} |B_k(p)|^2 |a_k|^{2n} |1-\bar{a}_k p|^{-2n} (1-|a_k|^2) a_k / (1-a_k p) \tag{3.4}$$

is uniformly bounded as $p \to 1$.

Now if we assume $\operatorname{Im} a_k > 1 - \operatorname{Re} a_k$, and $\operatorname{Re} a_k \geqq 0$, it follows that $\operatorname{Im} a_k > 2^{-\frac{1}{2}} |1-a_k|$ and hence that

$$\operatorname{Im}[a_k/(1-a_k p)] > 2^{-\frac{1}{2}} |1-a_k| / |1-a_k p|^2 .$$

Hence, from (3.4),

$$\sum_{k=1}^{\infty} |B_k(p)|^2 |a_k|^{2n} (1-|a_k|^2) |1-a_k| / |1-\bar{a}_k p|^{2n+2}$$

is uniformly bounded as $p \to 1$. From this (1.2) follows easily. This proves the induction step of Corollary 1 in case N is even.

If N is odd, the induction hypothesis again implies that (3.2) holds. Thus Lemma 1 implies that $B(r)$ has $N-1$ continuous derivatives on $(0, 2)$. The assertion (1.2) in this case follows directly from Theorem 3 (ii). This proves Corollary 1 and Theorem 3 (i).

Proof of Corollary 2. This is now a direct consequence of Corollary 1 and Lemma 1.

4.

In this section, we obtain a version of Theorem 3 above for arbitrary functions in the unit ball of H^∞. Let $F(z)$ be such a function; F is then of the form

$$F(z) = B(z) \exp\left(-\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} dm(t)\right) \tag{4.1}$$

where B is a Blaschke product and m is a positive Borel measure; [7, p. 67]. We will say that G is a *divisor* of F if G is in the unit ball of H^∞ and if there exists a function $H(z)$ in the unit ball of H^∞ such that $F = GH$. This is equivalent to a representation of type (4.1) for G, with a measure $m_1 \leqq m$ and with a Blaschke product B_1 which is a subproduct of B.

Just as with $B(z)$, the right side of (4.1) converges and represents an analytic function for $|z| > 1$, $z \neq \bar{a}_k^{-1}$, where B is given by (0.1). If we denote this function also by $F(z)$, it is easily verified that we again have the functional equation $F(z)\bar{F}(\bar{z}^{-1}) = 1$.

The following generalization of Theorem 3 above also extends results of M. Riesz [9] upon which Frostman's original results were based.

THEOREM 4. (i) *Let N be even and positive. Suppose $m(\{x\}) = 0$. Necessary and sufficient that $G^{(N)}(re^{ix})$ be bounded as $r \to 1-0$ for every divisor G of F is that*

$$\sum_{k=1}^{\infty} \frac{1-|a_k|}{|e^{ix}-a_k|^{N+1}} + \int_0^{2\pi} \frac{dm(t)}{|e^{ix}-e^{it}|^{N+1}} < \infty \tag{4.2}$$

hold.

(ii) *Let N be odd. Necessary and sufficient that*

$$\lim_{r\to 1-0} F^{(j)}(re^{ix}) = L_j$$

exist for $j = 0,\ldots,N-1$, that $F^{(N)}(re^{ix})$ be bounded as $r \to 1-0$, and that

$$L_j = \lim_{R\to 1+0} F^{(j)}(Re^{ix})$$

for $0 \leqq j \leqq N-1$ is that (4.2) *hold.*

The proof of Theorem 4 is similar to that of Theorem 3 and we shall only sketch it. First, we need the analogue of the results for the case $N = 0$ (of Frostman [6] and Cargo [3]) for $F(z)$.

LEMMA 3. *Let F be given by* (4.1) *with $m(\{x\}) = 0$. Then the following are equivalent.*

(a) (4.2) *holds* (*with $N = 0$*).

(b) *Every divisor of F has a radial limit of modulus* 1 *at e^{ix}.*

(c) *Every divisor of F has a radial limit at e^{ix}.*

PROOF. (a) implies (b) is proved precisely as in Frostman's theorem; (b) implies (c) is a logical inclusion; so we shall merely show that, if (a) fails, some divisor of $F(z)$ fails to have a radial limit at e^{ix}.

If the sum in (4.2) is infinite, this is a result of Cargo [3], so let us assume it is the integral which diverges. Thus, we may assume, for example, that

$$\int_0^{\pi/4} |1-e^{it}|^{-1} dm(t) = \infty$$

and prove that some divisor of F fails to have a radial limit at 1. Actually, if v is any positive Borel measure on $[0,\pi/4]$ with no mass at 0, such that

$$\int_0^{\pi/4} |1-e^{it}|^{-1}\, dv(t) = \infty\,, \tag{4.3}$$

and if $K(z)$ is defined by (4.1) with m replaced by v, we have

$$\arg K(r) = 2r\int_0^{\pi/4} \frac{\sin t}{|r-e^{it}|^2}\, dv(t) \geqq 2rc\int_\varepsilon^{\pi/4} \frac{t}{|r-e^{it}|^2}\, dv(t)$$

for some c and any $\varepsilon > 0$. Thus

$$\liminf_{r\to 1-0} \arg K(r) \geqq c'\int_\varepsilon^{\pi/4} \frac{dv(t)}{|1-e^{it}|}$$

since $t|1-e^{it}|^{-1}$ is bounded from 0. Hence $\arg K(r)$ tends to ∞ and the lemma will be proved if we can produce a measure v on $[0,\pi/4]$ such that $v \leqq m$, $|K(r_k)| \to 1$ (for some sequence $r_k \to 1$) and such that (4.3) holds. The construction is similar to Cargo's [3] and will be omitted.

Remark. The restriction $m(\{x\})=0$ was not used in the proof that (a) implies (b). However, the function $F(z)=\exp[-(1+z)/(1-z)]$ satisfies (c) but neither (a) nor (b).

For the proof of Theorem 4, we shall also need to set down the analogue of Lemma 1.

Lemma 4. *Let $F(z)$ be given* (4.1) *and satisfy* (4.2). *Then*

$$\lim_{r\to 1-0} F^{(N)}(re^{ix}) \quad \textit{and} \quad \lim_{R\to 1+0} F^{(N)}(Re^{ix})$$

exist and are equal.

The proof is an obvious generalization of that of Lemma 1.

PROOF OF THEOREM 4. The sufficiency of condition (4.2) is Lemma 4. Thus we shall only prove necessity. As before, part (ii) will be proved first.

We proceed as in the proof of Theorem 3 and consider the function $k_w(z)$ defined by (3.1) with B replaced by F, and with $N=2n+1$. We shall be able to apply Lemma 2 just as above if we can prove that the numerator, $g_p(z)$, of $k_p(z)$ and its first n derivatives vanish at $z=1/p$. To see this, take a sequence B_j of Blaschke products converging uniformly to F on compact subsets of $|z|<1$. Let $k_{p,j}$ be the kernel function defined by (3.1) with B replaced by B_j. By hypothesis, we see that F can have only a finite number of real zeros and we assume p is not one of these. Since F and B_j satisfy the same functional equation, B_j converges to F in a neighborhood of $1/p$. It follows that $g_{p,j}$, the numerator of $k_{p,j}$, converges uniformly to g_p in a neighborhood of $1/p$. Now the fact that g_p vanishes to order $n+1$ at $1/p$ follows from that property of $g_{p,j}$. Now an application of Lemma 2 shows the nth derivative of k_p to be bounded as $p \to 1-0$:

$$\left| \frac{d^n}{dz^n} k_p(z) \Big|_{z=p} \right| \leqq K .$$

Since B_j tends uniformly to F on compact subsets of $|z|<1$, we must have

$$\left| \frac{d^n}{dz^n} k_{p,j}(z) \Big|_{z=p} \right| \leqq K+1$$

if j is sufficiently large, say $j \geqq j_p$. It follows from Theorem 3.1 of [1], that there is a constant K_1 (independent of p) such that

$$\sum_{k=1}^{\infty} (1-|a_k^{(j)}|^2)/|1-\bar{a}_k^{(j)}p|^{N+1} \leqq K_1, \quad j \geqq j_p ,$$

where $\{a_k^{(j)}\}$ are the zeros of B_j. By Lemma 4.2 of [1], we may take the sequence B_j such that

$$\sum_{k=1}^{\infty} \frac{1-|a_k^{(j)}|^2}{|1-\bar{a}_k^{(j)}p|^{N+1}} \to \sum \frac{1-|a_k|^2}{|1-\bar{a}_k p|^{N+1}} + 2\int_0^{2\pi} \frac{dm(t)}{|1-pe^{it}|^{N+1}} \tag{4.4}$$

as $j \to \infty$. Therefore the right side of (4.4) is bounded (by K_1) as $p \to 1$ and (4.2) follows.

To obtain Theorem 4(i), we prove the analogue of Corollary 1 above. In this case, Lemma 3 proves that (4.2) holds for $N=0$ and Theorem 4(ii) provides the first step in an induction on N. If N is even, the result follows from Theorem 4(ii), so we may assume N to be odd. In this case, we once again take Blaschke products B_j tending to F and we repeat the type of argument used in 4(ii) to see that Theorem 3.1 and Lemma 4.2 of [1] imply the desired result. This proves Theorem 4.

REMARK. Again with regard to the restriction $m(\{0\})=0$ in Theorem 4(i) note that if $F(z)=\exp[-(1+z)/(1-z)]$, then for any N and any divisor G of F, $G^{(N)}(r)$ has a limit, but Theorem 4(i) fails for F.

REMARK. In case the measure m in (4.1) is singular with respect to Lebesgue measure, the above proof may be considerably simplified. One may then apply the Douglas–Shapiro–Shields Theorem of [5] and the integral representation theorem of [2].

REMARK. We close by remarking that, both in the situation of Theorem 4, and of Theorem 3 (in particular in Frostman's Theorem 1) it is necessary to add more than just $\lim_{r\to 1-0} f^{(N)}(re^{ix})$ exists for $f=B$ (resp. F) to obtain (1.2) (resp. (4.2)). For example, Samuelsson [10] has given examples of functions $F(z)$ for which $F, F', \ldots, F^{(N)}$ have radial limits equal to 0 at 1 (and $m(\{0\})=0$). Thus arbitrarily many radial derivatives may exist, yet (4.2) fails, even for $N=0$.

To obtain examples of Blaschke products with many radial derivatives, yet with (1.2) failing, it suffices to consider $(s-c)(1-\bar{c}s)^{-1}$, where $s(z)=\exp[-(1+z)/(1-z)]$, and c is a suitable complex number satisfying $|c|<1$.

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