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SEVERI'S CONJECTURE ON 0-CYCLES FOR A COMPLETE INTERSECTION

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We consider non-singular, complete connected surfaces F over the complex numbers C. Severi has introduced the concept of rational equivalence of algebraic cycles on F, and for this we refer to Mumford's recent paper [4] as well as Chevalley [1].

Let $A(F)$ denote the abelian group of cycles on F modulo rational equivalence. Grading it by the codimension,

$$A(F) = A^0(F) \oplus A^1(F) \oplus A^2(F) ,$$

where $A^0(F) \cong \mathsf{Z}$, $A^1(F) \cong \mathrm{Pic}(F)$ and $A^2(F)$ is the group of 0-cycles. The map associating to a 0-cycle $\sum n_P P$ its degree $\sum n_P$ lifts to the degree map $d: A^2(F) \to \mathsf{Z}$. Denote by $A_0(F)$ the kernel of d.

Severi conjecture 1 (disproved by Mumford).

$$A_0(F) \text{ is ``finite-dimensional''}.$$

Severi conjecture 2.

$$A_0(F) \cong (0) \quad \Leftrightarrow \quad F \text{ is rational} .$$

More precisely, one has concerning the first conjecture

THEOREM M (Mumford).

$$p_g F > 0 \quad \Rightarrow \quad A_0(F) \textit{ is not ``finite dimensional''} .$$

Here $p_g F = \dim_{\mathsf{C}} H^2(F, \mathcal{O}_F)$ denotes the geometric genus of F, and "finite dimensional" means that there exists an integer n such that every $C \in A^2(F)$ can be represented as a difference $C = C_+ - C_-$, where C_+ and C_- are effective cycles of degree at most n.

Below we shall prove the second conjecture in the case where F is birationally equivalent to a complete intersection, by means of a direct computation and using Theorem M.

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Recall that for non-singular complete surfaces over C the cohomology is a birational invariant (see [6, Lecture 5]), and that two non-singular complete surfaces over C are birationally equivalent if and only if they are dominated by the same projective surface, obtained by blowing up a finite number of closed points [6, Lecture 4].

Lemma. *If F' is the blow-up of a closed point $P \in F$, then*

$$A^2(F) \overset{\sim}{\to} A^2(F')\,.$$

Proof. Denote by E the exceptional divisor of F', and let for any scheme X, $|X|$ be the set of closed points of X. We shall construct group homomorphisms φ and ψ, making the following diagram commutative:

$$\begin{array}{ccc} \bigoplus_{|F|}\mathsf{Z} & \xrightarrow[\varphi]{} & \bigoplus_{|F'|}\mathsf{Z} \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle j'} \\ A^2(F) & \xrightarrow[\psi]{} & A^2(F')\,. \end{array}$$

Here j and j' are the natural surjections. We choose a point P' in $|E|$ and define φ by letting it be the identity outside the component over P, Z_P, and letting it map Z_P identically onto $\mathsf{Z}_{P'}$. The composed map $j' \circ \varphi$ is surjective because $E \overset{\sim}{\to} \mathsf{P}^1$ and $A^1(\mathsf{P}^1) = 0$.

We now claim that $j' \circ \varphi$ factors through j, giving rise to a surjection $\psi\colon A^2(F) \to A^2(F')$: Denoting by S^nF the symmetric product of F taken n times (see [4, p. 195] for details), one has that two 0-cycles A_1 and A_2 on F are rationally equivalent if and only if 1) they have the same degree n, 2) there exists a 0-cycle B of degree m such that $A_1 + B$ and $A_2 + B$ are effective, corresponding to the points x_1 and x_2 on $S^{n+m}F$, and 3) there exists a morphism $f\colon \mathsf{P}^1 \to S^{n+m}F$ with $f(0) = x_1$, $f(\infty) = x_2$.

Let C be the image of f, endowed with the reduced structure. In the non-trivial case, C is a rational curve connecting x_1 and x_2. We may furthermore suppose that the closed points of C do not all contain one and the same fixed coordinate, because in that case f would factor through a closed subscheme of $S^{n+m}F$ isomorphic to $S^{n+m-1}F$, and by a change of B we could get another morphism $\mathsf{P}^1 \to S^{n+m-1}F$ defining the equivalence between A_1 and A_2. Defining $U = F \setminus \{P\}$, $S^{n+m}U$ is a dense, open subset of $S^{n+m}F$ and of $S^{n+m}F'$. Further, $C_1 = C \setminus S^{n+m}U$ is a dense open subset of C, and its closure C' in $S^{n+m}F'$ is a rational curve. The restriction $f_{|f^{-1}(C_1)}$ extends in a unique way to a morphism f': $\mathrm{P}^1 \to C'$, as P^1 is non-singular and C' is complete. The blow-up $F' \to F$ defines a proper, birational morphism π: $S^{n+m}F' \to S^{n+m}F$ which satisfies $\pi \circ f' = f$, and whose effect on a $(n+m)$-tuple $(P_1, \ldots, P_{n+m})$ in $|S^{n+m}F'|$ is to change all coordinates in $|E|$ to P, and hold fixed the remaining ones. Choose $x_1' \in \pi^{-1}(x_1)$ and $x_2' \in \pi^{-1}(x_2)$. Then x_1' corresponds to a cycle A_1' on F' and x_2' to a cycle A_2'. According to the previous remarks $A_1' - A_2' - (\varphi(A_1) - \varphi(A_2))$ is a 0-cycle on E of degree zero, thus rationally equivalent to zero on F'. As $j'(A_1') = j'(A_2')$, this shows that $j' \circ \varphi(A_1) = j' \circ \varphi(A_2)$, and so completes the proof that $j' \circ \varphi$ factors through j.

To finish the proof we have to show that ψ is injective. So, let A_1 and A_2 be 0-cycles on F such that $\psi \circ j(A_1) = \psi \circ j(A_2)$. Then there exists a 0-cycle B' on F' such that $\varphi(A_1) + B'$ and $\varphi(A_2) + B'$ are effective of degree n, and there exists a morphism f': $\mathrm{P}^1 \to S^nF'$ with $f'(0) = x_1'$, $f'(\infty) = x_2'$, x_i' corresponding to $\varphi(A_i) + B'$. We decompose $B' = B'' + B'''$, where B''' is a cycle on E of degree m, and B'' a cycle on $F' \setminus E$. Then, with $B = B'' + m \cdot P$, the cycles $A_1 + B$ and $A_2 + B$ are effective of degree n and correspond to the points $x_1 = \pi(x_1')$ and $x_2 = \pi(x_2')$ on S^nF. Now either the image C of P^1 by $\pi \circ f'$ is a point, thus $A_1 = A_2$, or C is a rational curve on S^nF connecting x_1 and x_2. This proves that in either case $j(A_1) = j(A_2)$, which ends the proof of the lemma.

Corollary. *The group of 0-cycles $A^2(F)$ is a birational invariant for non-singular complete surfaces F.*

Proof. An immediate consequence of the lemma and the remarks preceding it.

As $A^2(\mathrm{P}^2) = (0)$ is trivial, the corollary trivializes one sense of the bi-implication in conjecture 2. Concerning the other sense we now state:

Theorem. *If F is a complete non-singular surface, birationally equivalent to a complete intersection, then*

$$A_0(F) = (0) \quad \Leftrightarrow \quad F \textit{ is rational.}$$

Proof. By the previous corollary we may assume that F is a complete intersection of multi-degree $(a_1, \ldots, a_r)$ in P^{r+2}, that is, given by r homogeneous polynomials in $\mathrm{C}[T_0, \ldots, T_{r+2}]$ of degrees $a_1, \ldots, a_r$.

According to Theorem M, if $p_g F > 0$, then $A_0(F) \neq (0)$, so we only want to focus on surfaces F satisfying $p_g F = 0$. As

$$\chi(F) = 1 - \dim_{\mathbf{C}} H^1(F, \mathcal{O}_F) + p_g F > 1$$

implies $p_g F > 0$, we shall determine which complete intersections have $\chi(F) \leqq 1$ and then discuss case by case.

Let $V^n_{(a_1,\ldots,a_r)}$ denote a complete intersection in $\mathbf{P}^{n+r}$ of multidegree $(a_1,\ldots,a_r)$. Hirzebruch's formula [2, p. 160] says

$$\sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \chi(V^n, \Omega^n) y^p z^{n+r} = \frac{1}{(1-zy)(1-z)} \prod_{i=1}^{r} \frac{(1+zy)^{a_i} - (1-z)^{a_i}}{(1+zy)^{a_i} + y(1-z)^{a_i}},$$

where $V^n = V^n_{(a_1,\ldots,a_r)}$, and $\chi(V^n, \Omega^p)$ is the Euler characteristic of the sheaf of p-forms on V^n.

Calculating the coefficient of z^{2+r}, one finds with $a = a_1 \cdots a_r$,

$$\chi(V^2) = a\left[\tfrac{1}{8}\left(\sum_{i=1}^{r} (a_i - 1)\right)^2 + \tfrac{1}{24} \sum_{i=1}^{r} (a_i - 1)^2 - \tfrac{2}{3} \sum_{i=1}^{r} (a_i - 1) + 1\right].$$

Using that the unit-ball in $\mathbf{R}^n$ is strictly convex, which gives that $\chi(V^2)$ for a fixed sum $\sum(a_i - 1)$ is minimal precisely when all a_i's are equal, one gets the following estimates:

1) If $r = 1$ and $a \geqq 2$, then $\chi(V^2) \geqq 1$ with equality only for $a = 2$ or $a = 3$.
2) If $r \geqq 2$ and $a_i \geqq 2$, then $\chi(V^2) \geqq 1$ with equality only for $r = 2$ and $a_1 = a_2 = 2$.

Now, the only cases of complete intersections we have to check are the following:

(i) $\mathbf{P}^2$.
(ii) Cubic hypersurface in $\mathbf{P}^3$.
(iii) Intersection of two quadrics in $\mathbf{P}^4$.

$\mathbf{P}^2$ is rational, so there is nothing to see. A non-singular cubic hypersurface in $\mathbf{P}^3$ is the blow-up of 6 points in $\mathbf{P}^2$, see [5], hence rational. Finally, according to [3, Chap. XIII, § 11], any irreducible component of the intersection of two quadrics whose vertices do not meet, is rational. In our case the intersection is connected and non-singular, therefore irreducible and an intersection point of the vertices would necessarily be singular.

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