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ON THE BOHR TOPOLOGY IN AMENABLE TOPOLOGICAL GROUPS

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Introduction.

In [1] E. M. Alfsen and P. Holm have characterized the Bohr compactification of a topological group $(G,\mathscr{T})$ as the completion of G with respect to a group topology $\mathscr{T}_B$ (the Bohr topology) which is coarser than $\mathscr{T}$. The purpose of this note is to prove that the general description of $\mathscr{T}_B$ can be simplified in amenable groups, i.e. groups admitting an invariant mean on the space of bounded left uniformly continuous functions. The result can be read as follows: W is a $\mathscr{T}_B$-neighbourhood of e if and only if there is a symmetric relatively dense $\mathscr{T}$-neighbourhood V of e with $V^7 \subset W$. Though stated in another way, this has earlier been proved by E. Følner for abelian groups ([2, Theorem 1] and [3]), and his ideas are used extensively.

Section 1 contains the needed results concerning the Bohr compactification; with a slight modification of proof we get a simpler characterization of $\mathscr{T}_B$ than in [1]. The connections between the upper and the lower mean values, invariant means and distinguished subsets of a group have been studied by E. Følner and in the abelian case by P. Tomter. Section 2 is devoted to this. In Section 3 the characterization of the Bohr topology in amenable topological groups is given.

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1. The Bohr compactification.

In [1] the Bohr compactification $\hat{G}$ of a topological group $(G,\mathscr{T})$ is obtained as the Hausdorff completion of G with respect to the finest uniform structure $\mathscr{U}$ on G satisfying

(1.1) $\mathscr{U}$ is totally bounded,

(1.2) $\mathscr{U}$ is compatible with the group structure, i.e. the group operations are uniformly continuous,

(1.3) $\mathscr{U}$ defines a topology on G coarser than $\mathscr{T}$.

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Proposition 1 of [1] tells us that a uniform structure satisfying (1.2) is completely determined by the associated group topology on G, and therefore it suffices to study the finest group topology on G satisfying the analogous of (1.1), (1.2), and (1.3).

Recall that a subset A of G is called *left* (*right*) *relatively dense* if there is a finite set $\{a_1,\ldots,a_n\}$ in G such that $G=\bigcup_{i=1}^n a_iA$ $(G=\bigcup_{i=1}^n Aa_i)$. If A is both left and right relatively dense, A is called relatively dense. The right uniform structure of a topological group is totally bounded if and only if the left uniform structure is, and this is the case if and only if each neighbourhood of e is relatively dense. It is well known that in this case the left and right uniform structures coincide. A proof of this fact is not so easily traced in the literature, so we include one for completeness.

LEMMA 1. *If $(G,\mathscr{T})$ is a totally bounded topological group, $\mathscr{U}_l$ $(\mathscr{U}_r)$ the left (right) uniform structure, then $\mathscr{U}_l=\mathscr{U}_r$, and the group operations are uniformly continuous.*

PROOF. It is an easily established fact that the group operations are uniformly continuous if and only if $\mathscr{U}_l=\mathscr{U}_r$, and this is the case if and only if G admits a fundamental system of neighbourhoods of e whose members V are all *invariant* in the sense that $xVx^{-1}=V$ for every x in G.

Let U be an arbitrary neighbourhood of e. Choose a symmetric neighbourhood V of e such that $V^3\subset U$. Now $G=\bigcup_{i=1}^n a_iV$ for some $a_1,\ldots,$ $a_n\in G$. Let

$$V_1 = \bigcap_{i=1}^{n} a_i V a_i^{-1} \quad \text{and} \quad W = \bigcup_{x\in G} x^{-1} V_1 x \, .$$

Then W is an invariant neighbourhood of e. If $y\in V_1$ and x is arbitrary, we have $x\in a_iV$ for some i. Now

$$x^{-1}yx \in (a_iV)^{-1}(a_iVa_i^{-1})a_iV = V^3 \subset U ,$$

so $W\subset U$, and the lemma is proved.

The problem of finding the finest uniform structure on G satisfying (1.1), (1.2), and (1.3) is therefore reduced to find the finest group topology on G coarser than the original one such that each neighbourhood is relatively dense. Following the proof of [1, Theorem 1] we now conclude:

THEOREM 1. *For a topological group $(G,\mathscr{T})$ let $\mathscr{T}_B$ be the finest group topology on G coarser than $\mathscr{T}$ defining a totally bounded uniform struc-*

ture. This uniform structure (denoted $\mathscr{U}_B$) is the finest satisfying (1.1), (1.2), *and* (1.3), *and the neighbourhood system of e associated with $\mathscr{T}_B$ consists of those subsets V of G which admit a sequence $\{V_n\}$ of sets such that:*

(1.4) $V_1^2 \subset V$ *and* $V_{n+1}^2 \subset V_n$ *for* $n=1,2,\ldots$.

(1.5) *Every V_n is a symmetric and relatively dense $\mathscr{T}$-neighbourhood of e.*

The topology $\mathscr{T}_B$ is called the Bohr topology on G, and from [1] it is known that the $\mathscr{U}_B$-uniformly continuous functions are exactly the almost periodic functions on G.

2. Invariant means and related subsets of the group.

On BR(G) (= the set of bounded real valued functions on G) we define the right upper mean value $\overline{M}$ by

$$\overline{M}(f) = \inf\{\sup_{x\in G}\textstyle\sum_i \alpha_i f(xa_i) : a_i \in G,\ \alpha_i > 0, \sum_i \alpha_i = 1\}.$$

The right lower mean value $\underline{M}$ is defined by $\underline{M}(f) = -\overline{M}(-f)$.

Lemma 2. *The right upper mean value $\overline{M}$ has the following properties:*

$$\inf_{x\in G} f(x) \leqq \overline{M}(f) \leqq \sup_{x\in G} f(x). \tag{2.1}$$

$$\overline{M}(\lambda f) = \lambda \overline{M}(f) \quad \textit{for } \lambda \geqq 0. \tag{2.2}$$

$$\overline{M}(f_a) = \overline{M}(f) \quad \textit{for } a \in G;\ f_a \textit{ is the function } f_a(x) = f(xa). \tag{2.3}$$

$$\overline{M}(f - f_a) \leqq 0. \tag{2.4}$$

$$\overline{M}(f+g) \leqq \overline{M}(f) + \overline{M}(g) \quad \textit{if } G \textit{ is abelian}. \tag{2.5}$$

Proof. Only part (2.4) needs a proof. Take $a_1 = a$, $a_{k+1} = a_k a$. Then

$$\begin{aligned}\overline{M}(f - f_a) &\leqq \sup_{x\in G} n^{-1} \sum_{i=1}^{n} (f - f_a)(xa_i) \\ &= \sup_{x\in G} n^{-1}(f(xa_1) - f(xa_{n+1})) \leqq 2n^{-1}\|f\|_\infty\end{aligned}$$

This holds for any n, and (2.4) follows.

If A is a subset of G and χ_A is its characteristic function, it is easy to see that A is right relatively dense if and only if $\underline{M}(\chi_A) > 0$. (This observation is due to Følner.) Sets with positive upper mean value have been studied by P. Tomter [6] in the abelian case, and we will transfer his ideas to arbitrary groups.

DEFINITION. A subset A of G is called *left* (*right*) *relatively accumulating* if there is a positive integer n_0 such that for any non-negative integer m, at least $m+1$ of any mn_0+1 left (right) translates of A have a common, non-empty intersection. If A is left and right relatively accumulating, A is called relatively accumulating.

Following [6, pp. 26–27] it is easily seen that

(a) if A is left relatively dense, then A is right relatively accumulating,

(b) if A is right relatively accumulating, then $A^{-1}A$ is right relatively dense,

(c) A is right relatively accumulating if and only if $\bar{M}(\chi_A) > 0$.

(b) was first proved by Følner.

REMARK. In connection with (a), note that a left relatively dense subset is not necessarily *left* relatively accumulating. An example of von Neumann can be used, take G to be the free group of two generators a and b, and let A be the set of elements beginning with a or a^{-1} when written as reduced words. $G = A \cup aA$, so A is left relatively dense. But A is not left relatively accumulating, for instance any two distinct members of the collection $\{A, bA, \ldots, b^n A\}$ have empty intersection.

Now let E be some linear space of complex valued functions on G which contains the constants and is closed under complex conjugation and right translations (that is, $f \in A$, $a \in G \Rightarrow f_a \in E$). A linear functional m on E is called a *right invariant mean* (RIM) if

$$m(\bar{f}) = \overline{m(f)}\,, \tag{2.6}$$

$$\inf_{x \in G} f(x) \leqq m(f) \leqq \sup_{x \in G} f(x) \quad \text{for any real valued } f \in E\,, \tag{2.7}$$

$$m(f_a) = m(f)\,. \tag{2.8}$$

Left invariant means are defined analogously, and if m is both left and right invariant, it is called an invariant mean. If m is a RIM, and if $f \in E$ is real valued, we have

$$m(f) = m(\sum \alpha_i f_{a_i}) \leqq \sup_{x \in G} \sum \alpha_i f(x a_i)$$

for any convex combination $\sum \alpha_i f_{a_i}$ of translates of f. Thus $m(f) \leqq \bar{M}(f)$, and we can conclude that

$$\underline{M}(f) \leqq m(f) \leqq \bar{M}(f)\,. \tag{2.9}$$

If $\bar{M}$ is subadditive on E' ($=$the real functions in E), the Hahn–Banach theorem implies the existence of a linear functional m satisfying $m(f) \leqq$

$\bar{M}(f)$ for $f \in E'$. Applying (2.4) we find that m is a RIM on E', and m can uniquely be extended to a RIM on E. In particular the space of all bounded complex valued functions on an abelian group will admit an invariant mean.

DEFINITION. A topological group G is called *amenable* if there is a RIM on $\mathrm{UCB}_l(G)$ (= the set of left uniformly continuous bounded complex valued functions on G). (A complex valued function f is left uniformly continuous if for each $\varepsilon > 0$ there is a neighbourhood U of e such that $|f(x) - f(y)| < \varepsilon$ if $x^{-1}y \in U$.)

A locally compact group is usually called amenable if there is a RIM on $L^\infty(G)$, but it is known that in this case the two definitions coincide. The results in Section 3 are valid not only for locally compact groups, and our choice of definition of amenability is motivated only by what is needed there.

A RIM is usually not strictly positive on positive, non-zero continuous functions. This is the case if and only if G is totally bounded.

Take $P(G)$ to be the space of linear combinations of continuous positive definite functions. Over $P(G)$ and over the space of almost periodic functions a RIM m coincides of course with the unique invariant mean defined on these spaces. If φ is positive definite, it is proved in [5, p. 59] that

$$(2.10) \qquad m(\varphi) = \inf \{\textstyle\sum_{i,j=1}^{n} a_i a_j \varphi(s_i^{-1}s_j) : s_i \in G,\ a_i > 0,\ \sum_i a_i = 1\}.$$

3. The Bohr topology in amenable topological groups.

We are now going to show that the characterization of the Bohr neighbourhoods given in Theorem 1 can be improved in amenable groups, in fact we shall prove that it suffices to have a finite chain of subsets of the sort described. As before, $\mathscr{T}_B$ denotes the Bohr topology.

THEOREM 2 A. *Let $(G, \mathscr{T})$ be a topological group satisfying the following condition:*

(A) *The right upper mean value $\bar{M}$ is subadditive over the space of real valued functions in* $\mathrm{UCB}_l(G)$.

Then a subset W of G is a $\mathscr{T}_B$-neighbourhood of e if and only if there is a right relatively accumulating subset E of G and a $\mathscr{T}$-neighbourhood V of e such that $(V^{-1}E^{-1}EV)^2 \subset W$.

THEOREM 2 B. *Suppose* $(G, \mathscr{T})$ *is a topological group satisfying*:

(B) *There is a right invariant mean on* $\mathrm{UCB}_l(G)$, *that is, G is amenable.*

Then a subset W of G is a $\mathscr{T}_B$*-neighbourhood of e if and only if there is a right relatively dense subset E of G and a* $\mathscr{T}$*-neighbourhood V of e such that* $(V^{-1}E^{-1}EV)^2 \subset W$.

PROOFS. From Theorem 1 it is easy to see that in both cases the condition is necessary. Conversely, suppose E and V have the stated properties. Following Følner [2] we shall construct a non-zero almost periodic function vanishing outside $(V^{-1}E^{-1}EV)^2$, and this will give the conclusion.

There is a left uniformly continuous function $h: G \to [0,1]$ with $h(e)=1$ and $h(x)=0$ for $x \notin V$.

Define $j(x) = \sup_{y \in E} h(y^{-1}x)$. Then $j \in \mathrm{UCB}_l(G)$, $j(x)=1$ for $x \in E$ and $j(x)=0$ for $x \notin EV$.

If (A) is satisfied, the subadditivity of $\overline{M}$ implies (via the Hahn–Banach theorem) that there is a right invariant mean m on $\mathrm{UCB}_l(G)$, and m can be chosen such that $m(j) = \overline{M}(j)$, cf. [3]. Further

$$m(j) = \overline{M}(j) \geqq \overline{M}(\chi_E) > 0 ,$$

since E is right relatively accumulating in this case.

If (B) is satisfied, we have

$$m(j) \geqq \underline{M}(j) \geqq \underline{M}(\chi_E) > 0 ,$$

since E is right relatively dense in case (B).

Hence in both cases we have a right invariant mean m on $\mathrm{UCB}_l(G)$ with $m(j) > 0$.

The left uniform continuity of j implies that the function φ defined by

$$\varphi(x) = m(j_x j) = m_t[j(tx)j(t)] .$$

is continuous, and straight forward calculations show that φ is positive definite. Further, $\varphi(x) \geqq 0$ for any x, and $\varphi(x)=0$ for $x \notin V^{-1}E^{-1}EV$.

We want to show that $m(\varphi) > 0$, and use the expression (2.10). If $\{a_i\}_1^n$ are positive numbers with $\sum_1^n a_i = 1$ and $\{s_i\}_1^n$ are elements from G, then by the right invariance of m we find that

$$\textstyle\sum_{i,j} a_i a_j \varphi(s_i^{-1}s_j) = m_t[(\sum_i a_i j(ts_i))^2] \geqq \big(m_t[\sum_i a_i j(ts_i)]\big)^2 = m(j)^2 .$$

Thus $m(\varphi) \geqq m(j)^2 > 0$.

Over $P(G)$ the functional m can be used to define a convolution

$$f*g(x) = m_t[f(t)g(t^{-1}x)] .$$

R. Godement has proved that $f*g$ will be almost periodic [5, p. 63]. Thus the function $\psi=\varphi*\varphi$ will be positive definite and almost periodic. Further $\psi(x)\geqq 0$ for all x, $\psi(x)=0$ for $x\notin(V^{-1}E^{-1}EV)^2$ and

$$\psi(e) = m(|\varphi|^2) \geqq |m(\varphi)|^2 > 0 .$$

The set

$$W_0 = \{x\in G : |\psi(x)-\psi(e)|<\psi(e)\}$$

is a $\mathcal{T}_B$-neighbourhood of e, since ψ is almost periodic. Further

$$W_0 \subset (V^{-1}E^{-1}EV)^2 \subset W ,$$

so W is a $\mathcal{T}_B$-neighbourhood of e.

Theorem 2 B can be given in a weaker form which makes clear the connection with Theorem 1.

COROLLARY 1. *If $(G,\mathcal{T})$ is an amenable topological group, then a subset W is a $\mathcal{T}_B$-neighbourhood of e if and only if there is a symmetric, relatively dense $\mathcal{T}$-neighbourhood V of e with $V^7\subset W$.*

PROOF. If $V^7\subset W$, take $E=V$ and let U be a $\mathcal{T}$-neighbourhood of e satisfying $UU^{-1}\subset V$. Apply Theorem 2 B with E and U, and the conclusion follows from

$$(U^{-1}E^{-1}EU)^2 \subset U^{-1}E^{-1}EVE^{-1}EU \subset V^7 \subset W .$$

In abelian groups we can simplify even more, and since condition (A) always holds in this case, we have:

COROLLARY 2. *If $(G,\mathcal{T})$ is an abelian topological group, a subset W is a $\mathcal{T}_B$-neighbourhood of 0 if and only if there is a symmetric, relatively accumulating $\mathcal{T}$-neighbourhood U of 0 such that $U^5\subset W$.*

PROOF. Let V be a symmetric neighbourhood of 0 with $V^4\subset U$, and take $E=U$ in Theorem 2 A.

If G is a discrete group, we may take $V=\{e\}$ in Theorem 2 A. In this case the conditions (A) and (B) are equivalent [4, Theorem 1], and we have

COROLLARY 3. *If G is an amenable discrete group, then a subset W is a $\mathcal{T}_B$-neighbourhood if and only if there is a right relatively accumulating subset E of G with $E^{-1}EE^{-1}E\subset W$.*

For an amenable topological group let n be the minimal number such that V^n is a Bohr neighbourhood whenever V is a symmetric, relatively dense neighbourhood of e. We have seen that in general $n \leqq 7$, $n \leqq 5$ for abelian groups and $n \leqq 4$ for discrete groups. A natural question is whether this number can be reduced for some special groups. The following example pointed out to us by J. F. Aarnes, shows that in general we have $n > 1$.

Take the discrete group of integers Z, and let $V = \{0, \pm 1, \pm 3, \pm 5, \ldots\}$. This set is symmetric and relatively dense. Since the characters on a group are almost periodic, the subset

$$U = \{n \in \mathsf{Z} : |e^{n\pi i} - 1| < 1\} = \{0, \pm 2, \pm 4, \pm 6, \ldots\}$$

is a Bohr neighbourhood of 0. Since $U \cap V = \{0\}$, V is not a Bohr neighbourhood. Hence for Z we have $2 \leqq n \leqq 4$. For the real numbers with the usual topology a similar argument shows that $2 \leqq n \leqq 5$.

Another question naturally arises, if G is not amenable: will then such an n exist, or perhaps the finite chain characterization of the Bohr neighbourhoods (at least for locally compact groups) is equivalent to amenability? The answers to these questions are not known to the author.

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