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THE MULTIPLIERS FOR FUNCTIONS WITH FOURIER TRANSFORMS IN L_p

R. LARSEN

1. Introduction.

In a previous paper [12] the author, together with T. S. Liu and J. K. Wang, began the study of certain subspaces of the group algebra $L_1(G)$ of a locally compact Abelian group G. These subspaces $A_p(G)$, $1 \leqq p < \infty$, were defined to be the spaces of all $f \in L_1(G)$ whose Fourier transforms $\hat{f}$ belong to $L_p(\hat{G})$, where $\hat{G}$ denotes the dual group of the locally compact Abelian group G and $L_p(\hat{G})$ is the space of complex valued functions on $\hat{G}$ whose pth powers are integrable with respect to Haar measure on $\hat{G}$. If for each p, $1 \leqq p < \infty$, we set

$$\|f\|^p = \|f\|_1 + \|\hat{f}\|_p, \quad f \in A_p(G) ,$$

where

$$\|f\|_1 = \int_G |f(t)|\, dt , \quad \|\hat{f}\|_p = \left(\int_{\hat{G}} |\hat{f}(\gamma)|^p\, d\gamma\right)^{1/p} ,$$

and dt and $d\gamma$ denote integration with respect to Haar measures on G and $\hat{G}$ respectively, then $A_p(G)$ is a Banach algebra with the indicated norm and the usual convolution product. It is then possible to study the relationship between various Banach algebra properties of $L_1(G)$ and $A_p(G)$, $1 \leqq p < \infty$, as was done in [12]. Since the appearance of [12] a number of writers have further investigated the algebras $A_p(G)$ or their generalizations [6], [8], [9], [10], [11], [13], [14], [15], [16], [19]. In particular, we asserted in [12] that if G is a *noncompact* locally compact Abelian group, then the multipliers for the algebras $A_p(G)$, $1 \leqq p < \infty$, correspond precisely to the Fourier–Stieltjes transforms of bounded regular Borel measures on G, that is, the multipliers for $A_p(G)$ are the same as those for $L_1(G)$ [17, p. 73]. The proof of this assertion given in [12] is defective, but a correct proof has subsequently been given in [6]. However, as indicated in [12], when G is *compact* there in general exist multipliers for $A_p(G)$ different from those defined by Fourier–Stieltjes

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transforms of measures. The main purpose of this article is to investigate more fully the nature of the multipliers for $A_p(G)$, $1 \leqq p < \infty$, when G is a compact Abelian group.

Let us first recall that a multiplier for $A_p(G)$ is a bounded linear operator T on $A_p(G)$ which commutes with translation, that is, $T(\tau_s f) = \tau_s(Tf)$ for each $f \in A_p(G)$ and $s \in G$, where $\tau_s f(t) = f(ts^{-1})$. Clearly the translation operators τ_s are themselves multipliers of norm one. Since $A_p(G)$ is semi-simple [12] to each multiplier T there corresponds, a unique bounded continuous function φ on $\hat{G}$ such that $(Tf)^\wedge = \varphi \hat{f}$ for each $f \in A_p(G)$ [18, p. 1135]. It is well known that these two descriptions of a multiplier are equivalent and so we shall interchange them at will.

When G is compact and $1 \leqq p \leqq 2$ it is elementary to prove that every bounded function φ on $\hat{G}$ defines a multiplier for $A_p(G)$. But if $p > 2$ the subspace of $A_p(G)$ whose Fourier transforms are invariant under multiplication by all bounded functions is the space $A_2(G)$, and this implies that not every bounded function defines a multiplier for $A_p(G)$. In this case, however, we shall show that the space of multipliers for $A_p(G)$ is linearly isomorphic to a proper subspace of the space of pseudo-measures on G [1], [2], [3], and that this subspace properly contains the space of all bounded regular Borel measures on G. Finally, for each $p > 2$ we shall norm the linear space $A_1(G)$ in such a way that its completion is a Banach space of continuous functions such that there exists a continuous linear isomorphism from the space of multipliers for $A_p(G)$ onto the dual space of this Banach space.

Throughout the paper M_p will denote the Banach space of multipliers for $A_p(G)$, $1 \leqq p < \infty$, and $\|T\|_p$ will denote the norm of the multiplier T as an operator on $A_p(G)$ [18]. $C(G)$ is the space of complex valued (bounded) continuous functions on the compact group G, and $L_p(G)$, $1 \leqq p < \infty$, the space of complex valued functions on the compact group G whose pth powers are integrable with respect to Haar measure on G. If $S \subset L_1(G)$, then $\hat{S}$ will denote the set of Fourier transforms of elements of S. It will always be assumed that the Haar measures on G and $\hat{G}$ are chosen in such a way that the Fourier inversion formula is valid. General results from harmonic analysis which are used in the body of the paper can all be found in [17].

Remarks. a) We collect here several facts about the spaces $A_p(G)$ which we shall use in the sequel, generally without explicit reference. If G is compact it follows from [12] that $A_2(G) = L_2(G)$ as linear spaces, and applications of the Hausdorff–Young Theorem [20, p. 190] reveal that for $1 < p' < 2$, $1/p' + 1/p = 1$, we have

$$A_1(G) \subset A_{p'}(G) \subset L_p(G) \subset L_2(G) = A_2(G) \subset L_{p'}(G) \subset A_p(G) \subset L_1(G)\,.$$

In general, if $q<p$, then $A_q(G)$ is a norm dense ideal in $A_p(G)$ [12]. It is also easy to establish that as linear spaces $M_p \subset M_q$ when $q<p$. Finally, for compact G, it is easily seen by means of the Fourier inversion formula that

$$A_1(G) = \hat{L}_1(\hat{G}) \subset C(G)\,.$$

Moreover $A_1(G)$ is supremum norm dense in $C(G)$ and $\hat{A}_1(G)$ is norm dense in $L_p(\hat{G})$, $1 \leqq p < \infty$.

b) If G is finite, then $\hat{G}$ is finite, and it is evident that $L_1(G)=A_p(G)$, $1\leqq p<\infty$. In this case nothing remains to be examined. However, this is the only instance in which the spaces $A_p(G)$ and $L_1(G)$ are identical when G is compact. More generally the following easily established theorem is valid.

THEOREM 1. *Let G be a locally compact Abelian group. Then the following are equivalent.*

(i) *G is nondiscrete.*

(ii) $A_p(G) \neq L_1(G)$, $1 \leqq p < \infty$.

(iii) $A_p(G) \neq A_q(G)$, $1 \leqq p, q < \infty$, $p \neq q$.

Thus we shall restrict our attention to *infinite* compact Abelian groups.

2. Spaces invariant under multiplication by bounded functions.

It is well known that every bounded measurable function φ on $\hat{G}$ defines a multiplier for $L_2(G)$ [4, p. 496], and hence, when G is compact, for $A_2(G)$. Moreover an elementary argument shows that, if φ defines a multiplier for $A_2(G)$, then it does so also for $A_p(G)$, $1 \leqq p < 2$. Thus we obtain the following theorem.

THEOREM 2. *Let G be an infinite compact Abelian group. Then every $\varphi \in C(\hat{G})$ defines a multiplier for $A_p(G)$, $1 \leqq p \leqq 2$.*

Since, as noted previously, every multiplier for $A_p(G)$ corresponds to a bounded continuous function, the preceding theorem shows that the multipliers for $A_p(G)$, $1 \leqq p \leqq 2$, correspond precisely to $C(\hat{G})$ when G is compact.

The situation for $p>2$ and compact G is quite different. In this case we set

$$(A_p(G))_0 = \{f \mid f \in A_p(G),\ \varphi\hat{f} \in \hat{A}_p(G),\ \varphi \in C(\hat{G})\}\,,$$

and prove the following result.

THEOREM 3. *Let G be an infinite compact Abelian group and $p>2$. Then $(A_p(G))_0=A_2(G)=L_2(G)$.*

PROOF. Let $f\in(A_p(G))_0$. Then $f\in L_1(G)$ and $\varphi\hat{f}\in\hat{A}_p(G)\subset\hat{L}_1(G)$ for each $\varphi\in C(\hat{G})$. However, the set of such functions in $L_1(G)$ is precisely $L_2(G)$ [7, p. 244], and hence $(A_p(G))_0\subset L_2(G)$. Conversely, if $f\in L_2(G)=A_2(G)$, then by the preceding result $\varphi\hat{f}\in\hat{A}_2(G)$ for each $\varphi\in C(\hat{G})$. But $\hat{A}_2(G)\subset\hat{A}_p(G)$ and so $A_2(G)\subset(A_p(G))_0$.

Therefore $(A_p(G))_0=A_2(G)=L_2(G)$.

COROLLARY. *Let G be an infinite compact Abelian group and $p>2$. Then not every function $\varphi\in C(\hat{G})$ defines a multiplier for $A_p(G)$.*

3. Multipliers, measures and pseudomeasures.

As indicated previously, when G is compact the linear space $A_1(G)=\hat{L}_1(\hat{G})$. This linear space is a Banach space when equipped with the norm $\|f\|_A=\|\hat{f}\|_1$, $f\in\hat{L}_1(\hat{G})$, and its dual space is the space of pseudomeasures on G, which we denote by $P(G)$ [3, p. 259]. The next result asserts that for $p>2$ the space of multipliers M_p can be identified with a subspace of $P(G)$. Notation and results on pseudomeasures used below can all be found in [3].

THEOREM 4. *Let G be an infinite compact Abelian group and $p>2$. Then there exists a continuous linear injective mapping from M_p into $P(G)$.*

PROOF. Let $T\in M_p$. Since $C(G)\subset L_2(G)=A_2(G)\subset A_p(G)$ we see that T defines a linear mapping from $C(G)$ to $L_1(G)$ which commutes with translation. Furthermore, suppose f_n, $f\in C(G)$ and $\lim_n\|f_n-f\|_\infty=0$, where $\|\cdot\|_\infty$ denotes the usual supremum norm in $C(G)$. Because G is compact it follows that $\lim_n\|f_n-f\|_1=\lim_n\|f_n-f\|_2=0$, and hence by the Plancherel Theorem $\lim_n\|f_n-f\|^2=0$. But $p>2$ implies that $T\in M_2$, and so $\lim_n\|Tf_n-Tf\|^2=0$, from which we conclude that $\lim_n\|Tf_n-Tf\|_1=0$. Thus the mapping defined by T is continuous.

Consequently there exists a unique pseudomeasure $\nu\in P(G)$ such that $Tf=\nu*f$ for each $f\in C(G)$ [3, p. 260]. Define $\alpha(T)=\nu$. Clearly α is a linear mapping from M_p into $P(G)$. That α is injective follows immediately from the denseness of $A_1(G)$ in $A_p(G)$. Moreover, since for each $\gamma\in\hat{G}$ the Fourier transform of the continuous character $(\cdot,\gamma)$ is the characteristic function of the set $\{\gamma\}$, and $T(\cdot,\gamma)=\varphi(\gamma)(\cdot,\gamma)$ where $\varphi\in C(\hat{G})$ is the unique function for which $(Tf)\hat{}=\varphi\hat{f}$ for each $f\in A_p(G)$, we see that

$$|\hat{\nu}(\gamma)| = \|\hat{\nu}(\cdot,\gamma)^{\wedge}\|_{\infty}$$
$$\leqq \|\nu*(\cdot,\gamma)\|_1 = \|T(\cdot,\gamma)\|_1 = |\varphi(\gamma)| \leqq \|\varphi\|_{\infty} \leqq \|T\|_p .$$

The validity of the last inequality follows from [18, p. 1135] and the semisimplicity of $A_p(G)$. But the Fourier transform for pseudomeasures is an isometry [3, p. 260], and hence

$$\|\alpha(T)\|_P = \|\nu\|_P = \|\hat{\nu}\|_{\infty} \leqq \|T\|_p ,$$

where $\|\cdot\|_P$ denotes the norm in $P(G)$.

Therefore α is a continuous linear injective mapping from M_p into $P(G)$.

REMARKS. a) The norms $\|\cdot\|_A$ and $\|\cdot\|^1$ on $\hat{L}_1(\hat{G})$ are obviously equivalent.

b) The mapping α from M_p to $P(G)$ is not surjective. As if it were, then since the Fourier transform of pseudomeasures maps $P(G)$ onto $C(\hat{G})$ [3, p. 260], as G is compact, the composition of the Fourier transform with α would produce a surjective mapping from M_p onto $C(\hat{G})$. That is, every function in $C(\hat{G})$ would define a multiplier for $A_p(G)$, thereby contradicting the Corollary to Theorem 3.

c) On the other hand, the subspace $\alpha(M_p)$ of $P(G)$ properly contains the space $M(G)$ consisting of all bounded regular Borel measures on G. Thus, when $p>2$ there exist multipliers for $A_p(G)$ which are not defined by the Fourier–Stieltjes transform of any bounded regular Borel measure on G. To see this, given $p>2$, set $m=p/2$, $n=m/(m-1)$, and choose r such that $0<r<2$ and $rn>2$. Let $E\subset\hat{G}$ be any infinite Sidon set [17, p. 120] and choose $\varphi\in C(\hat{G})$ such that

i) $\varphi(\gamma) = 0,\ \gamma\notin E$,
ii) $\sum_{\gamma}|\varphi(\gamma)|^2=\infty$,
iii) $\sum_{\gamma}|\varphi(\gamma)|^{rn}<\infty$.

It is easy to see that such choices can always be made. An application of Hölder's inequality shows that $\varphi\hat{f}\in L_2(\hat{G})\subset L_p(\hat{G})$ for each $f\in A_p(G)$, and so φ defines a multiplier for $A_p(G)$. However, $\varphi\neq\hat{\mu}$ for any $\mu\in M(G)$ because φ is a Fourier–Stieltjes transform if and only if $\sum_{\gamma}|\varphi(\gamma)|^2<\infty$ [2, p. 841].

4. The space $B_p(G)$.

In the preceding section we saw for $p>2$ that M_p is linearly isomorphic to a proper subspace of the continuous linear functionals on a certain Banach space of continuous functions. However, it is not immediately obvious whether M_p can be considered as a dual space of such a Banach space. The development of this section will show that this is indeed possible. We begin by defining the normed spaces of continuous functions $B_p(G)$, and shall ultimately prove that there exists a continuous linear isomorphism from M_p onto $B_p{}'(G)$, the dual space of $B_p(G)$. Furthermore we shall show that the completion $\bar{B}_p(G)$ of $B_p(G)$ can be considered as a Banach space of continuous functions.

Consider a fixed $p>2$. For $T\in M_p$ we shall denote by φ the unique function in $C(\hat{G})$ such that $(Tf)\hat{}=\varphi\hat{f}$ for each $f\in A_p(G)$. If $T\in M_p$, then for each $f\in\hat{L}_1(\hat{G})$ we set

$$\beta(T)(f) = \int_{\hat{G}} (Tf)\hat{}(\gamma)\,d\gamma = \int_{\hat{G}} \varphi(\gamma)\hat{f}(\gamma)\,d\gamma\,.$$

For $f\in\hat{L}_1(\hat{G})$ we define

$$\|f\|_B = \sup\{|\beta(T)(f)| \mid T\in M_p,\ \|T\|_p\leqq 1\}\,.$$

These definitions make sense as $M_p\subset M_1$.

It is routine to verify that $\|\cdot\|_B$ is a norm on $\hat{L}_1(\hat{G})$, and the normed linear space so obtained will be denoted by $B_p(G)$. Moreover, from the preceding definitions it is apparent that each $\beta(T)$ defines a continuous linear functional on $B_p(G)$. Thus β defines a mapping from M_p into $B_p{}'(G)$. It is not difficult to show that β is a continuous linear injective mapping from M_p to $B_p{}'(G)$. For example, if $f\in B_p(G)$, then

$$\begin{aligned}|\beta(T)(f)| = \left|\int_{\hat{G}} (Tf)\hat{}(\gamma)\,d\gamma\right| &= \left|\|T\|_p\int_{\hat{G}} (Tf)\hat{}(\gamma)/\|T\|_p\,d\gamma\right| \\ &= \|T\|_p\,|\beta(T/\|T\|_p)(f)| \leqq \|T\|_p\,\|f\|_B\,.\end{aligned}$$

Hence $\|\beta(T)\|\leqq\|T\|_p$, where $\|\cdot\|$ denotes the norm in $B_p{}'(G)$.

The theorem we wish to establish is the following one.

THEOREM 5. *Let G be an infinite compact Abelian group and $p>2$. Then β is a continuous linear bijective mapping from M_p to $B_p{}'(G)$.*

In light of the preceding discussion we need only prove that β is surjective. Before turning to the proof proper of this fact we shall establish several technical lemmas.

LEMMA 1. *Let G be an infinite compact Abelian group, $p>2$ and $f,g\in B_p(G)$.*

(i) *If* $1 \leqq r \leqq \infty$ *and* $1/r + 1/r' = 1$, *then* $\|f*g\|_B \leqq \|\hat{f}\|_r \|\hat{g}\|_{r'}$.
(ii) $\|f*g\|_B \leqq \|f\|^p \|g\|_\infty$.

PROOF. Clearly $f*g \in B_p(G)$ as $\hat{L}_1(\hat{G}) = A_1(G)$ is an algebra under convolution. Using [18, p. 1135] we see that for each $T \in M_p$,

$$|\beta(T)(f*g)| = \left| \int_{\hat{G}} \varphi \hat{f}(\gamma) \hat{g}(\gamma)\, d\gamma \right|$$

$$\leqq \|\varphi \hat{f}\|_r \|\hat{g}\|_{r'} \leqq \|\varphi\|_\infty \|\hat{f}\|_r \|\hat{g}\|_{r'} \leqq \|T\|_p \|\hat{f}\|_r \|\hat{g}\|_{r'} .$$

The application of Hölder's inequality is justified since $L_1(\hat{G}) \subset L_q(\hat{G})$, $1 \leqq q \leqq \infty$. Hence $\|f*g\|_B \leqq \|\hat{f}\|_r \|\hat{g}\|_{r'}$.

To prove (ii) we observe that for $T \in M_p$ we have

$$|\beta(T)(f*g)| = \left| \int_{\hat{G}} [T(f*g)]\hat{}\,(\gamma)\, d\gamma \right| = \left| \int_{\hat{G}} (Tf)\hat{}\,(\gamma) \hat{\tilde{g}}(\gamma^{-1})\, d\gamma \right| ,$$

where $\tilde{g}(t) = g(t^{-1})$. However Tf, $g \in A_1(G) \subset L_2(G)$ as $M_p \subset M_1$. Thus we may apply Parseval's formula to obtain

$$|\beta(T)(f*g)| = \left| \int_G Tf(t)\, \tilde{g}(t)\, dt \right|$$

$$\leqq \|Tf\|_1 \|\tilde{g}\|_\infty \leqq \|Tf\|^p \|g\|_\infty \leqq \|T\|_p \|f\|^p \|g\|_\infty .$$

Therefore $\|f*g\|_B \leqq \|f\|^p \|g\|_\infty$.

LEMMA 2. *Let G be an infinite compact Abelian group and $p > 2$. Suppose $F \in B_p'(G)$, $f \in B_p(G)$ and define $F_f(\hat{g}) = F(f*g)$ for each $g \in B_p(G)$. Then F_f defines a continuous linear functional on $L_p(\hat{G})$.*

PROOF. It is evident that F_f defines a linear functional on $\hat{B}_p(G) \subset L_p(\hat{G})$. Moreover from the first portion of Lemma 1 we see for each $g \in B_p(G)$ that

$$|F_f(\hat{g})| = |F(f*g)|$$

$$\leqq \|F\| \|f*g\|_B \leqq \|F\| \|\hat{f}\|_{p'} \|\hat{g}\|_p ,$$

where $1/p + 1/p' = 1$. Thus F_f is continuous on $\hat{B}_p(G)$ considered as a subspace of $L_p(\hat{G})$. Moreover $\hat{B}_p(G)$ is norm dense in $L_p(\hat{G})$.

Therefore F_f can be uniquely extended to a continuous linear functional on all of $L_p(\hat{G})$.

Given F_f as in the previous lemma we denote by $\hat{h}$ the unique element of $L_{p'}(\hat{G})$, $1/p + 1/p' = 1$, such that for each $\hat{g} \in \hat{B}_p(G)$

$$F_f(\hat{g}) = \langle\tilde{\hat{h}},\hat{g}\rangle = \int_{\hat{G}} \hat{h}(\gamma)\hat{g}(\gamma)\, d\gamma\, .$$

Since $1 < p' < 2$, the Hausdorff–Young theorem [20, p. 190] implies the existence of a unique $h \in L_p(G)$ whose Fourier transform is $\hat{h}$. Thus, given $F \in B_p'(G)$, for each $f \in B_p(G)$ we define $Tf = h$, where h is chosen as above. Clearly T is a linear transformation from the linear subspace $A_1(G) = B_p(G)$ of $A_p(G)$ to $A_{p'}(G) \subset A_p(G)$.

LEMMA 3. *Let G be an infinite compact Abelian group, $p > 2$ and $F \in B_p'(G)$. If T is defined as above, then T is a continuous linear transformation from the subspace $A_1(G)$ of $A_p(G)$ to $A_p(G)$.*

PROOF. Suppose $f \in A_1(G)$ and $1/p + 1/p' = 1$. Then, since $\hat{B}_p(G) \subset L_{p'}(\hat{G}) \subset L_p(\hat{G})$ and $\hat{B}_p(G)$ is norm dense in $L_{p'}(\hat{G})$, we conclude that

$$\begin{aligned}
\|(Tf)\hat{}\|_p = \|\hat{h}\|_p = \|\tilde{\hat{h}}\|_p \\
&= \sup \{|\langle\tilde{\hat{h}},\hat{g}\rangle| \mid \hat{g} \in \hat{B}_p(G), \|\hat{g}\|_{p'} \leqq 1\} \\
&= \sup \{|F_f(\hat{g})| \mid \hat{g} \in \hat{B}_p(G), \|\hat{g}\|_{p'} \leqq 1\} \\
&= \sup \{|F(f*g)| \mid \hat{g} \in \hat{B}_p(G), \|\hat{g}\|_{p'} \leqq 1\} \\
&\leqq \sup \{\|F\|\,\|f*g\|_B \mid \hat{g} \in \hat{B}_p(G), \|\hat{g}\|_{p'} \leqq 1\} \\
&\leqq \sup \{\|F\|\,\|\hat{f}\|_p\,\|\hat{g}\|_{p'} \mid \hat{g} \in \hat{B}_p(G), \|\hat{g}\|_{p'} \leqq 1\} \leqq \|F\|\,\|\hat{f}\|_p\, .
\end{aligned}$$

The penultimate inequality is due to Lemma 1(i).

Furthermore, using the fact that $B_p(G)$ is supremum norm dense in $C(G)$ and Parseval's formula we have

$$\begin{aligned}
\|Tf\|_1 &= \sup \{|\langle Tf,g\rangle| \mid |g \in C(G), \|g\|_\infty \leqq 1\} \\
&= \sup \{|\textstyle\int_G h(t)g(t^{-1})dt| \mid g \in B_p(G), \|g\|_\infty \leqq 1\} \\
&= \sup \{|\textstyle\int_G \hat{h}(\gamma)\hat{g}(\gamma)d\gamma| \mid g \in B_p(G), \|g\|_\infty \leqq 1\} \\
&= \sup \{|F_f(\hat{g})| \mid g \in B_p(G), \|g\|_\infty \leqq 1\} \\
&\leqq \sup \{\|F\|\,\|f*g\|_B \mid g \in B_p(G), \|g\|_\infty \leqq 1\} \\
&\leqq \sup \{\|F\|\,\|f\|^p\,\|g\|_\infty \mid g \in B_p(G), \|g\|_\infty \leqq 1\} \leqq \|F\|\,\|f\|^p\, .
\end{aligned}$$

The penultimate inequality is now due to Lemma 1(ii).

Combining these estimates we see at once that for each $f \in A_1(G)$,

$$\|Tf\|^p \leqq 2\|F\|\|f\|^p .$$

Hence T is continuous from $A_1(G) \subset A_p(G)$ to $A_p(G)$.

PROOF OF THEOREM 5. As mentioned before, we need only show that β is surjective. Given $F \in B_p{}'(G)$, let T be the operator defined preceding Lemma 3. In view of Lemma 3 this operator can be uniquely extended to a bounded linear operator on all of $A_p(G)$, since $A_1(G)$ is norm dense in $A_p(G)$. We shall denote this extension by T.

If $f,g \in A_1(G)$ and $s \in G$, then

$$\begin{aligned}\int_{\hat{G}} [T(\tau_s f)]\hat{}(\gamma)\hat{g}(\gamma)\,d\gamma &= F(\tau_s f * g)\\ &= F(f*\tau_s g)\\ &= \int_{\hat{G}} (Tf)\hat{}(\gamma)(\tau_s g)\hat{}(\gamma)\,d\gamma\\ &= \int_{\hat{G}} [Tf*\tau_s g]\hat{}(\gamma)\,d\gamma = \int_{\hat{G}} [\tau_s(Tf)]\hat{}(\gamma)\hat{g}(\gamma)\,d\gamma .\end{aligned}$$

Since $\hat{A}_1(G)$ is norm dense in $L_{p'}(\hat{G})$, $1/p + 1/p' = 1$, and $(Tf)\hat{} \in L_p(\hat{G})$ for each $f \in A_1(G)$, we conclude that $[T(\tau_s f)]\hat{} = [\tau_s(Tf)]\hat{}$ per each $f \in A_1(G)$ and $s \in G$. The semisimplicity of $A_1(G)$, the continuity of T and the norm denseness of $A_1(G)$ in $A_p(G)$ combine to imply that $T\tau_s = \tau_s T$ for each $s \in G$. Thus $T \in M_p$.

Moreover, if $f,g \in A_1(G)$, then

$$\begin{aligned}\beta(T)(f*g) &= \int_{\hat{G}} [T(f*g)]\hat{}(\gamma)\,d\gamma\\ &= \int_{\hat{G}} (Tf)\hat{}(\gamma)\hat{g}(\gamma)\,d\gamma = F_f(\hat{g}) = F(f*g) ,\end{aligned}$$

by the definition of T. But $\{f*g \mid f,g \in A_1(G)\}$ is norm dense in $B_p(G)$. Indeed, let $\{u_\alpha\} \subset A_1(G)$ be an approximate identity for $A_1(G)$. Then in particular we have $\lim_\alpha \|\hat{f} - \hat{f}\hat{u}_\alpha\|_1 = 0$ for each $f \in A_1(G)$. Furthermore

$$\begin{aligned}\|f - f*u_\alpha\|_B &= \sup\{|\beta(T)(f - f*u_\alpha)| \mid T \in M_p, \|T\|_p \leqq 1\}\\ &= \sup\{|\textstyle\int_{\hat{G}} \varphi(\gamma)[\hat{f}(\gamma) - \hat{f}\hat{u}_\alpha(\gamma)]d\gamma| \mid T \in M_p, \|T\|_p \leqq 1\}\\ &\leqq \|\hat{f} - \hat{f}\hat{u}_\alpha\|_1 ,\end{aligned}$$

as $\|\varphi\|_\infty \leqq \|T\|_p \leqq 1$ by [18, p. 1135]. Hence $\lim_\alpha \|f - f*u_\alpha\|_B = 0$, and $\{f*g \mid f,g \in A_1(G)\}$ is norm dense in $B_p(G)$.

Therefore $\beta(T)=F$, and β is surjective.

The next result shows that the completion $\bar{B}_p(G)$ of $B_p(G)$ can be identified with a space of continuous functions.

THEOREM 6. *Let G be an infinite compact Abelian group. For each $p>2$ there exists a continuous linear injective mapping ι of $\bar{B}_p(G)$ onto a subspace of $C(G)$.*

PROOF. From the Fourier inversion formula we see that if $f \in B_p(G)$ then for each $t \in G$,

$$\begin{aligned} |f(t)| &= \left| \int_{\hat{G}} (t,\gamma)\hat{f}(\gamma)\, d\gamma \right| \\ &= \left| \int_{\hat{G}} (\tau_{t^{-1}}f)\hat{\ }(\gamma)\, d\gamma \right| \\ &= |\beta(\tau_{t^{-1}})(f)| \\ &\leqq \sup\{|\beta(T)(f)| \mid T \in M_p, \|T\|_p \leqq 1\} = \|f\|_B . \end{aligned}$$

Hence $\|f\|_\infty \leqq \|f\|_B$ for each $f \in B_p(G)$.

Considering the elements of $\bar{B}_p(G)$ as Cauchy sequences of elements of $B_p(G)$ it is apparent from the preceding inequality that, if $\{f_n\} \subset B_p(G)$ is a Cauchy sequence in $B_p(G)$, then there exists a unique function $f \in C(G)$ such that $\lim_n \|f_n - f\|_\infty = 0$. Setting $\iota(\{f_n\}) = f$ we obtain a well defined linear mapping from $\bar{B}_p(G)$ onto a subspace of $C(G)$. It follows at once from the previous estimate that ι is a continuous mapping. The proof that ι is injective can be taken *mutatis mutandis* from [4, p. 499].

The proof in [4] carried over to the present context also immediately establishes the following corollary.

COROLLARY. *Let G be an infinite compact Abelian group. For each $p>2$ the space of finite linear combinations of the functionals $\{\beta(\tau_s) \mid s \in G\}$ is weak* dense in $B_p'(G)$.*

REMARK. It is clear that the development of this section owes a great deal to [4], where a similar characterization of the multipliers for $L_p(G)$ is studied. The work in [4] has also been extended in [5].

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WESLEYAN UNIVERSITY, CONNECTICUT, U.S.A.