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A PHRAGMÉN-LINDELÖF TYPE THEOREM FOR A CERTAIN CLASS OF GENERALIZED SUBHARMONIC FUNCTIONS

TORBJÖRN HEDBERG

We shall consider a class of functions introduced by Y. Domar [1]. The class contains the positive subharmonic functions and is defined as follows:

Let E be an open connected subset of the n-dimensional Euclidean space R^n and suppose that $B \geqq 1$ is a given number. The function u is then said to belong to the class $S(B)$ in E if it satisfies the following conditions:

(i) u is non-negative and measurable
(ii) u is bounded on every compact subset of E
(iii) for every n-dimensional sphere $S_R(x_0) \subset E$ with centre x_0 and radius R we have

$$u(x_0) \leqq \frac{B}{S_R} \int_{S_R(x_0)} u\,dx = BA(x_0,u,R)\,,$$

where S_R denotes the volume of the sphere.

In this paper we always take $n \geqq 2$ and $E = \mathsf{R}^n$ and we denote the points by $x = (x^1,x^2,\ldots,x^n)$ and we also write $|x| = ((x^1)^2 + \ldots + (x^n)^2)^{\frac{1}{2}}$.

The problem to be studied here consists in finding conditions on a function v such that if $u \leqq v$ and $u \in S(B)$ then $u = 0$.

The functions v that we are going to consider vanish outside a proper subset of E. In particular we shall in the case $E = \mathsf{R}^2$ study a function v vanishing outside a sector of the plane with its centre at the origin. This will give us a generalization of the well-known Phragmén–Lindelöf theorem for subharmonic functions (see e.g. [3, p. 44]).

We start by proving a theorem concerning functions v satisfying a more general condition. Similar but sharper results follow in the case $B = 1$ from the estimates of harmonic measures in [2].

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Let D be a region in R^n. We shall use the following definitions:

$$\theta_\xi = \{x \mid x \in D \text{ and } x^1 = \xi\},$$
$$\theta(\xi) = m\theta_\xi,$$

where m denotes the $(n-1)$-dimensional Lebesgue measure.

We put $p = (n-1)^{-1}$.

THEOREM 1. *Suppose that the region D satisfies the following conditions:*

(i) $\theta_\xi = \emptyset$ *for* $\xi < 0$ *and* $\theta(\xi) > 0$ *for* $\xi \geqq 0$;

(ii) *there exists some monotone positive increasing function h on R_+, such that $h(t)$ tends to 1 as t tends to 0, and such that*

$$\left[h\left(\frac{|\xi - \xi_0|}{\theta(\xi_0)^p}\right)\right]^{-1} \leqq \frac{\theta(\xi)}{\theta(\xi_0)} \leqq h\left(\frac{|\xi-\xi_0|}{\theta(\xi_0)^p}\right)$$

for all ξ and $\xi_0 > 0$ such that $|\xi - \xi_0| \leqq 2\,\theta(\xi_0)^p$.

For each B not less than 1 but smaller than some constant only depending on h and greater than 1 we can then choose $\lambda > 0$ such that, if

(iii) $u \in S(B)$ *in* R^n,

(iv) $u \leqq \exp\left(\lambda \int_0^{x^1} \theta^{-p}\, dt\right)$ *for x in D,*

(v) $u = 0$ *for x not in D,*

then $u = 0$ everywhere.

PROOF. Suppose that u satisfies the above conditions and let v be the function defined by the right hand side members of (iv) and (v). Let x_0 be an arbitrary point in D. For every $B \geqq 1$,

$$\text{(1)} \qquad BA(x_0, v, R) = v(x_0)\frac{B}{S_R}\int_{S_R(x_0)\cap D} \exp\left(\lambda\int_{x_0^1}^{x^1}\theta^{-p}\,dt\right)dx$$
$$\leqq Bv(x_0)\,\frac{m(S_R(x_0)\cap D)}{S_R}\,\exp\left(\lambda R \sup\theta(t)^{-p}\right),$$

where the supremum is taken over all t such that $|t - x_0^1| \leqq R$. We then take $R = 2\,\theta(x_0^1)^p$. It follows immediately from condition (ii) that

$$\text{(2)} \qquad \theta(x^1) \geqq \theta(x_0^1)\,h(2)^{-1} \quad \text{if } |x_0^1 - x^1| \leqq R.$$

Denote the intersection of the set $\{x \mid |x_0^1 - x^1| \leqq \delta R\}$ with $S_R(x_0)$ and with D by $S_R^\delta(x_0)$ and D^δ, respectively. The condition (ii) then gives us that the measure of the set D^δ satisfies

$$mD^\delta = 2R\delta\,\theta(x_0^1)(1+q_1(\delta)) = R^n 2^{2-n}\delta(1+q_1(\delta))\,,$$

where q_1 tends to 0 uniformly in x_0 as δ tends to 0. Furthermore we have that

$$mS_R^\delta(x_0) = 2R^n\delta\,\omega_{n-1}(1+q_2(\delta))\,,$$

where q_2 tends to zero with δ and where ω_{n-1} is the volume of the $(n-1)$-dimensional unit ball.

This shows that if $\delta > 0$ is chosen small enough we can find an $\varepsilon > 0$ such that the measure of the part of $S_R^\delta(x_0)$ not in D^δ is larger than εR^n and we have proved the existence of a $C < 1$ such that the following relation holds:

$$m(S_R(x_0) \cap D) < C S_R \quad \text{for all } x_0 \text{ in } D\,. \tag{3}$$

We can now prove the theorem for a coefficient B such that $1 \leqq B < 1/C$. We introduce numbers $B' > B$ and d such that $B'C < d < 1$ and choose $\lambda > 0$ such that

$$\exp(2\lambda h(2)^p) < d^{-1}\,. \tag{4}$$

Since $B' < d/C$, (1)–(4) show that for every $x \in D$ there is an R such that

$$B' A(x,v,R) < v(x)\,.$$

Put

$$M = \sup_x u(x)/v(x) < \infty\,.$$

If u were not identically vanishing there would exist an $x_0 \in D$ such that

$$u(x_0) > BB'^{-1} Mv(x_0)\,.$$

But on the other hand

$$u(x_0) \leqq BA(x_0,u,R) \leqq MBA(x_0,v,R) < BB'^{-1}Mv(x_0)$$

for some R chosen as above.

This contradiction proves the theorem.

Next we shall apply this theorem to a special region in the plane. For simplicity we now denote the points by $z = (x,y)$ and put $\arg z = \varphi$ and $|z| = r$. The region we shall consider is the region between two straight half-lines forming an angle β at the origin, $0 < \beta < 2\pi$, and we denote as before this region by D.

If we apply our theorem we find that if $B_0 > 1$ is small enough then for every B, $1 \leqq B \leqq B_0$, there exists a number $\alpha > 0$ such that, if

(i) $u \in S(B)$ in R^2,
(ii) $u = O(r^\alpha)$ uniformly in D for r tending to infinity,
(iii) $u = 0$ for z not in D,

then $u = 0$.

We assume that B is sufficiently close to 1 to guarantee the existence of one such α and we denote by $\alpha(B,\beta)$ the least upper bound of all such α.

We want to prove that $\alpha(B,\beta)$ is close to the corresponding value for subharmonic functions if B is close to 1. Before we do this we state without proof a trivial lemma, where D is as above, i.e. with angle β. We also suppose that the region is symmetric with respect to the positive x-axis.

LEMMA. *Let $\beta < \beta_2 < 2\pi$ and $\delta > 0$ be arbitrary and let*

$$v(z) = \max(r^{\pi/\beta_2 - \delta/2} \cos \pi\varphi/\beta_2, 0) .$$

Then there exists a number $B > 1$ such that for every $z \in D$ the inequality $v(z) > BA(z,v,R)$ holds for some R.

We can now prove our second theorem.

THEOREM 2. $\lim_{B \to 1} \alpha(B,\beta) = \alpha(1,\beta) = \pi/\beta$ *for every* β, $0 < \beta < 2\pi$.

PROOF. The Phragmén–Lindelöf theorem for subharmonic functions [3, p. 44] gives us that $\alpha(1,\beta) = \pi/\beta$. Let $\beta < \beta_1 < \beta_2 < 2\pi$ and let D and D_1 be regions of the above type, symmetric around the positive x-axis and with opening angles β and β_1 respectively. Assume moreover that the vertices are $(0,0)$ and $(-1,0)$ respectively. Let δ be a given number and choose v and B according to the lemma. Suppose that u is a function which satisfies

(i) $u \in S(B)$ in R^2,
(ii) $u = O(r^{\pi/\beta_2 - \delta})$ uniformly in D for r tending to infinity,
(iii) $u = 0$ for z not in D.

We can assume that u is continuous for if not we could reduce the problem by taking a convolution of u with some suitable function and then consider a slightly different region. Put

$$M = \sup_D \frac{u(x,y)}{v(x+1,y)} < \infty .$$

This supremum is attained at some finite point $z_0 = (x_0, y_0) \in D$. If $M > 0$, then we have by the lemma that, for some R,

$$u(z_0) = Mv(x_0+1, y_0) > MBA((x_0+1, y_0), v, R) \geqq BA((x_0, y_0), u, R) \geqq u(z_0)\,.$$

The contradiction proves that $u=0$ which implies that $\alpha(B,\beta) \geqq \pi/\beta_2 - \delta$. Since $\alpha(B,\beta)$ is non-increasing as a function of B and since δ and $\pi/\beta - \pi/\beta_2$ can be chosen arbitrarily small the theorem follows.

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UNIVERSITY OF UPPSALA, SWEDEN