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POWER SUMS OF INTEGERS WITH MISSING DIGITS

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1.

In this paper $\sum^*_n$ denotes summation over all n not containing the digits $s_1,\ldots,s_m$ when expressed in the scale of r. Let

$$S(\lambda,K;r;s_1,\ldots,s_m) = S(\lambda,K) = \textstyle\sum^*_{1\leqq n<r^K} n^\lambda ,$$
$$S(\lambda;r;s_1,\ldots,s_m) = S(\lambda) = \lim_{K\to\infty} S(\lambda,K) ,$$

where $S(\lambda)$ is defined only if the limit exists. It is well-known that $S(-1)$ exists (when $m\geqq 1$), cf. [1; *p.* 120]. In the sequel we find expressions for $S(\lambda,K)$, $\lambda\geqq 0$, λ integer. Further we find formulae giving upper and lower bounds for $S(\lambda)$, $\lambda<0$. In particular we give numerical bounds for $S(-1;r;r-1)$ when $3\leqq r\leqq 20$.

2.

Put

$$\begin{aligned} S'(\lambda,K) &= S(\lambda,K) && \text{when } \lambda > 0 ,\\ &= S(0,K)+1 && \text{when } \lambda = 0 . \end{aligned} \tag{2.1}$$

Let $\lambda\geqq 0$, $K\geqq 0$. Then

$$\begin{aligned} S'(\lambda,K+1) &= \textstyle\sum^*_{0\leqq a<r} \sum^*_{0\leqq b<r^K} (ar^K+b)^\lambda \\ &= \textstyle\sum^*_{0\leqq a<r} \sum^*_{0\leqq b<r^K} \sum^\lambda_{j=0} \binom{\lambda}{j} (ar^K)^{\lambda-j} b^j \\ &= \textstyle\sum^\lambda_{j=0} \binom{\lambda}{j} r^{K(\lambda-j)} \sum^*_{0\leqq a<r} a^{\lambda-j} \sum^*_{0\leqq b<r^K} b^j . \end{aligned}$$

Put, for $\lambda\geqq 0$,

$$u(\lambda) = S'(\lambda,1) = \textstyle\sum^*_{0\leqq n<r} n^\lambda = \sum^{r-1}_{n=0} n^\lambda - \sum^m_{i=1} s_i^\lambda . \tag{2.2}$$

Explicit expressions for the $u(\lambda)$'s may be found by using the well-known expressions for the ordinary power sums. In particular we note that

$$u(0) = r-m . \tag{2.3}$$

With this notation we get

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(2.4) $$S'(\lambda,K+1) = u(0)\,S'(\lambda,K)+\sum_{j=0}^{\lambda-1}\binom{\lambda}{j}u(\lambda-j)r^{K(\lambda-j)}S'(j,K)\,,$$

where the last sum is 0 when $\lambda=0$. Applying (2.4) repeatedly we obtain

(2.5) $$S'(\lambda,K+1) = u(0)^k S'(\lambda,K+1-k)+ \\ +\sum_{j=0}^{\lambda-1}\binom{\lambda}{j}u(\lambda-j)\sum_{L=0}^{k-1}u(0)^L r^{(K-L)(\lambda-j)}S'(j,K-L)\ .$$

Put

(2.6) $$T(\lambda,j,K) = \sum_{L=0}^{K-1}u(0)^L r^{(K-L)(\lambda-j)}S'(j,K-L)\,.$$

Then, with $k=K$ in (2.5), we get

(2.7) $$S'(\lambda,K+1) = u(0)^K u(\lambda)+\sum_{j=0}^{\lambda-1}\binom{\lambda}{j}u(\lambda-j)\,T(\lambda,j,K)\,.$$

In particular

(2.8) $$S'(0,K) = u(0)^K\,.$$

We now prove by induction the following theorem.

THEOREM 1. *For $\lambda \geqq 0$ and $K\geqq 0$ we have*

$$S'(\lambda,K) = \sum_{\nu=0}^{\lambda}\alpha(\lambda,\nu)\left(r^\nu u(0)\right)^K\,,$$

where the constants $\alpha(\lambda,\nu)$ are given by

$$\alpha(0,0) = 1;$$
$$\alpha(\lambda,0) = \frac{u(\lambda)}{u(0)}-\sum_{k=1}^{\lambda}\sum_{\nu=k}^{\lambda}\binom{\lambda}{k}\frac{u(k)}{u(0)}\alpha(\lambda-k,\nu-k)\frac{r^\nu}{r^\nu-1},\qquad \lambda>0;$$
$$\alpha(\lambda,\nu) =\sum_{k=1}^{\nu}\binom{\lambda}{k}\frac{u(k)}{u(0)}\alpha(\lambda-k,\nu-k)\frac{1}{r^\nu-1},\qquad \lambda>0,\ 0<\nu\leqq\lambda\,.$$

First we see that (2.8) proves the theorem for $\lambda=0$. By an easy induction we see that

(2.9) $$\sum_{\nu=0}^{\lambda}\alpha(\lambda,\nu) = 0 \quad \text{for } \lambda>0\,.$$

This proves the theorem for $K=0$ and all λ. Now suppose the theorem is true for all $\lambda'<\lambda$. Then, by (2.6),

$$\begin{aligned}T(\lambda,j,K) &= \sum_{L=0}^{K-1}u(0)^L\, r^{(K-L)(\lambda-j)}\sum_{\mu=0}^{j}\alpha(j,\mu)\left(r^\mu u(0)\right)^{K-L}\\ &= \sum_{\mu=0}^{j}\alpha(j,\mu)\,u(0)^K\sum_{L=0}^{K-1}r^{(\lambda-j+\mu)(K-L)}\,.\end{aligned}$$

Combining with (2.7) we get

$$S'(\lambda,K+1) \\ = u(0)^K u(\lambda)+\sum_{j=0}^{\lambda-1}\binom{\lambda}{j}u(\lambda-j)\sum_{\mu=0}^{j}\alpha(j,\mu)\,u(0)^K\frac{r^{(\lambda-j+\mu)(K+1)}-r^{\lambda-j+\mu}}{r^{\lambda-j+\mu}-1}\,.$$

Putting $k=\lambda-j$, $k=1,2,\ldots,\lambda$, and $\nu=k+\mu=\lambda-j+\mu$, $\nu=k,k+1,\ldots,\lambda$, we obtain

$$S'(\lambda,K+1)$$
$$=\frac{u(\lambda)}{u(0)}u(0)^{K+1}+\sum_{k=1}^{\lambda}\sum_{\nu=k}^{\lambda}\binom{\lambda}{k}\frac{u(k)}{u(0)}\alpha(\lambda-k,\nu-k)\frac{(r^{\nu}u(0))^{K+1}-r^{\nu}u(0)^{K+1}}{r^{\nu}-1}$$
$$=\sum_{\nu=0}^{\lambda}\alpha(\lambda,\nu)(r^{\nu}u(0))^{K+1}\,,$$

which proves the theorem for λ. This completes the induction.

3.

We shall need the following lemma, verification of which is easy and will be omitted.

LEMMA. *If $x\neq 0$ and $x+y\neq 0$, then*

$$\frac{1}{(x+y)^{\lambda}}=\sum_{\delta=0}^{\varepsilon}\binom{-\lambda}{\delta}\frac{y^{\delta}}{x^{\lambda+\delta}}+\sum_{u=1}^{\lambda}(-1)^{\varepsilon+u}\binom{\lambda+\varepsilon}{\lambda-u}\frac{y^{\varepsilon+u}}{x^{\lambda+\varepsilon}(x+y)^{u}}.$$

Put

$$\Omega(\lambda,K,L)=\sum\nolimits^{*}_{r^{L}\leqq n<r^{K}}n^{-\lambda}=S(-\lambda,K)-S(-\lambda,L)\,, \tag{3.1}$$

$$R(\varepsilon,K,l)=\sum\nolimits^{*}_{1\leqq a<r^{K}}\sum\nolimits^{*}_{0\leqq b<r^{lK}}\sum_{u=1}^{\lambda}\binom{\lambda+\varepsilon}{\lambda-u}(-1)^{u}\frac{b^{\varepsilon+u}}{(r^{lK}a)^{\lambda+\varepsilon}(ar^{lK}+b)^{u}}. \tag{3.2}$$

Then

$$S(-\lambda,kK)=\sum_{l=0}^{k-1}\Omega(\lambda,(l+1)K,lK)\,. \tag{3.3}$$

By the lemma,

$$\begin{aligned}\Omega(\lambda,(l+1)K,lK)&=\sum\nolimits^{*}_{1\leqq a<r^{K}}\sum\nolimits^{*}_{0\leqq b<r^{lK}}(ar^{lK}+b)^{-\lambda}\\&=\sum\nolimits^{*}_{1\leqq a<r^{K}}\sum\nolimits^{*}_{0\leqq b<r^{lK}}\left\{\sum_{\delta=0}^{\varepsilon}\binom{-\lambda}{\delta}\frac{b^{\delta}}{(ar^{lK})^{\lambda+\delta}}+\right.\\&\quad\left.+\sum_{u=1}^{\lambda}(-1)^{\varepsilon+u}\binom{\lambda+\varepsilon}{\lambda-u}\frac{b^{\varepsilon+u}}{(ar^{lK})^{\lambda+\varepsilon}(ar^{lK}+b)^{u}}\right\}\\&=\sum_{\delta=0}^{\varepsilon}\binom{-\lambda}{\delta}r^{-lK(\lambda+\delta)}S'(\delta,lK)S(-\lambda-\delta,K)+(-1)^{\varepsilon}R(\varepsilon,K,l)\,.\end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4) gives

$$\begin{aligned}S(-\lambda,kK)&=\Omega(\lambda,K,0)+\sum_{l=1}^{k-1}\Omega(\lambda,(l+1)K,lK)\\&=S(-\lambda,K)+\sum_{\delta=0}^{\varepsilon}\binom{-\lambda}{\delta}S(-\lambda-\delta,K)u(K,\delta,k)+\\&\quad+\sum_{l=1}^{k-1}(-1)^{\varepsilon}R(\varepsilon,K,l)\,,\end{aligned} \tag{3.5}$$

where

$$u(K,\delta,k) = \sum_{l=1}^{k-1} r^{-lK(\lambda+\delta)} S'(\delta,lK) = \sum_{l=1}^{k-1} r^{-lK(\lambda+\delta)} \sum_{\mu=0}^{\delta} \alpha(\delta,\mu)\big(r^{\mu}u(0)\big)^{lK} = \sum_{\mu=0}^{\delta} \alpha(\delta,\mu) \sum_{l=1}^{k-1}\big(r^{\mu-\lambda-\delta}u(0)\big)^{lK} . \tag{3.6}$$

Put

$$u(K,\delta) = \sum_{\mu=0}^{\delta}\alpha(\delta,\mu)\frac{\big(r^{\mu-\lambda-\delta}u(0)\big)^{K}}{1-\big(r^{\mu-\lambda-\delta}u(0)\big)^{K}}.$$

Then, by (3.6),

$$u(K,\delta) = \lim_{k\to\infty} u(K,\delta,k) . \tag{3.7}$$

Let

$$\begin{aligned} \underline{P}(v,u,K) &= S(-v-u,K) && \text{when } u \text{ is odd },\\ &= \textstyle\sum^{*}_{1\leqq a<r^K} a^{-v}(a+1)^{-u} && \text{when } u \text{ is even}; \end{aligned} \tag{3.8}$$

$$\begin{aligned} \overline{P}(v,u,K) &= \textstyle\sum^{*}_{1\leqq a<r^K} a^{-v}(a+1)^{-u} && \text{when } u \text{ is odd },\\ &= S(-v-u,K) && \text{when } u \text{ is even}; \end{aligned} \tag{3.9}$$

$$\begin{aligned} \underline{P}(v,u) &= \lim_{K\to\infty}\underline{P}(v,u,K),\\ \overline{P}(v,u) &= \lim_{K\to\infty}\overline{P}(v,u,K) . \end{aligned} \tag{3.10}$$

We note that

$$ar^{lK} \leqq ar^{lK}+b < (a+1)r^{lK} \quad \text{when } 0\leqq b<r^{lK} . \tag{3.11}$$

Hence, by (3.2), (3.8), and (3.9),

$$\sum_{l=1}^{k-1} R(\varepsilon,K,l) \leqq \sum_{l=1}^{k-1}\sum_{u=1}^{\lambda}\binom{\lambda+\varepsilon}{\lambda-u}(-1)^{u} S'(\varepsilon+u,lK)\overline{P}(\lambda+\varepsilon,u,K)r^{-lK(\lambda+\varepsilon+u)} , \tag{3.12}$$

$$\sum_{l=1}^{k-1} R(\varepsilon,K,l) \geqq \sum_{l=1}^{k-1}\sum_{u=1}^{\lambda}\binom{\lambda+\varepsilon}{\lambda-u}(-1)^{u} S'(\varepsilon+u,lK)\underline{P}(\lambda+\varepsilon,u,K)r^{-lK(\lambda+\varepsilon+u)} . \tag{3.13}$$

Let

$$r(\varepsilon,K) = \lim_{k\to\infty}\sum_{l=1}^{k-1} R(\varepsilon,K,l) . \tag{3.14}$$

Letting $k\to\infty$ in (3.5) we obtain, by (3.7), (3.12)–(3.14),

THEOREM 2. *For $\lambda>0$ and $K>0$ we have*

$$S(-\lambda) = S(-\lambda,K)+\sum_{\delta=0}^{\varepsilon}\binom{-\lambda}{\delta}S(-\lambda-\delta,K)u(K,\delta)+(-1)^{\varepsilon}r(\varepsilon,K) ,$$

where

$$\sum_{u=1}^{\lambda}(-1)^u\binom{\lambda+\varepsilon}{\lambda-u}\underline{P}(\lambda+\varepsilon,u,K)u(K,\varepsilon+u)$$
$$\leqq r(\varepsilon,K) \leqq \sum_{u=1}^{\lambda}(-1)^u\binom{\lambda+\varepsilon}{\lambda-u}\overline{P}(\lambda+\varepsilon,u,K)u(K,\varepsilon+u)\,.$$

Putting $K=1$ and $\varepsilon=0$ in Theorem 2 we obtain

THEOREM 3. *If for each r we have specified a set $\{s_1(r),\ldots,s_m(r)\}$ with m fixed, then*

$$S(-1;r;s_1(r),\ldots,s_m(r)) = r\log r+O(r) \quad \text{as } r\to\infty\,.$$

In particular,

$$S(-1;r;r-1) = r\log r+O(r) \quad \text{as } r\to\infty\,.$$

4.

We have done some numerical calculations on $S(-1;r,r-1)$. Note that in this particular case (3.11) may be improved, since

$$ar^{\lambda K}+b < \left(a+\frac{r-2}{r-1}\right)r^{\lambda K}\,.$$

This gives a better upper bound for $r(\varepsilon,K)$. We used $K=4$ and $\varepsilon=2$ in Theorem 2 with this improved bound and obtained the following lower and upper bounds, $\underline{S}$ and $\overline{S}$, for $S(-1;r;r-1)$.

r	$\underline{S}$	$\overline{S}$	r	$\underline{S}$	$\overline{S}$	r	$\underline{S}$	$\overline{S}$
3	2.679	2.683	9	19.534	19.672	15	40.276	40.596
4	5.154	5.170	10	22.774	22.941	16	43.994	44.345
5	7.749	7.783	11	26.111	26.308	17	47.773	48.156
6	10.494	10.551	12	29.536	29.763	18	51.611	52.026
7	13.382	13.464	13	33.042	33.300	19	55.504	55.953
8	16.399	16.509	14	36.624	36.913	20	59.449	59.928

The numerical calculations were performed on the IBM 360/50 H at the University of Bergen.

REFERENCE

1. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, At the Clarendon Press, Oxford, 1962.

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