MATH. SCAND. 28 (1971), 308–316

ON THE PRINCIPLE OF EQUIVALENCE OF SPARRE ANDERSEN

D. FOATA and M. P. SCHÜTZENBERGER[1]

1. Introduction.

The present paper is concerned with the algebraic study of the so-called *equivalence principle* of Sparre Andersen [1] and of the theorem of Bohnenblust as presented by Farrell [3]. We recall the former in its simplest form and, letting X be a set of n distinct real numbers, we consider the set F' of the $n!$ sequences obtained when permuting the elements of X in all possible manners. To each sequence $f=(x_1,x_2,\ldots,x_n) \in F'$, $\{x_1,x_2,\ldots,x_n\}=X$, we associate the sequence of the $n+1$ partial sums

$$s_0 = 0,\ s_1 = x_1\,,$$
$$s_k = s_{k-1}+x_k \quad \text{for } k=2,\ldots,n\,,$$

and we use it to define the following two numbers:

$L(f)$ = the number of strictly positive terms in the sequence $(s_0,s_1,\ldots,s_n)$;

$\Pi(f)$ = the index of the first maximum among the terms of the same sequence, that is, $\Pi(f)=m$ iff $s_j<s_m$ for $j<m$ and $s_j \leqq s_m$ for $m \leqq j$, $m,j=0,1,\ldots,n$.

Thus for any permutation $f \in F'$, both $L(f)$ and $\Pi(f)$ are natural numbers at most equal to n. In general they are different but Sparre Andersen has discovered the surprising fact that their distributions over the $n!$ permutations of F' are identical. This is essentially the equivalence principle. One of the proofs is due to Richards (quoted by Baxter in [2]). It consists in constructing a bijective map ϱ of F' to itself that is such that $\Pi(\varrho f)=L(f)$ identically.

Bohnenblust's theorem is not so easy to state here but its proof in-

Received October 3, 1968.

[1] Research on this paper was supported by Contract No AF 61 (052)-945, U.S. Air Force.

volves a similar idea. In the present paper we give a common algebraic formulation of both theorems and proofs and we exhibit a large class of cases in which Richard's construction leads to a generalized "Equivalence Principle". For this purpose we use the terminology of free monoids. Given a set X we identify the finite sequences of (non necessarily distinct) elements of X with the elements (the "words") of the free monoid X^* generated by X. Then a subset F' of X^* such that it contains together with any of its members every word obtained by permuting its letters will be called an *abelian subset* of X^*.

With these notations, instead of starting with a set of real numbers, we consider an abstract set X, a fixed morphism σ of X^* into the additive group of R and we identify any word of length $n, f=x_1x_2\ldots x_n \in X^*$, $x_1,x_2,\ldots,x_n \in X$, and the sequence $(\sigma x_1, \sigma x_2, \ldots, \sigma x_n)$ of R^n. The image by σ of the left factor f' of length m $f'=x_1x_2\ldots x_m$, $0 \leqq m \leqq n$, of the word f is precisely the partial sum $\sigma x_1+\sigma x_2+\ldots+\sigma x_m$. The empty word e is a factor of any word of X^* and $\sigma e=0$ since σ is a morphism.

Thus if PP^* denotes the subsemigroup of X^* consisting of all $g \in X^*$ such that $\sigma g>0$, we have that $L(f)$ is simply the *number of left factors of f that belong to PP^*.*

Further, let A^* denote the set of all words f such that $\sigma f' < \sigma f$ for any proper (that is, $\neq f$) left factor f' of f. For the corresponding sequence of partial sums, this means that the maximum is reached at the last term.

It is clear that A^* is a submonoid of X^* and that every word f has one well defined left factor a of maximum length in A^* (possibly, it is the empty word e, corresponding to an empty sequence). The number Πf is precisely the length of a and we have now the terminology needed to state with greater generality the

EQUIVALENCE PRINCIPLE OF SPARRE ANDERSEN. *Let F', PP^* and A^* be as above. There exists a bijection $\varrho: F' \to F'$ such that $\Pi(\varrho f)=L(f)$ identically.*

The sets PP^* and A^* have been defined here with the help of the morphism σ. We shall see that this can be done for a larger class of objects. Interestingly enough, the submonoids A^* which we shall encounter appear also in a quite different context as special instances of "synchronising variable length codes" ([4], [6]).

2. F-partition and F-factorisation.

We consider a fixed set ("alphabet") X and the free monoid X^* generated by X. The empty word is noted e and $XX^*=X^*\setminus\{e\}$ is the *free semigroup* generated by X. More generally, for any subset S of XX^*, S^* (resp. $SS^* = S^* \setminus \{e\}$) denotes the submonoid (resp. subsemigroup) of X^* generated by S. If M is a submonoid of X^*, the *basis* $(M \setminus \{e\}) \setminus (M \setminus \{e\})^2$ of M, is the least subset of XX^* that generates M.

We also consider a *fixed non empty abelian* subset F of XX^* having the property $(\mathscr{P})$:

$(\mathscr{P})$ F contains every left and every right factor $\neq e$ of any of its members.

DEFINITION. 1. A pair (P^*, Q^*) of submonoids of X^* is an *F-partition* iff

$$F \subset P^* \cup Q^*, \quad \emptyset = F \cap P^* \cap Q^* .$$

DEFINITION 2. A pair (A^*, B^*) of submonoids of X^* is an *F-factorization* iff every word f of F has exactly one factorization $f = ab$ with $a \in A^*$, $b \in B^*$.

In this section we discuss the relationship between F-partition and F-factorization. The assumption $(\mathscr{P})$ is only introduced for convenience since we can always take the minimal abelian subset $\bar{F}$ satisfying $(\mathscr{P})$ and containing F. Then clearly any $\bar{F}$-partition ($\bar{F}$-factorization) is an F-partition (F-factorization). Furthermore all our statements have a trivial symmetric counterpart obtained by exchanging P^* and Q^*, A^* and B^* and left and right.

1. *Let (A^*, B^*) be an F-factorization. Then A^* satisfies the condition*

$$a \in A^*,\ f \in X^*,\ af \in A^* \cap F, \quad \textit{imply} \quad f \in A^* .$$

PROOF. Let a and f as above. If $f = e$, the conclusion $f \in A^*$ is trivially verified. If f is not the empty word, we have $f \in F$ since F contains every factor of its members. Since (A^*, B^*) is a F-factorization we have $f = a'b'$ with $a' \in A^*$, $b' \in B^*$.

Thus $af = a'' \in A^*$ and $af = aa'b'$. Because of the unicity of the factorization, this implies $b' = e$, that is $f = a' \in A^*$.

We call *right F-prefix* any submonoid of X^* that satisfies the condition stated in 1. and, we call *left F-prefix* any submonoid B^* that satisfies the symmetric condition $b \in B^*$, $f \in X^*$, $fb \in B^* \cap F$ imply $f \in B^*$.

This terminology comes from the fact that, for $F = XX^*$, the right F-prefix submonoids are precisely the prefix submonoids of the theory of variable length codes.

2. *Let A be the basis of a right F-prefix monoid A^*. Every $f \in F$ has exactly one factorization in the form $f = a_1 a_2 \ldots a_m c$ with $m \geqq 0$; $a_1, a_2, \ldots, a_m \in A$; $c \in X^* \setminus AX^*$.*

PROOF. We proceed by induction on the largest $m \geqq 0$ such that $f \in A^m X^*$. If $m = 0$, there is nothing to prove. If $m > 0$, suppose $f = a_1 g = a_1' g'$ with $a_1, a_1' \in A$. One of a_1 and a_1' must be a left factor of the other, say $a_1 = a_1' h$, $h \in X^*$. We have $a_1 \in F$, and since A^* is right F-prefix, it follows that $h \in A^*$. By the hypothesis $a_1, a_1' \in A =$ the basis of A^*; this implies $h = e$ that is $a_1 = a_1'$.

3. *A* n.a.s.c. *that (A^*, B^*) be an F-factorization is that A^* and B^* be submonoids that satisfy the following three conditions*:

3.1. $\emptyset = A^* \cap B^* \cap F$;

3.2. *A^* is right F-prefix and B^* is left F-prefix*;

3.3. $F \subset A^* B^*$.

PROOF. The necessity of these conditions follows from the definition for 3.1. and 3.3. and from 1. for 3.2. To prove that they are sufficient we have only to show that under 3.1., 3.2. and 3.3. any relation $ab = a'b' \in F$ $(a, a' \in A^*,\ b, b' \in B^*)$ implies $a = a'$, $b = b'$. Indeed, if $ab = a'b'$ the word a must be a left factor of a' or a' must be a left factor of a, say $a = a'h$ for instance. Then $b' = hb$. We have $h \in \{e\} \cup F$. Thus $h \in A^*$ since A^* is right F-prefix. Also $h \in B^*$ since B^* is left F-prefix. By 3.1. we conclude that $h = e$.

4. *Let (A^*, B^*) be an F-factorization. Then*:

4.1. *Every right factor of a word of $A^* \cap F$ is in A^**;

4.2. *Every proper left factor of a word of $A \cap F$ is in B^**;

4.3. $(B^m A^n) \cap F \subset A \cup A^2 \cup \ldots \cup A^n \cup B \cup B^2 \cup \ldots \cup B^m$, $0 \leqq n, m$.

PROOF. Consider a word $a = fg \in A^* \cap F$. Clearly g is in A^* if either f or g is the empty word. If f and g are different from e, we have $f, g \in F$ hence $f = a'b'$, $g = a''b''$, $a', a'' \in A^*$, $b', b'' \in B^*$. Further $a'b'a'' \in F$, hence $a'b'a'' = a_1 b_1$, $a_1 \in A^*$, $b_1 \in B^*$. Thus we can write $a = ae \in A^* B^*$ and $a = a_1 b_1 b'' \in A^* B^*$. Because of the unicity of the factorization, this implies $b_1 b'' = e$, that is $g = a'' \in A^*$ and it proves 4.1. In similar fashion, we have $b'a''b'' = a_2 b_2 \in A^* B^*$, and from $a = ae = a' a_2 b_2$ we conclude that $b_2 = e$. Since $g \neq e$ and $b'' = e$, it implies $a'' \neq e$, hence $a_2 \neq e$ because of $b_2 = e$. Thus $a = a' a_2$ belongs to the basis A only if $a' = e$, that is, only if $f = b' \in B^*$ and 4.2. is proved since the empty left factor of a obviously belongs to B^*.

Now let $a \in A$, $b \in B$ be such that $ba \in F$. We have $ba = a_3 b_3$, $a_3 \in A^*$, $b_3 \in B^*$. By 4.1., either $b_3 = e$ or b_3 is *not* a right factor of a_3, that is, it admits a as a proper right factor. Symmetrically either $a_3 = e$ or it admits b as a proper left factor. Thus one of a_3 and b_3 must be the empty word e. Suppose for instance $b_3 = e$. We can write $a_3 = a'a''$ where $a' \in A$, $a'' \in A^*$. By the symmetric version of 4.1. and $ba = a_3 = a'a''$ we see that b must be a *proper* left factor of a', that is $a' = bh$, where $h \neq e$ and where $h \in A^*$ by 4.1. However $ba = bha''$ shows that $a = ha''$. Since $a \in A$, 4.2. asserts that $h \in B$ or $h = a$. Since the first case is excluded, we have $a = h$, hence $a'' = e$ and finally $ba = a' \in A$. This proves $BA \cap F \subset A \cup B$ and 4.3. follows by induction on m and n.

Recall that two words $g, g' \in X^*$ are *conjugate* iff one can find $h, h' \in X^*$ satisfying $g = hh'$; $g' = h'h$. Clearly conjugacy is an equivalence relation. It is in fact the restriction to X^* of the usual conjugacy relation in the free group generated by X.

5. *Let (A^*, B^*) be an F-factorization. A word $f \in F$ has a conjugate in A^* iff it has no conjugate in B^*.*

PROOF. Let $f \in F$. We have $f = ab$, where $a \in A^*$, $b \in B^*$. By 4.3. the conjugate ba of f belongs to A^* or to B^*. Thus it suffices to show that none of the conjugates of a word $a \in A^* \cap F$ belongs to B^*. Indeed we can write $a = a_1 a_2 \ldots a_m$, $m > 0$, $a_1, a_2, \ldots, a_m \in A$, and any conjugate of a has the form $f' = h' a_{k+1} a_{k+2} \ldots a_m a_1 a_2 \ldots a_{k-1} h$, where $h' \neq e$ and $hh' = a_k \in A$. By 4.1. and 4.2. we know that $h' \in A^*$ and that $h \in BB^*$ unless $h = e$, in which case $f' \in A^*$. Thus $f' \in AA^*BB^* \neq B^*$.

We now relate F-factorization and F-partition. To this effect, given a submonoid M of X^*, we call *right* (resp. *left*) *associate* of M the set of all words in X^* such that any of their right (resp. left) factors belongs to M.

The reader can verify that the monoid A^* mentioned in the introduction is the right associate of the monoid $P^* = \{e\} \cup \{f \in X^* : 0 < \sigma f\}$.

6. *The right associate of a submonoid P^* is a right prefix monoid whose basis A is such that $X^* \setminus AX^*$ is contained in the submonoid $(X^* \setminus P^*)^*$ generated by $X^* \setminus P^*$.*

PROOF. Let a and a' belong to the right associate of P^*. Any right factor of aa' is a right factor of a' or a product ha' where h is a right factor of a. Since P^* is a monoid this shows that its right associate A^*

is also a monoid. Further A^* is right prefix since by definition it satisfies the stronger condition that $f'f \in A^*$ implies $f \in A^*$ for any $f,f' \in X^*$. Finally if $f \in XX^* \setminus AX^*$, it does not belong to A^*. Thus it has a right factor $f'' \neq e$ which belongs to $X^* \setminus P^*$ and letting $f=f'f''$ we have also $f' \in X^* \setminus AX^*$. Induction on the length of f completes the proof.

7. *A* n.a.s.c. *that* (A^*,B^*) *be an* F*-factorization is that there exists an* F*-partition* (P^*,Q^*) *such that* $F \cap A^*$ *and* $F \cap B^*$ *coincide respectively with the intersections with* F *of the right associate of* P^* *and of the left associate of* Q^*.

Proof. Let (A^*,B^*) be an F-factorization; we set $P^*=$ the submonoid generated by $X^* \setminus B^*$ and $Q^*=B^*$.

We have $F \subset P^* \cup Q^*$. Let f belong to $F \cap P^*$. By the definition of P^* we have $f=f_1f_2\ldots f_m$ where $m>0$ and $f_1,f_2,\ldots,f_m \in F \setminus B^*$. Since (A^*,B^*) is an F-factorization we have $f_1=ab$ with $a \in AA^*$ and $b \in B^*$. Thus $f \in aA^*B^*$ and accordingly $f \notin B^*$. This proves that $\emptyset=F \cap P^* \cap Q^*$ and consequently that (P^*,Q^*) is an F-partition.

The fact that $B^* \cap F$ is the intersection of F with the left associate of $Q^*(=B^*)$ follows from the symmetric version of 4.1.

By 4.1. and $A^* \cap F \subset P^*$, $A^* \cap F$ is contained in the right associate of P^*. Finally let $f \in F$ belong to the right associate of P^*. We have $f \notin A^*BB^*$ since every word of A^*BB^* has a right factor in B^* and since $B^* \cap F \subset F \setminus P^*$. Thus $f \in A^*$ since $f \setminus A^*BB^* \subset A^*$ and the necessity of the condition is proved.

Reciprocally let (P^*,Q^*) be an F-partition and let A and B be the basis of the associated monoids. We show that (A^*,B^*) satisfies the conditions of 3.

First $F \cap A^* \cap B^* \subset F \cap P^* \cap Q^*$. Since this last intersection is empty this gives 3.1. Condition 3.2 follows from 6. and its symmetric. Thus to verify 3.3 it suffices by induction on the length to consider a word f satisfying the condition $f \in F \setminus AX^*$ and to show that it belongs to B^*. Indeed, the condition $f \notin AX^*$ implies $f' \notin AX^*$ for any left factor f' of f. Thus, by 6., f and any of its left factors belong to $(X^* \setminus P^*)^*$. Since $F \setminus P^* \subset Q^*$ because (P^*,Q^*) is an F-partition, we see that f and any of its left factors belong to Q^*, that is, that $f \in B^*$ by definition.

3. Richards' construction.

We keep the same notations and the same set F. We let (P^*,Q^*) be a fixed F-partition and (A^*,B^*) be the associated F-factorization.

We introduce the restrictive assumption that $P^* \cap F$ (hence $Q^* \cap F$)

are abelian sets. (Counter examples show that Richards' map ϱ is not always bijective without this hypothesis.)

DEFINITION 3. Let the map ϱ of $\{e\} \cup F$ to itself be defined by induction on the length by:

$$\varrho e = e \text{ and for } f = f'x \in F,\ x \in X\ ,$$
$$\varrho f = x\varrho f' \text{ or } = (\varrho f')x \text{ depending upon } f \in P^* \text{ or } f \in Q^*\ .$$

8. *The map ϱ is a bijection.*

PROOF. It is clear that ϱf is a word obtained by permutation of the letters of f. Thus by our assumption that $P^* \cap F$ is abelian, f and ϱf always belong to the same monoid P^* or Q^*.

Assume the result proved for every word shorter than $f \in F$. If $\varrho f = g \in P^*$, we know that $f \in P^*$ and there exists one and only one pair $(x,f') \in X \times X^*$ such that $g = xg'$, $g' = \varrho f'$ and $f = f'x$. In similar fashion, if $\varrho f = g \in Q^*$ we have $f = f'x$ with $\varrho f'x = g$ in a unique manner.

9. *For any $f \in F$, the number $L(f)$ of left factors in PP^* of f is equal to the length $\Pi(\varrho f)$ of a in the factorization $g = ab$, $a \in A^*$, $b \in B^*$, of $g = \varrho f$.*

PROOF. The result is true for $f \in F \cap X$ and we can suppose that it is proved for any word shorter than f.

Let f_1 be the left factor of maximal length of f that belongs to Q^* or to P^* depending upon $f \in P^*$ or $f \in Q^*$. If $f = f_1 h$, it is a straightforward consequence of the definition of ϱ that $\varrho f = \bar{h}\varrho f_1$ for $f \in P^*$ and $\varrho f = \varrho f_1 h$ for $f \in Q^*$, where $\bar{h} = x_m x_{m-1} \ldots x_1$ if $h = x_1 x_2 \ldots x_m$, $m > 0$, $x_1, x_2, \ldots, x_m \in X$. Further any left factor $h' \neq e$ of h belongs to the same monoid P^* or Q^* as f does. Thus by our definition of A^* and B^* as the associated monoids of P^* and Q^*, we have $\bar{h} \in A^*$ (resp. $h \in B^*$) for $f \in P^*$ (resp. $\in Q^*$). Now letting λh be the length of h and recalling 4, we have

$$L(f) = L(f_1) + \lambda h \text{ and } \Pi(\varrho f) = \lambda h + \Pi(\varrho f_1) \quad \text{for } f \in P^*\ ,$$
$$L(f) = L(f_1),\ \Pi(\varrho f) = \Pi(\varrho f_1) \qquad \text{for } f \in Q^*\ ,$$

4. Concluding remarks.

This completes our proof of the generalized equivalence principle. For F consisting of the words in which each letter of X appears at most once, the reader will recognize in $P^* \cap F$ the set $\varepsilon^{-1}0$ of Bohnenblust and Farrell, for a function ε taking only values 0 or 1. The general case follows since one can always represent the "set function" ε used by these authors as finite sums $\varepsilon = \sum r_i \varepsilon_i$ where r_i is real and the range of the set function ε_i is the set $\{0, 1\}$ for all i.

For this reason it would be quite interesting to be able to give explicitly all the abelian F-partitions of an arbitrary abelian subset F containing the factors of its members. We limit ourselves here to the case of $F = XX^*$, that is to the case where Richards' construction gives the validity of the equivalence principle for *any* abelian subset of X^*. To simplify notations we suppose $\operatorname{Card} X = k$ finite and we recall Hahn's Theorem [5].

THEOREM. *Let M be a submonoid of R^k and $\leq$ a total preorder on M. There exists a morphism $\nu: \mathsf{R}^k \to \mathsf{R}^k$ and a lexicographic order $\leq$ on R^k such that for $m, m' \in M$ one has $m \leq m'$ iff $\nu m \leq \nu m'$.*

We prove

10. *A* n.a.s.c. *that (P^*, Q^*) be an abelian XX^*-partition is that there exists a morphism μ of X^* into the additive group R^k and a lexicographic order $\leq$ on R^k such that $P^* = \{f \in X^*: 0 \leq f\}$; $QQ^* = \{f \in X^*: \mu f < 0\}$.*

PROOF. The condition is sufficient. Any lexicographic order $\leq$ on R^k is compatible with the additive group structure (that is, $r \leq r'$ implies $r + r'' \leq r' + r''$, identically for r, r', $r'' \in \mathsf{R}^k$). Thus in particular $\{r \in \mathsf{R}^k: 0 \leq r\}$ $(= R_1)$ and $\{r \in \mathsf{R}^k: r < 0\}$ $(= R_2)$ are respectively a submonoid and a subsemigroup. These two sets are disjoint and, since $\leq$ is a total order, their union is R^k. It follows that $P^* = \mu^{-1} R_1$ and $Q^* = \mu^{-1} R_2$ satisfy $P^* \cap Q^* = \{e\}$ and $P^* \cup Q^* = X^*$. Finally, P^* and Q^* are abelian subsets since they are inverse images by a morphism μ into a commutative monoid.

The condition is necessary. Let α be the canonical homomorphism of X^* onto the free *abelian* monoid X^+ generated by X and suppose that the abelian submonoids P^* and Q^* give an XX^*-partition. Then αP^* and αQ^* are submonoids of X^+ such that $\alpha P^* \cup \alpha Q^* = X^+$ and $\alpha P^* \cap \alpha Q^* = \{0\}$. Thus we are left to show that there exists a morphism $\theta: X^+ \to \mathsf{R}^k$ and a lexicographic order $\leq$ on R^k such that $\alpha P^* = \{a \in X^+: 0 \leq \theta a\}$ and $\alpha Q^* \setminus \{0\} = \{a \in X^+: \theta a < 0\}$. First we define a binary relation $\leq$ on X^+ by letting $a \leq a'$ iff for any $b \in X^+$, $a + b \in \alpha P^*$ implies $a' + b \in \alpha P^*$. Clearly $\leq$ is a preorder and we can then find a morphism $\theta: X^+ \to \mathsf{R}^k$ and a preorder $\leq$ on R^k such that $a \leq a'$ iff $\theta a \leq \theta a'$ in R^k.

Now we have $\alpha P^* = \{a \in X^+: 0 \leq a\}$ since on one hand, $0 \leq a$ and

$0 \in \alpha P^*$ imply $a+0 \in \alpha P^*$ proving $\{a \in X^+ : 0 \leq a\} \subset \alpha P^*$ and on the other hand, for any $c \in \alpha P^*$ we have $0 \leq c$ because $0+b \in \alpha P^*$ implies $b \in \alpha P^*$ and $c+b \in \alpha P^*$. Thus we can write $\alpha P^* = \{a \in X^+ : 0 \leq \theta a\}$ and we have only to show that the preorder $\leq$ (on X^+, hence on R^k) is total. Again this is equivalent with the statement that for $a, a' \in X^+$, not $a \leq a'$ implies $a' \leq a$ that is $a+b \in \alpha P^*$ for any b such that $a'+b \in \alpha P^*$. Suppose not $a \leq a'$ and $a'+b \in \alpha P^*$. The first relation entails the existence of at least one $c \in X^+$ such that $a+c \in \alpha P^*$ and $a'+c \notin \alpha P^*$. Thus $a'+b+a+c \in \alpha P^*$. Since $a'+c \notin \alpha P^*$ implies $a'+c \in \alpha Q^* \setminus \{0\}$ and since αQ^* is a submonoid we cannot have $a+b \in \alpha Q^*$ because it would give $a'+b+a+c \in \alpha Q^* \setminus \{0\}$ in contradiction with $a'+b+a+c \in \alpha P^*$ and the relation $\alpha P^* \cap \alpha Q^* = \{0\}$.

REFERENCES

1. E. Sparre Andersen, *The equivalence principle in the theory of fluctuations of sums of random variables*, Colloquium on Combinatorial Methods in Probability Theory, [August 1962], Aarhus Universitet, Matematisk Institut, 13–16.
2. G. Baxter, *Notes for a seminar in stochastic processes*, Department of Mathematics, University of Minnesota, 1957.
3. R. H. Farrell, *Notes on a combinatorial theorem of Bohnenblust*, Duke Math. J. 32 (1965), 333–339.
4. S. W. Golomb and B. Gordon, *Codes with bounded synchronization delay*, Information and Control 8 (1965), 355–372.
5. H. Hahn, *Über die nichtarchimedischen Grössensysteme*, S. B. Akad. Wiss. Wien IIa 116 (1907), 601–655.
6. M. P. Schützenberger, *Sur une question concernant certains sous-monoïdes libres*, C. R. Acad. Sci. Paris 261 (1965), 2414–2420.

DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DE MONTRÉAL, CANADA

ET

INSTITUT DE RECHERCHE MATHÉMATIQUE AVANCÉE, UNIVERSITÉ DE STRASBOURG, FRANCE

FACULTÉ DES SCIENCES, UNIVERSITÉ DE PARIS, FRANCE