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INJECTIVE MODULES OVER KRULL ORDERS

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0. Introduction.

The purpose of this article is to investigate injective modules over maximal orders over Krull domains.

For the moment consider a Krull domain A, that is a commutative integral domain A with a nonempty set of minimal nonzero prime ideals, Z say, possessing the properties

i) If $p \in Z$, then A_p is a discrete rank one valuation ring
ii) $A = \bigcap_{p \in Z} A_p$
iii) If $f \in A^*$, then f is a unit in A_p for almost all $p \in Z$.

Let $\mathscr{M}$ denote the category of A-modules. Let $\mathscr{M}_i$, $i=1,2$, denote the Serre subcategories of $\mathscr{M}$ consisting, respectively, of torsion A-modules for $i=1$ (that is, those M for which $K \otimes_A M = 0$ where K is the field of quotients of A) and pseudo-nul A-modules for $i=2$ (that is, those $M \in \mathscr{M}_1$ for which $M_p = 0$ for $p \in Z$). Then both $\mathscr{M}_1$ and $\mathscr{M}_2$ are localizing subcategories in the sense of Gabriel [6]. (See Claborn-Fossum [3] for the construction of the adjoint in case $i=2$.) The category $\mathscr{M}/\mathscr{M}_1$ is naturally equivalent to the category of K-modules.

As for $\mathscr{M}/\mathscr{M}_2$, it is seen that A is a noetherian generator of $\mathscr{M}/\mathscr{M}_2$. Furthermore, the Krull dimension of $\mathscr{M}/\mathscr{M}_2$ is one (when $A \neq K$). I. Beck has investigated this category with the idea that it behaves much like the category of modules over a Dedekind domain. In fact, Beck shows that the global injective dimension of $\mathscr{M}/\mathscr{M}_2$ is at most one. It is probably the case that the noetherian objects in $\mathscr{M}/\mathscr{M}_2$ have a decomposition which is every bit as good as that for the modules of finite type over a Dedekind domain. However that is a subject for further investigation.

The same phenomenon occurs, as is to be seen here, if one considers the category of modules over a maximal order over a Krull domain. In particular let A and K be as above. Let Σ be a central separable K-algebra. An A-*order* in Σ is an A-subalgebra Λ of Σ such that

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i) $1 \in \Lambda$
ii) $K\Lambda = \Sigma$
iii) Each element of Λ is integral over A.

Such an order always exists in Σ. An A-order is *maximal* if it is not properly contained in an A-order. I have shown that maximal A-orders exist (Fossum [5]), and in fact that any A-order is contained in a maximal A-order.

Now a maximal A-order Λ can be characterized by the following properties

i) Λ is an A-order
ii) $\Lambda = \bigcap_{p \in Z} \Lambda_p$
iii) Each Λ_p is a maximal A_p-order in Σ.

Another class of orders, the *tame orders*, are defined to be those A-orders satisfying i), ii) and

iii′) Each Λ_p is an hereditary A_p-order.

An order satisfying ii) is called a *divisorial* Λ-order.

For further properties of maximal orders which are needed in the sequel, the reader should refer to Fossum [5]. From now on Λ will denote an A-order in $\Sigma = K \otimes_A \Lambda$, a central separable K-algebra. I leave to the reader the interpretation of these results for orders in semi-simple finite dimensional K-algebras.

Let ${}_\Lambda\mathcal{M}$ denote the category of left Λ-modules considered to be a subcategory of $\mathcal{M}$. Let ${}_\Lambda\mathcal{M}_1$ and ${}_\Lambda\mathcal{M}_2$ denote the categories ${}_\Lambda\mathcal{M} \cap \mathcal{M}_1$ and ${}_\Lambda\mathcal{M} \cap \mathcal{M}_2$ respectively. It is seen that ${}_\Lambda\mathcal{M}/{}_\Lambda\mathcal{M}_1 \cong {}_\Sigma\mathcal{M}$. I wish to investigate the category ${}_\Lambda\mathcal{M}/{}_\Lambda\mathcal{M}_2$ and its associated category ${}_\Lambda\mathcal{M}^2$, the full exact subcategory of ${}_\Lambda\mathcal{M}$ which is the image in ${}_\Lambda\mathcal{M}$ of ${}_\Lambda\mathcal{M}/{}_\Lambda\mathcal{M}_2$ under the adjoint to the canonical $T\colon {}_\Lambda\mathcal{M} \to {}_\Lambda\mathcal{M}/{}_\Lambda\mathcal{M}_2$.

For this investigation some definitions are required. In particular, a left ideal I in Λ is said to be *divisorial* provided $I = \bigcap_{p \in Z} I_p$. Note that Λ has the ascending chain condition on divisorial left ideals. (In fact Λ is a divisorial lattice in the sense of Claborn–Fossum [3] and I is a divisorial sublattice. Any divisorial lattice has the ascending chain condition on divisorial sublattices.) Note that (0) is divisorial in this sense. Now a left Λ-module M is called *codivisorial* provided $\operatorname{Ann}_\Lambda x = \{\lambda \in \Lambda : \lambda x = 0\}$ is divisorial for all $x \in M$. The first result then states:

$M \in {}_\Lambda\mathcal{M}^2$ if and only if M is codivisorial.

Note that Λ is codivisorial, as is Σ. To see this, let $\sigma \in \Sigma$. Clearly

$\mathrm{Ann}_A\sigma \subseteqq \bigcap_{p\in Z}(\mathrm{Ann}\,\sigma)_p$. Suppose that $w \in \bigcap_{p\in Z}(\mathrm{Ann}_A\sigma)_p$. Now there is the exact sequence

$$0 \to \mathrm{Ann}_A\sigma \to \Lambda \to \Lambda\sigma \to 0\ ,$$

which yields

$$0 \to \mathrm{Ann}_{\Lambda_p}\sigma \to \Lambda_p \to \Lambda_p\sigma \to 0\ .$$

Hence $(w/1)\cdot\sigma = 0$, so there is a $t \notin p, t \in A$, such that $twr = 0$. But Σ is A-torsion free, so $w\sigma = 0$. Hence $w \in \mathrm{Ann}_A\,\sigma$.

The second major result is to show that localization with respect to $p \in Z$ preserves injective envelopes. This is the most important result, for it has as a corollary, that the global homological dimension of ${}_\Lambda\mathscr{M}/{}_\Lambda\mathscr{M}_2$, in terms of injective resolutions, is at most one. Further results lead to the classification of the indecomposable injectives in ${}_\Lambda\mathscr{M}/{}_\Lambda\mathscr{M}_2$.

Needless to say, most of the results reported here are anticipated by the corresponding results which hold for Krull domains. I am grateful to Dr. Istvan Beck for many stimulating discussions concerning these ideas. His results for the commutative case, and his methods, are used repeatedly here. In addition I thank Professor John A. Riley, whose invitation to speak at the George H. Hudson Symposium at the State University College at Plattsburgh, New York, stimulated me to think about orders again. In particular this paper was presented at this symposium in April, 1970.

1. Codivisorial modules.

As in the introduction, A denotes a Krull domain with field of quotients K. Z denotes the prime ideals of height one in A. Σ is a central separable K-algebra and Λ is a divisorial A-order in Σ. It is understood that Λ is a tame or maximal order if this hypothesis is needed. A left Λ-module M is said to be a *torsion* module if $K \otimes_A M = 0$, and it is *pseudo-nul* if $M_p = A_p \otimes_A M = 0$ for all $p \in Z$. When there is need to use symbols, ${}_\Lambda\mathscr{M}$, ${}_\Lambda\mathscr{M}_1$ and ${}_\Lambda\mathscr{M}_2$ denote respectively the category of all left-modules, torsion left Λ-modules and pseudo-nul left Λ-modules.

Let $\mathscr{R}$ denote the collection of those left Λ-modules M such that $\mathrm{Hom}_\Lambda(N, M) = 0$ for all pseudo-nul N. $\mathscr{R}$ is a full subcategory of ${}_\Lambda\mathscr{M}$.

PROPOSITION 1.1. *The following statements are equivalent for a left Λ-module M, where Λ is a divisorial Λ-order.*

i) *$M \in \mathscr{R}$.*
ii) *If M' is a pseudo-nul submodule of M, then $M' = 0$.*
iii) *$\mathrm{Ann}_\Lambda x$ is divisorial for all $x \in M$*, that is, $\mathrm{Ann}_\Lambda x = \bigcap_{p\in Z}(\mathrm{Ann}_\Lambda x)_p$.

Proof. i) $\Rightarrow$ ii) is clear. Suppose ii). Let $x \in M$, and let $I = \mathrm{Ann}_\Lambda x$. Let $I' = \bigcap_{p \in Z} I_p$, and let $\lambda \in I'$. Now $\Lambda\lambda x$ is a submodule of M which has the property that $(\Lambda\lambda x)_p = 0$ for all $p \in Z$. For $(\Lambda\lambda x)_p \subseteqq I_p x = 0$. Hence $\Lambda\lambda x$ is pseudo-nul, and thus $\Lambda\lambda x = 0$, so $\lambda x = 0$. Thus I is divisorial, and ii) $\Rightarrow$ iii).

Now suppose iii). Let N be pseudo-nul and

$$\alpha\colon N \to M \in \mathrm{Hom}_\Lambda(N, M) .$$

Let $x \in \mathrm{Im}\,\alpha$. Then $(\Lambda x)_p = \Lambda_p x = 0$ for all $p \in Z$, since $\mathrm{Im}\,\alpha$ is pseudo-nul. Since $\mathrm{Ann}_\Lambda x$ is divisorial, it follows that

$$\mathrm{Ann}_\Lambda x = \bigcap_{p \in Z} \Lambda_p = \Lambda$$

so $x = 0$. Hence $\alpha = 0$. So iii) $\Rightarrow$ i).

A left-module $M \in \mathscr{R}$ is called a *codivisorial* module.

Corollary 1.2. *If I is a left ideal in Λ, then I is divisorial if and only if Λ/I is codivisorial.*

It is clear that $\mathscr{R}$ is closed under the operation of taking injective envelopes. In fact, if $M' \in \mathscr{R}$ and $M' \to M''$ is an essential extension, then $M'' \in \mathscr{R}$. Furthermore, if

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence of left Λ-modules, and if M' and M'' are codivisorial, then M is codivisorial. This follows from the left exactness of $\mathrm{Hom}_\Lambda(N, -)$ and the definition of $\mathscr{R}$. However the converse of this statement does not hold, so $\mathscr{R}$ is not abelian.

2. The reflection functor.

Let M be a left Λ-module. There is a canonical homomorphism $M \to M_p$ for each $p \in Z$, which induces a homomorphism

$$d_M\colon M \to \textstyle\prod_{p \in Z} M_p .$$

It is clear that $\mathrm{Ker}\, d_M$ is pseudo-nul, and in fact is the maximal pseudo-nul submodule of M. Also $d_M(M)$ is codivisorial. For if N is pseudo-nul, $N \subseteqq d(M)$, then $d_M{}^{-1}(N)$ is pseudo-nul, so $d_M{}^{-1}(N) = \mathrm{Ker}\, d_M$ and hence $N = 0$.

Let $\alpha\colon M \to M'$. Then α induces $\alpha_p\colon M_p \to M_p{}'$, and

$$\textstyle\prod \alpha_p\colon \prod M_p \to \prod M_p{}' .$$

Thus a commutative diagram is obtained

$$\begin{array}{ccccc}
0 \to \operatorname{Ker} d_M \to & M & \xrightarrow{d_M} & \prod M_p & \\
& \Big\downarrow \alpha & & \Big\downarrow \prod\alpha_p & \\
0 \to \operatorname{Ker} d_{M'} \to & M' & \xrightarrow{d_{M'}} & \prod M_p{}' & .
\end{array}$$

So α induces $d(\alpha)\colon d(M) \to d(M')$. Thus d becomes a functor $d\colon {}_{\Lambda}\mathscr{M} \to \mathscr{R}$ which is the reflector of the inclusion functor $\mathscr{R} \to \mathscr{M}$.

If M is a torsion left Λ-module, then $\operatorname{Im} d_M = d(M)$ is actually a submodule of $\coprod_p M_p$. For if $x \in M$, there is a $0 \neq t$, $t \in \Lambda$ such that $tx = 0$. Hence $x/1 = 0$ in M_p for all $p \in Z$ except possibly those p with $t \in p$. So $d_M x \in \coprod_p M_p$.

The next two results are due to Beck [1] in the commutative case.

LEMMA 2.1. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of left Λ-modules. If M is codivisorial and M'' is pseudo-nul, then $M' \to M$ is an essential extension.*

PROOF. Let $x \in M$, $0 \neq x$. Let $\bar{x}$ denote the image of x in M''. Let $I = \operatorname{Ann}_\Lambda x$, $J = \operatorname{Ann}_\Lambda \bar{x}$. Now $J = \{\lambda \in \Lambda\colon \lambda x \in M'\}$, so $I \subseteqq J$. Since M'' is pseudo-nul, $\Lambda\bar{x} \cong \Lambda/J$ is pseudo-nul. Hence $J_p = \Lambda_p$ for all $p \in Z$. Thus $I \neq J$ (since $I = \bigcap I_p$) and thus there is an element $\lambda \in J$ such that $\lambda x \in M'$ and $\lambda x \neq 0$.

COROLLARY 2.2. *If M is a torsion left Λ-module, then $d(M) \to \coprod_{p\in Z} M_p$ is an essential extension.*

PROOF. Let $L = \operatorname{Coker}(d(M) \to \coprod_p M_p)$. Then there results the exact sequence

$$0 \to \operatorname{Ker} d_M \to M \to \coprod M_p \to L \to 0\,.$$

Applying $\Lambda_q \otimes_\Lambda -$ for $q \in Z$ yields the exact sequence

$$0 \to (\operatorname{Ker} d_M)_q \to M_q \to \coprod (M_p)_q \to L_q \to 0\,.$$

If now $p \neq q$, then $(M_p)_q \cong K \otimes_\Lambda M = 0$. Hence this sequence is just

$$0 \to 0 \to M_q \cong M_q \to 0 \to 0\,.$$

So L is pseudo-nul. By the lemma, d_M is essential, and the corollary is established.

Suppose M is a left Λ-submodule of a left Σ-module V such that $KM = V$ (that is, M spans V). Call M a *Λ-lattice* in V. Then V/M is a torsion Λ-module. Since $V_p = V$ for $p \in Z$, it follows that $(V/M)_p = V/M_p$. Hence

$$d(V/M) \stackrel{e}{\subset} \coprod_{p\in Z} V/M_p\,.$$

Observe that $\operatorname{Ker} d_M = \bigcap_{p \in Z} M_p/M$. Combining these remarks with the corollary above yields the next result.

PROPOSITION 2.3. *If M is a divisorial left Λ-lattice in V* (that is, $M = \bigcap_{p \in Z} M_p$), *then V/M is codivisorial and*

$$V/M \hookrightarrow \coprod_{p \in Z} V/M_p$$

is an essential extension whose cokernel is pseudo-nul.

3. Injective codivisorial modules.

Let E be an injective codivisorial left Λ-module. Since Λ is a divisorial A-order, it has the ascending chain condition on divisorial left ideals. Thus E is Σ-injective, that is, $E^{(T)}$ is injective for any set T (see Faith [4]). Thus I have the first result.

PROPOSITION 3.1. *An injective codivisorial left Λ-module is Σ-injective.*

It thus follows from results due to Cailleau [2] that any injective codivisorial left Λ-module is the direct sum of indecomposable injective modules.

The converse of this also holds. Any direct sum of injective codivisorial left Λ-modules is injective. It is to establish this result that the remainder of this section is devoted. There is included a proof of Cailleau's result in this special case.

For the remainder of this section I will assume that Λ is a tame A-order in Σ. The next task is to determine the indecomposable injective codivisorial left Λ-modules.

PROPOSITION 3.2. *Let E be an A-torsion free left Λ-module. E is Λ-injective if and only if E is a Σ-module. Then E is an indecomposable injective left Λ-module if and only if E is a simple left Σ-module* (*which is then an indecomposable injective left Σ-module*).

PROOF. Any non-zero element in A is regular in Λ. If E is injective as a Λ-module, it is then divisible as an A-module, uniquely since it is A-torsion free. Hence E is a K-module in a unique manner, and thus E is a left Σ-module. Since Σ is simple artinian, E is thus an injective Σ-module. E decomposes as a Λ-module. Note that Σ is an injective Λ-module, so any Σ-module is an injective Λ-module.

COROLLARY 3.3. *E is an indecomposable injective A-torsion free left Λ-module if and only if E is isomorphic to a simple Σ-module.*

This takes care of the case of Λ-torsion free injective Λ-modules. They are just Σ-modules.

LEMMA 3.4. *Let E be a left Λ_p-module. Then E is an injective left Λ-module if and only if E is an injective left Λ_p-module.*

PROOF. Let I be a left ideal of Λ. Consider the following commutative diagram, where $f\colon I \to E$ is either given or is the composite with

$$I \to I_p \xrightarrow{f_p} E$$

for some given $f_p\colon I_p \to E$. (Since E is an Λ_p-module $f\colon I \to E$ extends uniquely to $f_p\colon I_p \to E$.)

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[rr] \arrow[dd] \arrow[dr, "f"] & & \Lambda \arrow[dd] \arrow[dl, dashed, "g"'] \\
& & E & \\
0 \arrow[r] & I_p \arrow[rr] \arrow[ur, "f_p"] & & \Lambda_p \arrow[ul, dashed, "g_p"']
\end{tikzcd}$$

If $f\colon I \to E$ is given and E is an injective Λ_p-module, then f_p can be extended to a map $g_p\colon \Lambda_p \to E$. Composing this with $\Lambda \to \Lambda_p$ gives a $g\colon \Lambda \to E$ which restricts to f on I.

If E is an injective Λ-module, then f extends to g which extends to $g_p\colon \Lambda_p \to E$ and $g_p \mid I_p = f_p$.

PROPOSITION 3.5. (a) *If M is a maximal divisorial left ideal, then $E(\Lambda_p/M_p)$ is an indecomposable injective codivisorial Λ-module.*

(b) *Let E be a torsion codivisorial Λ-module. If E is an indecomposable injective Λ-module, then there is a maximal divisorial left ideal M such that E is isomorphic to $E(\Lambda/M)$.*

PROOF. (a) By Corollary 1.2, Λ/M is codivisorial. It is also a torsion module. Hence $\Lambda/M \to \coprod(\Lambda/M)_q$ is essential. But since M is a maximal divisorial left Λ-ideal, it follows that $\Lambda_q = M_q$ for all $q \in Z$ except one, say $p = q$, and in this case M_p is a maximal left Λ_p-ideal. So $\Lambda/M \to (\Lambda/M)_p$ is an essential extension. Now the injective envelope of Λ_p/M_p as a Λ_p-module is an injective Λ-module, so it contains $E_\Lambda(\Lambda/M)$ as a direct summand. And $E_\Lambda(\Lambda/M)$, being the maximal essential extension of Λ/M in $E_{\Lambda_p}(\Lambda_p/M_p)$, contains Λ_p/M_p and is an essential extension of Λ_p/M_p as a Λ-module. Thus $E_\Lambda(\Lambda/M) = E_{\Lambda_p}(\Lambda_p/M_p)$ which is indecomposable as a Λ-module and so as a Λ-module (since M_p is a maximal left ideal). This proves (a).

(b) Since E is codivisorial and torsion, $E \to \coprod_p E_p$ is an essential extension. If E is injective, then this map is an isomorphism. Hence $E = E_p$ for exactly one p, and $E_q = 0$ if $p \neq q$. Now E is indecomposable as a Λ_p-module if and only if E is indecomposable as a Λ-module. If it is indecomposable as a Λ_p-module, then $E = E_{\Lambda_p}(\Lambda_p/M_p)$ for some maximal left ideal M_p of Λ_p. Let $M = \Lambda \cap M_p$. Then $E = E(\Lambda/M)$ where M is a maximal divisorial left Λ-ideal. This proves (b).

Now all torsion indecomposable codivisorial injective left Λ-modules have been found.

PROPOSITION 3.6. *Let $\{E_\alpha\}_{\alpha \in \mathscr{A}}$ be a family of torsion codivisorial indecomposable injective left Λ-modules. Then $\coprod_{\alpha \in \mathscr{A}} E_\alpha$ is an injective Λ-module.*

PROOF. Let $E = \coprod_{\alpha \in \mathscr{A}} E_\alpha$. Then $E_p = \coprod_{\alpha \in \mathscr{A}} (E_\alpha)_p$ is a direct sum of $E(\Lambda_p/M_p)$ which is Σ-injective (since Λ_p is noetherian). Hence each E_p is injective. Let I be a left ideal in Λ. Let $f \colon I \to E$. Suppose $J = \operatorname{Ker} f$. Since $K \otimes_\Lambda E = 0$, it follows that $KJ = KI$. Hence $J_q = I_q$ for almost all $q \in Z$. Now

$$J_q = \operatorname{Ker} f_q = \operatorname{Ker}(I_q \to E_q) .$$

Thus $\operatorname{Im} f$ is contained in only a finite number of the E_p. (Note that $E = \coprod_{p \in Z} E_p$.) Since this finite direct sum is injective, f can be extended to Λ. Hence E is injective.

THEOREM 3.7. *Let E be a codivisorial left Λ-module. Then E is injective if and only if E is the direct sum of indecomposable injective codivisorial left Λ-modules.*

PROOF. It has been seen that the direct sum of torsion indecomposable injective codivisorial left Λ-modules is injective. If one adds to such a module an A-torsion free injective left Λ-module, the sum is still injective.

I now give a proof of Cailleau's result in my special case. Suppose on the other hand that E is an injective codivisorial left Λ-module. Let $0 \neq x \in E$ be an element such that $\operatorname{Ann}_\Lambda x$ is maximal among the set of divisorial ideals $\{\operatorname{Ann}_\Lambda y;\ 0 \neq y \in E\}$. Then I claim that $E(\Lambda x) \cong E(\Lambda/\operatorname{Ann}_\Lambda x)$ is indecomposable ($E(\Lambda x)$ is taken to be a submodule of E). For suppose $E(\Lambda x) = E' \oplus E''$. Let $x = e' + e''$ with $e' \in E'$, $e'' \in E''$. Now

$$\operatorname{Ann}_\Lambda x \subseteq \operatorname{Ann}_\Lambda e' \cap \operatorname{Ann}_\Lambda e'' .$$

Hence $\operatorname{Ann}_\Lambda x = \operatorname{Ann}_\Lambda e' = \operatorname{Ann}_\Lambda e''$ or else $e' = 0$ or $e'' = 0$ (in which cases $E' = 0$ or $E'' = 0$). Since $E(\Lambda x)$ is an essential extension of Λx, it follows that if $e' \neq 0$, there are $\lambda, \tau \in \Lambda$ such that $\lambda e' = \tau x \neq 0$ (that is, $0 \neq \lambda e' \in \Lambda x$). Now

$$\tau x = \tau e' + \tau e'' = \lambda e' .$$

Hence $\tau e'' \in E'' \cap E' = 0$. So $\tau \in \mathrm{Ann}_\Lambda e''$. If $e'' \neq 0$, then $0 = \tau x = \lambda e'$, a contradiction. Hence $e' \neq 0$ implies $e'' = 0$. Now $e'' = 0$ implies that $E'' = 0$. Hence $E(\Lambda x)$ is indecomposable.

Now the usual argument shows that E is equal to the submodule obtained by taking a maximal direct sum of indecomposable injective submodules. So E is such a direct sum.

Note: This proof works for any Σ-injective module.

4. The module $E(\Sigma/\Lambda)$.

Let Λ be a tame A-order in this section.

Consider the torsion codivisorial left and right Λ-module Σ/Λ. Now $\Sigma/\Lambda \to \coprod_p \Sigma/\Lambda_p$, as both left and right Λ-modules, is an essential extension, as was seen in section 2. Since Λ_p is hereditary, and Σ is an injective left and right Λ_p-module, it follows that Σ/Λ_p is an injective left and right Λ-module. Hence

$$I = E(\Sigma/\Lambda) = \coprod_p \Sigma/\Lambda_p$$

considered either as a left Λ-module or as a right Λ-module is injective, by the results of the last section.

Let N be a left Λ-module. Consider the right module $\mathrm{Hom}_\Lambda(N,I)$. (If α: $N \to I$ and $\lambda \in \Lambda$, then $x(\alpha\lambda) = (x\alpha)\lambda$ for $x \in N$.) $\mathrm{Hom}_\Lambda(N,I)$ is a codivisorial right Λ-module. For if α: $N \to I$, then $\mathrm{Ann}_\Lambda \alpha = \bigcap_{x \in N} \mathrm{Ann}_\Lambda x\alpha$.

If N is a left Λ-module, there is induced a canonical homomorphism

$$q = q_N: \quad N \to \mathrm{Hom}_\Lambda(\mathrm{Hom}_\Lambda(N,I), I)$$

of left Λ-modules given by $(xq)\alpha = x\alpha$ for $x \in N$, $\alpha \in \mathrm{Hom}_\Lambda(N,I)$.

Let me check that this is a homomorphism of left Λ-modules. Let $x \in N$, $\lambda \in \Lambda$ and α: $N \to I$. Now

$$(\lambda x)q\,\alpha = (\lambda x)\alpha = \lambda(x\alpha) = \lambda\cdot(xq\alpha) = (\lambda\cdot xq)\alpha .$$

Hence $(\lambda x)q = \lambda\cdot xq$. (Note: I am using the convention of letting maps act on the side of an element opposite to the side on which a ring element acts.)

It is now possible to add another condition to those in Proposition 1.1.

Proposition 4.1. *Let Λ be a tame A-order. Then a left Λ-module N is pseudo-nul if and only if* $\mathrm{Hom}_\Lambda(N,I) = 0$.

PROOF. If N is pseudo-nul, then $\mathrm{Hom}_\Lambda(N,I)=0$ since I is codivisorial. Suppose, on the other hand, that N is not pseudo-nul. Then $d(N)\neq 0$. Let $E=E(d(N))$. $d(N)$ is codivisorial, so $E(d(N))$ is codivisorial. Let $E(d(N))=\coprod E_\alpha$ where E_α are indecomposable injective Λ-modules. Let $p\in Z$, $\alpha\in\mathscr{A}$ be such that $(E_\alpha)_p\neq 0$. Then there is a nonzero homomorphism $E_\alpha=(E_\alpha)_p\to\Sigma/\Lambda_p$. To see this, suppose that E_α is A-torsion-free. Then E_α is a simple Σ-module so isomorphic to a minimal left ideal of Σ. This minimal left ideal cannot go to zero in the homomorphism $\Sigma\to\Sigma/\Lambda_p$. If E_α is A-torsion, then $E_\alpha=E_{\Lambda_p}(\Lambda_p/M_p)$ is an indecomposable injective Λ_p-module, where M_p is a maximal left ideal of Λ_p. This is seen to have a nonzero map into Σ/Λ_p by the structure of Λ_p as an hereditary Λ_p-order. Thus the composite

$$N\to d(N)\to E_\alpha\to\Sigma/\Lambda_p$$

is not zero so $\mathrm{Hom}_\Lambda(N,I)\neq 0$.

REMARK. The last part of the above proof consisted in remarking that I contains a copy of each torsion indecomposable injective codivisorial left Λ-module.

PROPOSITION 4.2. *Let Λ be a tame A-order. The following are equivalent.*
i) *M is codivisorial.*
ii) *$q_M\colon M\to\mathrm{Hom}_\Lambda(\mathrm{Hom}_\Lambda(M,I),I)$ is an injection.*

PROOF. It has been observed above that $\mathrm{Hom}_\Lambda(N,I)$ is codivisorial. Suppose that M is codivisorial. Let $0\neq x\in M$. Now Λx is not pseudo-nul, so there is an $0\neq\bar\alpha\colon\Lambda x\to I$. Extend $\bar\alpha$ to $\alpha\colon M\to I$. Then $x\alpha\neq 0$ so $xq_M\neq 0$. This can be done for each $x\in M$, $x\neq 0$ so q_M is an injection.

REMARK. The most of these results could be established for a divisorial A-order Λ such that $\mathrm{inj\,dim}_\Lambda\Lambda\leqq 1$.

It may be enlightening to examine the relation of the category ${}_\Lambda\mathscr{M}/{}_\Lambda\mathscr{M}_2$ to a certain subcategory of the left modules over the ring

$$\prod_p\mathrm{Hom}_{\Lambda_p}(\Sigma/\Lambda_p,\Sigma/\Lambda_p)=\mathrm{Hom}_\Lambda(I,I)\,.$$

Such an investigation, along the lines suggested by Roos in his work on locally noetherian categories [7], could shed much more light on the category ${}_\Lambda\mathscr{M}/{}_\Lambda\mathscr{M}_2$.

It is clear, for example, that $\mathrm{Hom}_\Lambda(\mathrm{Hom}_\Lambda(\cdot,I),I)$ induces an exact functor from ${}_\Lambda\mathscr{M}/{}_\Lambda\mathscr{M}_2$ to this category of modules. Which subcategory is the image?

5. Essential extensions of pseudo-nul modules are pseudo-nul.

The main object of this section is to show that the category of pseudo-nul left Λ-modules is closed under essential extension. This shows that, in particular, the injective envelope of a pseudo-nul module is pseudo-nul.

The A-order Λ is maximal in this section.

THEOREM 5.1. *Let $N' \to N$ be an essential extension of N'. If N' is pseudo-nul, then N is pseudo-nul.*

The first step is to show that N is an A-torsion module. This can be stated as a separate result.

PROPOSITION 5.2. *Let $N' \to N$ be an essential extension of the torsion left Λ-module N'. Then N is torsion.*

PROOF. Suppose $x \in N$ is such that $\mathrm{Ann}_A x = (0)$. Let $I = \mathrm{Ann}_\Lambda x$, so there is the exact sequence $0 \to I \to \Lambda \to \Lambda x \to 0$. Now tensor this with K to get

$$0 \to K \otimes_A I \to \Sigma \to K \otimes_A \Lambda x \to 0$$

or $0 \to \Sigma I \to \Sigma \to \Sigma x \to 0$, with $\Sigma x \neq (0)$. Now $\Sigma I = \Sigma e$, e an idempotent. So, if $\lambda \in I$, then $\lambda e = \lambda$, and conversely. Now Λx is A-torsion free. For suppose $a\lambda x = 0$ for some $\lambda \in \Lambda$, $a \in A$, $a \neq 0$. Then $a\lambda e = a\lambda$, and hence $a(\lambda e - \lambda) = 0$. But Λ is A-torsion free, so $\lambda e = \lambda$. Thus $\lambda x = \lambda e x = 0$.

Since $\Lambda x \neq (0)$, and $N' \to N$ is essential $\Lambda x \cap N' \neq (0)$. But $\Lambda x \cap N'$ is a submodule of a torsion free module, so is torsion free. This is a contradiction, for N' is a torsion left Λ-module.

LEMMA 5.3. *Let $N' \to N$ be an essential extension of the pseudo-nul left Λ-module N'. If $N_p \neq (0)$, then there is an $x \in N$ such that*

$$\mathrm{Ann}_A x = \{a \in A : ax = 0\} = \mathrm{p}, \quad p \in Z .$$

PROOF. By the proposition above N is a torsion left Λ-module. If $N_p \neq (0)$, there is an $x \in N$ such that $(0) \neq \mathrm{Ann}_A x \subseteqq p$. By Proposition 3.2 of Beck [1], there is a $t \in A$ such that

$$p = \{f \in A : ft \in \mathrm{Ann}_A x\} .$$

Then $\mathrm{Ann}_A tx = p$. For if $f \in A$ is such that $f(tx) = 0$, then $ft \in \mathrm{Ann}_A x$, so $f \in p$. And $ptx \subseteqq (\mathrm{Ann}_A x)x = 0$.

LEMMA 5.4. *With the same $N' \to N$ as above, and $N_p \neq (0)$, there is a $y \in N$ such that*

$$\mathrm{Ann}_\Lambda(\Lambda y) = P = (\mathrm{Rad}\,\Lambda_p) \cap \Lambda .$$

PROOF. If $x \in N$ is choosen so that $\operatorname{Ann}_A x = p$, then $Q = \operatorname{Ann} \Lambda x = \{\lambda \in \Lambda \colon \lambda \Lambda x = 0\}$ is an ideal which intersects A to p. Now any such ideal is contained in P. For $Q_q = \Lambda_q$ for any $q \in Z$, $q \neq p$, and Q_p is an ideal in Λ_p. But Λ_p has a unique maximal ideal $\operatorname{Rad} \Lambda_p$. So

$$Q \subseteq \cap_{q \in Z} Q_q \subseteq \operatorname{Rad} \Lambda_p \cap \Lambda = P .$$

Now using an argument similar to Beck's in the proof of his Proposition 3.2 [ibid] I will demonstrate the lemma.

$P_p = \operatorname{Rad} \Lambda_p$ is a principal left and right ideal. Let $u \in \Lambda_p$ be such that $P_p = u\Lambda_p = \Lambda_p u$. u can be picked such that $u^{-1} \in \Lambda_q$ for all $q \neq p$. So $Pu^{-1} \subseteq \Lambda$ and $u^{-1}P \subseteq \Lambda$. Furthermore, $Q_p = P_p{}^r$ for some positive integer r.

Since $Pu^{-1}Q \subseteq Q$ and $u^{-1}Q \subseteq u^{-1}P \subseteq \Lambda$,

$$u^{-1}Q \subseteq \{\lambda \in \Lambda \colon P\lambda \subseteq Q\} .$$

Furthermore $u^{-1}Q \nsubseteq Q_p$. For otherwise, $u^{-1}Q \subseteq u^{-1}Q_p \subseteq Q_p$. Since Λ_p is a maximal order this says $u^{-1} \in \Lambda_p$, a contradiction ($P_p = \Lambda_p$ if $u^{-1} \in \Lambda_p$). Let $q \in Q$ be such that $q/t = u^r$ for some $t \notin p$ ($Q_p = u^r \Lambda_p = \Lambda_p u^r$, so there are $q \in Q$, $t \in A - p$ such that $u^r = q/t$). Hence $q = u^r t \in Q$ and $qu^{-1} = u^{r-1}t$ is not in Q, for otherwise $Q = u^{r-1}\Lambda_p$, a contradiction.

Now $P = \{\lambda \in \Lambda \colon \lambda \Lambda q u^{-1} \subseteq Q\}$. For clearly $P\Lambda u^{-1} = Pu^{-1}q \subseteq Q$. On the other hand, if $\lambda \Lambda u^{-1} q \subseteq Q$ then $\lambda \Lambda_p u^{-1} q \subseteq Q_p$ so $\lambda \Lambda_p t u^{r-1} \subseteq \Lambda_p u^r$, and hence $\lambda \Lambda_p \subseteq \Lambda_p u$, so $\lambda \in P_p$, $\lambda \in \Lambda$, so $\lambda \in P$.

Let $y = u^{-1}qx$. If $\lambda \Lambda y = \lambda \Lambda u^{-1} q x = 0$, then $\lambda \Lambda u^{-1} q \subseteq Q$, so $\lambda \in P$. Further

$$P\Lambda y = Pu^{-1}qx \subseteq Qx = 0 .$$

So $P = \operatorname{Ann} \Lambda y$.

Now I proceed with $\operatorname{Ann} \Lambda y$, $y \in N$ and P as in this lemma.

Since Λ_p is of finite type as an A_p-module, and since $P_p = \operatorname{Ann} \Lambda_p y$, there are a finite number of elements $\lambda_1, \ldots, \lambda_r \in \Lambda$ such that $P_p = \cap \operatorname{Ann}_{\Lambda_p} \lambda_i y$. (Condition (H) for the A_p-algebra Λ_p. See Gabriel [6].) I claim that $P = \cap_{i=1}^r \operatorname{Ann}_\Lambda \lambda_i y$.

For $P\lambda_i y \subseteq Py = 0$, so $P \subseteq \cap \operatorname{Ann}_\Lambda \lambda_i y$. If $\lambda \lambda_i y = 0$ for $i = 1, \ldots, r$, then $\lambda \lambda_i y = 0$ in $\Lambda_p y$ so $\lambda \in P_p$. Hence $\lambda \in P$. Thus there is an injection

$$\Lambda/P \to \coprod_{i=1}^r \Lambda \lambda_i y \to \coprod_{i=1}^r \Lambda y .$$

Now Λy is an essential extension of $\Lambda y \cap N'$ and hence $\coprod_{i=1}^r \Lambda y$ is an essential extension of $\coprod_{i=1}^r \Lambda y \cap N'$. So

$$(\Lambda/P) \cap \coprod_{i=1}^r (\Lambda y) \cap N' \neq (0) .$$

But P is the annihilator of each nonzero submodule of Λ/P. On the other hand, since N' is pseudo-nul, an element $z \in N'$ always admits an element $t \in A - p$ such that $tz = 0$. But $P \cap A = p$. So $tz = 0 \in \operatorname{Ann} \Lambda y$, so $t \in p$ a contradiction.

Thus, if N' is pseudo-nul, so is N.

There are several consequences of this result.

COROLLARY 5.5. *If N is a pseudo-nul left Λ-module, then its injective envelope $E(N)$ is a pseudo-nul Λ-module.*

Let E be an injective left Λ-module. Then $\operatorname{Ker} d_E$ is the maximal pseudo-nul submodule of E. The injective envelope of $\operatorname{Ker} d_E$ is pseudo-nul, by the last Corollary, and can be taken in E. Hence $\operatorname{Ker} d_E$ is njective. Thus $E \cong \operatorname{Ker} d_E \oplus d(E)$. Hence $d(E)$ is injective. This gives

COROLLARY 5.6. *If E is an injective left Λ-module, then the maximal pseudo-nul submodule of E is an injective left Λ-module. The module*

$$d(E) = \operatorname{Im}(E \xrightarrow{d_E} \prod E_p)$$

is an injective codivisoral left Λ-module and

$$E \cong \operatorname{Im} d_E \oplus \operatorname{Ker} d_E .$$

Since $\operatorname{Im} d_E = d(E)$ is codivisorial, and injective, it is a direct sum of its indecomposable injective submodules. The torsion free indecomposable components can be collected together to give

$$d(E) = td(E) \oplus fd(E) ,$$

where $td(E)$ is the torsion submodule of $d(E)$ and $fd(E)$ is the torsion free submodule. (In fact $fd(E) \cong K \otimes_A d(E) \cong K \otimes_A E$.) Now $td(E)$, being a torsion A-module, has the module $\coprod (td(E))_p$ as an essential extension by Corollary 2.2. Thus each $td(E)_p$ is injective. Now

$$d(E)_p \cong td(E)_p \oplus (fd(E))_p = E_p ,$$

so I can conclude

COROLLARY 5.7. *If E is an injective left Λ-module, then E_p is an injective left Λ_p-module.*

COROLLARY 5.8. *Let M be a left Λ-module. Let*

$$0 \to M \to E_0 \to E_1 \to \dots$$

be an (minimal) injective resolution of M. Then

$$0 \to M_p \to (E_0)_p \to \dots$$

is an (minimal) injective resolution of M_p for each $p \in Z$. Hence E_i is pseudo-nul for each $i \geqq 2$, if the resolution is minimal.

PROOF. Each Λ_p is hereditary.

COROLLARY 5.9. *If Λ is a maximal A-order, then for all $N \in {}_\Lambda\mathscr{M}/{}_\Lambda\mathscr{M}_2$;*

(a) $\mathrm{Ext}^i_{{}_\Lambda\mathscr{M}/{}_\Lambda\mathscr{M}_2}(\cdot, N) = 0$ for $i > 2$,

and

(b) $hd({}_\Lambda\mathscr{M}/{}_\Lambda\mathscr{M}_2) \leqq 1$.

PROOF. Let $T\colon {}_\Lambda\mathscr{M} \to {}_\Lambda\mathscr{M}/{}_\Lambda\mathscr{M}_2$ be the quotient functor. By Corollary 5.5, ${}_\Lambda\mathscr{M}_2$ is closed under injective envelopes. So $T(E)$ is injective if E is injective by Cor. 3 to Prop. 6 of Chap. III, § 3 of Gabriel [6]. Let M be a left Λ-module,

$$0 \to M \to E_0 \to E_1 \to \ldots$$

exact in ${}_\Lambda\mathscr{M}$ with each E_i injective. Then

$$0 \to TM \to TE_0 \to TE_1 \to 0$$

is exact by Corollary 5.8, and each TE_i is injective. Hence (a) and (b) follow.

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