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ON MODELS WITH UNDEFINABLE ELEMENTS

A. EHRENFEUCHT and G. FUHRKEN

The following problem was posed by C. Ryll-Nardzewski (cf. [3]): Is there a complete theory T formulated in a first order language with only finitely many non-logical symbols and which has the following properties:

(i) T has a model $\mathfrak{A}$ every element of which is first order definable in $\mathfrak{A}$; hence $\mathfrak{A}$ is a prime model of T.

(ii) For every set $\mathfrak{X}$ of non-principal (dual) prime ideals of $F_1(T)$ — the Boolean algebra of formulas with v_0 as only free variable taken modulo equivalence in T — there is a model $\mathfrak{B}$ of T (which can be taken to be an elementary extension of $\mathfrak{A}$) such that the non-principal prime ideals of $F_1(T)$ which are realized in $\mathfrak{B}$ are exactly those in $\mathfrak{X}$; furthermore those prime ideals are realized by exactly one element each.

This problem is an extension of an earlier problem which had been solved in [2]. We give a partial answer to the extended problem in the following

THEOREM. *There is a complete extension T of the first order theory of linear orderings such that:* (i) *holds; and there is a set $\mathfrak{Y}$ of non-principal prime ideals of $F_1(T)$, $\mathfrak{Y}$ being of the power of the continuum, such that* (ii) *holds for subsets of $\mathfrak{Y}$.*

The theory T will be described as the elementary theory of a particular model $\mathfrak{A}$ which we are now going to describe. Let Q be the set of rational numbers; $\langle r_n : n \in \omega \rangle$ be an enumeration (without repetition) of Q; $\langle t_n : n \in \omega \rangle$ a family of positive irrational numbers which are linearly independent over the rationals. For $n \in \omega$ put

$$B_n = \{r_n - t_n \cdot (i+1)^{-1} : i \in \omega\} .$$

Note: (i) the sets B_n are pairwise disjoint; (ii) each B_n has order type ω; (iii) $\sup B_n = r_n$; (iv) for any real numbers x and y, if $x<y$ then there are arbitrarily large $n \in \omega$ such that for some $z \in B_n$, $x<z<y$.

Put $Q' = Q \times \{0\}$, and for $n \in \omega$, $B_n' = B_n \times \{1,\ldots,n+2\}$; finally put

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$A = Q' \cup \bigcup \{B_n' : n \in \omega\}$ and $\mathfrak{A} = \langle A, \prec \rangle$, where $\prec$ is the lexicographical ordering. Note:

(1.1) $\mathfrak{A}$ is a linearly ordered system.
(1.2) The subsystem of $\mathfrak{A}$ determined by Q' is isomorphic to the ordered system of the rationals.
(1.3) The sets B_n' are pairwise disjoint.
(1.4) Each B_n' has order type ω.
(1.5) For each $n \in \omega$, $\sup B_n' = \langle r_n, 0 \rangle$.
(1.6) For any $x, y \in A$, if $x \prec y$ and if the interval from x to y is not finite, then there are arbitrarily large n such that for some $z \in B_n'$ $x \prec z \prec y$.

By definition $\mathfrak{A}$ is a model of T. Hence in order to establish part (i) of the theorem it suffices to show that every element of A is first order definable in $\mathfrak{A}$. This is seen as follows:

(2.1) For each $n \in \omega$, the set B_n' is first order definable in $\mathfrak{A}$, viz. by the property of belonging to a maximal discrete subset of power $n+2$.
(2.2) For each $n \in \omega$, each element of B_n' is first order definable in $\mathfrak{A}$; this follows from (2.1) and (1.4).
(2.3) For each $n \in \omega$, $\langle r_n, 0 \rangle$ is first order definable in $\mathfrak{A}$; this follows from (2.1) and (1.5).

In order to show part (ii) of the theorem we shall proceed as follows: Let I be a set of irrational numbers. Proceed as in the construction of $\mathfrak{A}$ except for taking $Q' = (Q \cup I) \times \{0\}$. Call the resulting model $\mathfrak{B}$. We shall show:

1° $\mathfrak{B}$ is a model of T.

2° For each $i \in I$, the prime ideal defined by $\langle i, 0 \rangle$ in $\mathfrak{B}$ is non-principal.

3° For each $i, j \in I$, if $i \neq j$ then the prime ideals defined by $\langle i, 0 \rangle$ and by $\langle j, 0 \rangle$ are distinct.

Part (ii) of the theorem will then be proved by taking I the set of all irrational numbers and Y the set of all non-principal prime ideals realized in the corresponding $\mathfrak{B}$. Note that these are not all non-principal prime ideals of $F_1(T)$; in fact there are continuum many others. Those have the property that if they are realized in a model, then they are realized by infinitely many elements.

Let us recall Fraïssé's relation $\equiv_n$, $n \in \omega$. Let $\mathfrak{A}$ and $\mathfrak{B}$ be as above. For $m \in \omega$, $a \in {}^m A$ and $b \in {}^m B$, put

$$a \equiv_0 b \quad \text{iff} \quad \begin{cases} \{\langle a_i, b_i\rangle : i \in m\} \text{ establishes an isomorphism be-} \\ \text{tween } \mathfrak{A} \restriction \{a_0, \ldots, a_{m-1}\} \text{ and } \mathfrak{B} \restriction \{b_0, \ldots, b_{m-1}\}; \end{cases}$$

$$a \equiv_{n+1} b \quad \text{iff} \quad \begin{cases} \text{for each } x \in A \text{ there is a } y \in B \text{ such that} \\ a\langle x\rangle \equiv_n b\langle y\rangle \text{ and for each } y \in B \text{ there is a} \\ x \in A \text{ such that } a\langle x\rangle \equiv_n b\langle y\rangle. \end{cases}$$

The two important properties of this relation which we need are (e.g. see [1]):

(3.1) If for each $n \in \omega$, $\emptyset \equiv_n \emptyset$, then $\mathfrak{A} \equiv \mathfrak{B}$.

(3.2) For each formula φ in $F_1(T)$ there is a $n \in \omega$ (which is given by the quantifier depth of φ) such that for any $x, y \in A$, if $\langle x\rangle \equiv_n \langle y\rangle$, then x satisfies φ in $\mathfrak{A}$ if and only if y satisfies φ in $\mathfrak{A}$.

For the formulation and proof of the next lemma it is convenient to expand the language of T by the following defined symbols:

(i) $Sv_0v_1 = v_0 \prec v_1 \wedge \neg\exists v_2[v_0 \prec v_2 \wedge v_2 \prec v_1]$,

(ii) $S^n v_0 v_1 = \exists v_2 \ldots \exists v_n[Sv_0v_2 \wedge Sv_2v_3 \wedge \ldots \wedge Sv_nv_1]$, $\quad n = 2, 3, \ldots$,

(iii) $L_n v_0 = \exists v_1 S^n v_1 v_0$, $\quad n = 1, 2, \ldots$,

(iv) $R_n v_0 = \exists v_1 S^n v_0 v_1$, $\quad n = 1, 2, \ldots$,

(v) $Dv_0v_1 = v_0 \prec v_1 \wedge R_1v_0 \wedge L_1v_1 \wedge \forall v_2[v_0 \prec v_2 \wedge v_2 \prec v_1 \to L_1v_2 \wedge R_1v_2]$.

(The intuitive meaning of the latter is that the interval from v_0 to v_1 is discrete; in $\mathfrak{A}$ or $\mathfrak{B}$ this implies that the interval is finite.) For $k \in \omega$, put $\mathfrak{A}_k = (A, S_i^{\mathfrak{A}}, L_i^{\mathfrak{A}}, R_i^{\mathfrak{A}})_{i \in k}$ and define $\mathfrak{B}_k$ similarly.

LEMMA. *Let* $n, m \in \omega$, $a \in {}^mA$, $b \in {}^mB$, $k = 3^n$. *Assume*

(i) $\{\langle a_i, b_i\rangle : i \in \omega\}$ *establishes an isomorphism between* $A_k \restriction \{a_0, \ldots, a_{m-1}\}$ *and* $B_k \restriction \{b_0, \ldots, b_{m-1}\}$; *say, the* a_i*'s are in increasing order.*

(ii) *For each* $z \in B_0' \cup \ldots \cup B'_{k-1}$ *the following conditions hold*: (a) *if* $z \leqq a_0$ *or* $z \leqq b_0$, *then* $a_0 = b_0$; (b) *if* $a_{m-1} \leqq z$ *or* $b_{m-1} \leqq z$, *then* $a_{m-1} = b_{m-1}$; (c) *if, for* $i = 0, \ldots, m-2$, $a_i \leqq z \leqq a_{i+1}$ *or* $b_i \leqq z \leqq b_{i+1}$, *then* $a_i = b_i$ *and* $a_{i+1} = b_{i+1}$.

Under these conditions $a \equiv_n b$.

The proof is by induction on n. We shall only treat a typical case. Assume the lemma holds for n (and all m). Given a and b satisfying conditions (i) and (ii) with $k = 3^{n+1}$ and $x \in A$ with $a_i \prec x \prec a_{i+1}$. (Other cases are: $x \prec a_0$; $a_{m-1} \prec x$; $x = a_i$; and the cases with the roles of x and y interchanged.) We shall find a $y \in B$ such that $a\langle x\rangle$ and $b\langle y\rangle$ satisfy conditions (i) and (ii) with $k = 3^n$. There are various possibilities:

(I) $x \in B_p'$, for some $p < k$, or more generally, $a_i = b_i$ and $a_{i+1} = b_{i+1}$. In this case take $y = x$.

(II) $x \in B_p'$, for some $p \geqq k$. Again various possibilities have to be distinguished:

(IIa) $\langle a_i, x\rangle \in D^{\mathfrak{A}}$. If there are at most 3^n elements between a_i and x, say h of them, take as y the $(h+1)$st element to the right of b_i; if there are more than 3^n elements between a_i and x take as y the 3^nth element to the right of b_i.

(IIb) $\langle x, a_{i+1}\rangle \in D^{\mathfrak{A}}$ and the previous case does not apply. Proceed similarly.

(IIc) Neither (IIa) nor (IIb) holds though (II) holds. Let x be the hth term of its discrete component. If $h \leqq 3^n$ take as y the hth term of a discrete component between b_i and b_{i+1}, using (1.6); if $p-h \leqq 3^n$ proceed similarly; in the remaining case take as y the $[p/2]$th term of a discrete component between b_i and b_{i+1}, again using (1.6).

(III) $x \in Q'$. Take as y any element between b_i and b_{i+1} belonging to the Q' (of $\mathfrak{B}$).

Corollary. (i) $\mathfrak{A} \equiv \mathfrak{B}$, *hence* $\mathfrak{B}$ *is a model of* T.

(ii) *Let* $y \in B$, $y = \langle i, 0\rangle$, *where* $i \in I$. *Then the prime ideal of* $F_1(T)$ *which is defined by* y *in* $\mathfrak{B}$ *is non-principal.*

Proof. Part (i) follows from the lemma and (3.1). For part (ii), let φ be an element of $F_1(T)$ satisfied by y in $\mathfrak{B}$. Let n be the number obtained for φ from (3.2). Put $k = 3^n$ and let $x \in Q'$ such that for no $z \in B_0' \cup \ldots \cup B_{k-1}'$, z is between x and y. Let χ be a formula with a single free variable which defines according to (2.3) x in $\mathfrak{A}$ and hence in $\mathfrak{B}$. Then by the lemma and (3.2) y satisfies $\varphi \wedge \neg\chi$ in $\mathfrak{B}$ while $\varphi \wedge \neg\chi$ is not equivalent in T with φ.

Finally let $i, j \in I$ and $i \neq j$, say $i < j$. Let $r \in Q$ with $i < r < j$. By (2.3), $\langle r, 0\rangle$ is definable in $\mathfrak{A}$ and hence in $\mathfrak{B}$, say by the formula χ. Then $\langle i, 0\rangle$ satisfies $\exists v_1 [v_0 < v_1 \wedge \chi(v_1)]$ in $\mathfrak{B}$ while $\langle j, 0\rangle$ does not. Hence the prime ideal defined by those elements are distinct.

This concludes the proof of the theorem. We may remark that every element of B is definable in B by formulas of $L_{\omega_1 \omega}$.

LITERATURE

1. A. Ehrenfeucht, *An application of games to the completeness problem for formalized theories*, Fund. Math. 43 (1960), 129–141.
2. G. Fuhrken, *A model with exactly one undefinable element*, Colloq. Math. 19 (1968), 183–185.
3. Autoreferat von [2], Zbl. Math. 176 (1969), 275.

UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, U. S. A.
UNIVERSITY OF MINNESOTA, MINNEAPOLIS, U. S. A.