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ESTIMATES NEAR DISCONTINUITIES FOR SOME DIFFERENCE SCHEMES

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Consider the initial-value problem

$$\frac{\partial u}{\partial t} = \varrho \frac{\partial u}{\partial x}, \quad t \geqq 0, \tag{1}$$

$$u(x,0) = v(x), \tag{2}$$

where ϱ is a real number, and a finite difference approximation consistent with (1),

$$E_k v(x) = \sum_{j=-\infty}^{\infty} a_j v(x-jh), \quad \sum_{j=-\infty}^{\infty} |a_j| < +\infty, \tag{3}$$

where $k/h=\lambda$ is constant. We shall discuss the approximation of the exact solution $u(x,t)=\big(E(t)v\big)(x)=v(x+\varrho t)$ for $t=nk$ of (1), (2) by $E_k^n v$ near a discontinuity of $u(x,t)$. More precisely, we shall assume that v vanishes for positive x, so that $E(t)v$ vanishes for $x>-\varrho t$. The general case of an isolated discontinuity at $x=0$ can be reduced to this case by subtracting a smooth initial function. Under this assumption we shall estimate $(E_k^n v)(x)$ in terms of the distance δ of x parallel to the x-axis to the characteristic through the origin. To be specific, if χ_y denotes the characteristic function of the interval (y,∞) we shall estimate for $p=2$ and ∞,

$$\|\chi_{\delta-\varrho t} E_k^n v\|_p ,$$

where $\|\cdot\|_p$ denotes the L_p-norm,

$$\|v\|_p = \left(\int_{-\infty}^{\infty} |v(x)|^p \, dx\right)^{1/p}, \quad 1 \leqq p < \infty,$$

$$\|v\|_\infty = \operatorname{ess\,sup}_x |v(x)| .$$

Notice that such estimates can also be interpreted as results on the domain of influence of the initial-values at a point. This problem has been

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treated previously by Hedstrom [3] and Apelkrans [1] when $v=1-\chi_0$. Hedstrom gave very precise estimates and his approach would, although quite technical, permit more general conclusions. The technique used by Apelkrans, which is due to Kreiss and Lundqvist [4], is simpler and also suitable for variable coefficients, but his results were less precise. Our aim here is to show that this latter technique can be used to give sharper results even for more general initial functions v. Our paper can be considered a continuation of our previous paper [2] and in the proofs below we shall depend on the technique developed there.

Our results will be expressed in terms of the symbol of E_k, the absolutely convergent Fourier series

$$a(\xi) = \sum_{j=-\infty}^{\infty} a_j e^{ij\xi} .$$

We shall assume that $a(\xi)$ is analytic in a strip around the real axis. In applications a is generally the quotient of two trigonometric polynomials; if the operator is explicit, a is a trigonometric polynomial.

We shall make the following general assumptions about the symbol: The symbol can be written in the form

$$a(\xi) = \exp(-i\lambda\varrho\xi + \psi(\xi)) ,$$

where $\psi(\xi)$ is also analytic in a strip around the real axis and

$$\psi(\xi) = i\beta\xi^r(1+o(1)), \quad \beta \neq 0, \ r>1, \qquad \text{as } \xi \to 0 , \tag{4}$$

$$\operatorname{Re}\psi(\xi) \leqq -\gamma\xi^s, \quad \gamma>0, \ |\xi| \leqq \tfrac{3}{2}\pi . \tag{5}$$

These assumptions mean that E_k is accurate of order $r-1$ and dissipative of order s. If $r \neq s$, β has to be real.

The main idea in the proofs is to introduce for $\eta>0$ the operator defined by

$$E_{k,\eta}v = e^{\eta x}E_k(e^{-\eta x}v) ,$$

and to notice that since $\chi_{\delta-\varrho t}(x) \leqq \exp(\eta(x+\varrho t-\delta))$, for any such η, we have for $p=2$ and ∞,

$$\begin{aligned} \|\chi_{\delta-\varrho t}E_k^n v\|_p &\leqq e^{\eta(\varrho t-\delta)}\, \|E_{k,\eta}^n(e^{\eta x}v)\|_p \\ &\leqq e^{\eta(\varrho t-\delta)}\, \|E_{k,\eta}^n\|_p\, \|e^{\eta x}v\|_p . \end{aligned} \tag{6}$$

In appraising the right hand side we shall use that the symbol of $E_{k,\eta}$ is $a(\xi-ih\eta)$. We shall therefore need estimates for the symbol $a(\xi)$ in a neighborhood of the real line.

LEMMA 1. *There are positive constants* c, C, *and* η_0 *such that with* $\tau = s(s-r+1)^{-1}$,

$$\operatorname{Re}\psi(\xi - i\eta) \leqq -c\xi^s + C\eta^\tau, \quad 0 \leqq \eta \leqq \eta_0, \ |\xi| \leqq \tfrac{3}{2}\pi, \tag{7}$$

and such that if r *is odd and* $\beta < 0$,

$$\operatorname{Re}\psi(\xi - i\eta) \leqq -c\eta\xi^{r-1} + C\eta^r, \quad 0 \leqq \eta \leqq \eta_0, \ |\xi| \leqq \tfrac{3}{2}\pi. \tag{8}$$

PROOF. Since by (5), for any $\varepsilon > 0$ and small η,

$$\sup\{\operatorname{Re}\psi(\xi - i\eta);\ \varepsilon \leqq |\xi| \leqq \tfrac{3}{2}\pi\} < 0,$$

it is sufficient to prove (7) and (8) for small ξ and η. We have for such ξ and η,

$$\operatorname{Re}(\psi(\xi - i\eta) - \psi(\xi)) \leqq C\eta(|\xi|^{r-1} + \eta^{r-1}),$$

and hence by (5),

$$\operatorname{Re}\psi(\xi - i\eta) \leqq -\gamma\xi^s + C\eta|\xi|^{r-1} + C\eta^r,$$

which clearly implies (7). When r is odd and $\beta < 0$, we find

$$\operatorname{Re}\psi(\xi - i\eta) \leqq \operatorname{Re}(\psi(\xi - i\eta) - \psi(\xi)) \leqq \tfrac{1}{2}r\beta\eta\xi^{r-1} + C(\eta^2|\xi|^{r-2} + \eta^r),$$

and since the coefficient of the first term on the right is negative we can estimate the middle term by the others.

We notice that Lemma 1 implies that there are positive constants C and η_0 such that

$$|a(\xi - i\eta)| \leqq \exp(-\lambda\varrho\eta + C\eta^\tau), \quad 0 \leqq \eta \leqq \eta_0, \ \xi \text{ real}, \tag{9}$$

with

$$\tau = \begin{cases} r, & \text{for } r \text{ odd and } \beta < 0, \\ s(s-r+1)^{-1}, & \text{otherwise.} \end{cases} \tag{10}$$

Apelkrans [1] calls operators for which (9) holds contractive of order τ.

In deducing the inequalities from (6) we shall want to choose η in an optimal way. We shall then use the following lemma.

LEMMA 2. *Let* λ, δ, η_0, *and* C_0 *positive,* α *real, and* $\tau > 1$ *be given. Then there are positive constants* c *and* C (*with* $C = 1$ *for* $\alpha = 0$) *such that with* $\varkappa = \tau/(\tau - 1)$, $\delta_t = \delta t^{-1}$, $t = nh\lambda$,

$$\inf_{0 \leqq h\eta \leqq \eta_0} (h\eta)^\alpha \exp(-\delta\eta + C_0 n(h\eta)^\tau) \leqq Cn^{-\alpha/\tau}\exp(-cn\min\{\delta_t^\varkappa, \delta_t\}).$$

PROOF. Choose $\eta_1 \leqq \eta_0$ such that $C_0\lambda^{-1}\eta_1^{\tau-1} \leqq \tfrac{1}{2}$ and set

$$h\eta = \eta_2 \equiv \min\{\eta_1, (\tfrac{1}{2}\lambda\delta_t C_0^{-1})^{1/(\tau-1)}\} \quad \text{for } \alpha \geqq 0\,,$$
$$h\eta = \max\{n^{-1/\tau}, \eta_2\} \quad \text{for } \alpha < 0\,.$$

We begin with a simple estimate in L_2.

THEOREM 1. *There is a positive constant c such that for $v \in L_2$ vanishing for positive x,*

$$\|\chi_{\delta-\varrho t} E_k^n v\|_2 \leqq \exp(-cn\min\{\delta_t^{\varkappa}, \delta_t\})\,\|v\|_2\,,$$

where $nk=t$, $\delta_t = \delta t^{-1}$, and

$$\varkappa = \begin{cases} r/(r-1), & \text{if } r \text{ is odd and } \beta < 0\,, \\ s/(r-1), & \text{otherwise}\,. \end{cases} \tag{11}$$

PROOF. By (9) we have with τ defined in (10), for $0 \leqq h\eta \leqq \eta_0$,

$$\|E_{k,\eta}\|_2 \leqq \exp(-\lambda\varrho h\eta + C(h\eta)^\tau)\,,$$

and hence by (6), since $\exp(\eta x) \leqq 1$ where v is non-zero,

$$\|\chi_{\delta-\varrho t} E_k^n v\|_2 \leqq \exp(-\delta\eta + Cn(h\eta)^\tau)\,\|v\|_2\,.$$

The result now follows by Lemma 2 with $\alpha = 0$.

We now turn to maximum-norm estimates. As was the case in [2], our main technical tool will be the following form of the Carleson–Beurling inequality.

LEMMA 3. *Let $\varphi \in C_0^\infty$ have support in $(-\tfrac{3}{2}\pi, \tfrac{3}{2}\pi)$ and be identically 1 for $|\xi| \leqq \pi$. Then there is a constant C such that for any operator E_k of the form (3) with symbol a, the following inequality holds:*

$$\|E_k\|_\infty \leqq C\|\varphi a_\alpha\|_2^{\frac{1}{2}}\,\|D(\varphi a_\alpha)\|_2^{\frac{1}{2}}, \qquad D = D_\xi = d/d\xi\,,$$

where $a_\alpha(\xi) = \exp(i\alpha\lambda\xi)a(\xi)$ and α is an arbitrary real number.

PROOF. See [2, Lemmas 2.1, 2.5 and 2.7].

REMARK. Lemma 2.7 in [2] is somewhat exaggeratedly claimed to be a trivial consequence of the closed graph theorem. The proof by the closed graph theorem depends on the fact that if a is 1-periodic and $\eta \in \mathscr{C}_0$ is such that $\eta = 1$ in an interval of length 1 then $a\eta \in M_p$ implies $a \in M_p$. By means of a finite periodic partition of unity this in turn can be reduced to the following statement: If $\varphi \in M_p$ has its support in an interval of length < 1 and if

$$\tilde{\varphi}(x) = \sum_{j=-\infty}^{\infty} \varphi(x+j)$$

is its 1-periodic extension then $\tilde{\varphi} \in M_p$. For the purpose of this paper and in fact also in [2] it is sufficient to have this result for $p=\infty$. But in that case the result follows from Theorems 1.5.1 (a) and 2.7.6 in Rudin [5].

We can now state and prove the maximum-norm analogue of Theorem 1. In doing so we include the possibility of a discontinuity in a derivative of v rather than in v itself; we shall see that if v tends to zero at a certain rate as $x \to 0$, a corresponding improvement of the estimate can be obtained.

THEOREM 2. *Assume that v vanishes for positive x and that $|x|^{-\alpha} v(x) \in L_\infty$ for some $\alpha \geqq 0$. Then for given $T > 0$ there are positive constants c and C such that for $t \leqq T$,*

$$\|\chi_{\delta-\varrho t} E_k^n v\|_\infty \leqq C h^{\alpha/\varkappa} n^{\omega} \exp(-cn \min\{\delta_t^\varkappa, \delta_t\}) \, \||x|^{-\alpha} v\|_\infty$$

where $\varkappa$ is defined by (4) *and*

$$\begin{aligned} \omega &= 0, && \text{if } r \text{ is odd and } \beta < 0\,, \\ &= \tfrac{1}{2}(1 - r/s), && \text{otherwise}\,. \end{aligned}$$

PROOF. Let φ be the function in Lemma 3. Assume first r odd, $\beta < 0$. By (8) we then have for $\eta \leqq \eta_0$,

$$|\varphi(\xi)\, a_\varrho(\xi - i\eta)^n| \leqq C \exp(-cn\eta\xi^{r-1} + Cn\eta^r)\,,$$

$$|D_\xi(\varphi(\xi)\, a_\varrho(\xi - i\eta)^n)| \leqq Cn(\xi^{r-1} + \eta^{r-1}) \exp(-cn\eta\xi^{r-1} + Cn\eta^r)\,,$$

and hence

$$\|\varphi a_\varrho(\,.\, - i\eta)^n\|_2 \leqq C(n\eta)^{-\frac{1}{2}/(r-1)} \exp(Cn\eta^r)\,,$$

$$\|D_\xi(\varphi a_\varrho(\,.\, - i\eta)^n)\|_2 \leqq Cn(n\eta)^{-\frac{1}{2}/(r-1)}\,[(n\eta)^{-1} + \eta^{r-1}] \exp(Cn\eta^r)\,.$$

Consequently by Lemma 3,

$$\begin{aligned} \|E_{k,\eta}^n\|_\infty \leqq Cn^{\frac{1}{2}}\,(nh\eta)^{-\frac{1}{2}/(r-1)}\,[(nh\eta)^{-\frac{1}{2}} + \\ + (h\eta)^{\frac{1}{2}(r-1)}] \exp(-n\lambda h\eta\varrho + Cn(h\eta)^r)\,. \end{aligned}$$

For $\eta > 0$ we have

$$\|e^{\eta x} v\|_\infty \leqq \sup_{x<0}\{|x|^\alpha e^{\eta x}\}\, \||x|^{-\alpha} v\|_\infty = C\eta^{-\alpha}\, \||x|^{-\alpha} v\|_\infty\,,$$

and hence by (6) for $0 < h\eta \leqq \eta_0$,

$$\|\chi_{\delta-\varrho t}E_k^n v\|_\infty \leqq C\eta^{-\alpha}n^{\frac{1}{2}}(nh\eta)^{-\frac{1}{2}/(r-1)}\left[(nh\eta)^{-\frac{1}{2}}+(h\eta)^{\frac{1}{2}(r-1)}\right]\cdot$$
$$\cdot\exp\left(-\delta\eta+Cn(h\eta)^r\right).$$

The result now follows by Lemma 2 in this case. In the opposite case we have by (7) with $\tau=s(s-r+1)^{-1}$ for $\eta\leqq\eta_0$,

$$|\varphi(\xi)a_\varrho(\xi-i\eta)^n| \leqq C\exp\left(-cn\xi^s+Cn\eta^\tau\right),$$
$$|D_\xi(\varphi(\xi)a_\varrho(\xi-i\eta)^n)| \leqq Cn(|\xi|^{r-1}+\eta^{r-1})\exp\left(-cn\xi^s+Cn\eta^\tau\right),$$

and hence

$$\|\varphi a_\varrho(\,.\,-i\eta)^n\|_2 \leqq Cn^{-\frac{1}{2}/s}\exp\left(Cn\eta^\tau\right),$$
$$\|D_\xi(\varphi a_\varrho(\,.\,-i\eta)^n)\|_2 \leqq Cn^{1-\frac{1}{2}/s}(n^{-(r-1)/s}+\eta^{r-1})\exp\left(Cn\eta^\tau\right).$$

By Lemma 3 we therefore obtain for $h\eta\leqq\eta_0$,

$$\|E_{k,\eta}^n\|_\infty \leqq Cn^{\frac{1}{2}(1-r/s)}\left[1+n^{\frac{1}{2}(r-1)/s}(h\eta)^{\frac{1}{2}(r-1)}\right]\exp\left(-n\lambda h\eta\varrho+Cn(h\eta)^\tau\right),$$

and the result follows as above.

Above we have only considered the behavior to the right of the discontinuity. Clearly a corresponding analysis holds to the left. This case may be reduced to the one treated above by a change of sign in x. Notice that since the symbol of the corresponding operator is $a(-\xi)$, the coefficient of $i\xi^r$ for r odd then changes sign, so that the values of τ, $\varkappa$, and ω above are altered accordingly. Consequently for odd r, the estimates will be different on the two sides of the discontinuity.

We shall finally prove that when $r<s$, even if we are not in the case r odd, $\beta<0$, the factor $n^{\frac{1}{2}(1-r/s)}$ in Theorem 2 can be suppressed if v is smooth for $x<0$. The proof of this fact will be somewhat technical and will depend heavily on the presentation in [2]. The first step is the next lemma, the proof of which will be modelled after that of Theorem 5.1 in [2]. We shall express the result in terms of the standard norm on $\mathscr{C}^\alpha$,

$$\|v\|_{\mathscr{C}^\alpha} = \|v\|_\infty+\begin{cases}\sup_x|D^jv(x)|, & \alpha_0=0\,,\\ \sup_{x,y}|D^jv(x)-D^jv(y)|\,|x-y|^{-\alpha_0}, & 0<\alpha_0<1\,,\end{cases}$$

where α is positive and $\alpha=j+\alpha_0$, with j non-negative integer and $0\leqq\alpha_0<1$.

We assume from now on that $r<s$. In particular, β is real in (4).

LEMMA 4. *For given α with $\frac{1}{2}r<\alpha<r-\frac{1}{2}$ there are positive constants c*

and C *such that for* $v \in \mathscr{C}^\alpha$, *vanishing for positive* x, *we have for* $\delta_t \leqq 1$, $t = nk$, *with* $\varkappa = s/(r-1)$,

$$\|\chi_{\delta-\varrho t} E_k^n v\|_\infty \leqq C t h^{\alpha(1-1/r)} \exp(-cn\delta_t^\varkappa) \|v\|_{\mathscr{C}^\alpha} . \tag{12}$$

Proof. By Theorem 2 it is sufficient to prove (12) for v with compact support and it will then clearly be enough to consider v in $\mathscr{C}_0^\infty$. As in [2] the technical work will be carried out in a different norm. For α positive, let $\omega_\alpha(\zeta) = \zeta^\alpha$ be the branch which is positive for $\zeta = \xi$ positive and analytic for $\eta = \operatorname{Im}\zeta < 0$, $\zeta \neq 0$. Let then $L_{\infty,\alpha}^*$ be the completion of $\mathscr{C}_0^\infty$ with respect to the norm

$$\|v\|_{\infty,\alpha}^* = \|\mathscr{F}^{-1}(\omega_\alpha \hat{v})\|_\infty ,$$

where $\mathscr{F} v = \hat{v}$ denotes the Fourier transform of v and ω_α is taken along the real axis. Notice that although the definition of $L_{\infty,\alpha}^*$ is slightly modified as compared to that in [2] for α non-integer, the embedding and interpolation results described in Lemmas 2.9 and 2.10 in [2] and their proofs remain unchanged. Consequently, it is sufficient to prove (12) for $v \in \mathscr{C}_0^\infty$ and with $\|v\|_{\mathscr{C}^\alpha}$ replaced by $\|v\|_{\infty,\alpha}^*$. We shall use the multiplier norm $M(a) = M_\infty(a)$ defined by

$$M(a) = \sup_{0 \neq v \in \mathscr{C}_0^\infty} \frac{\|\mathscr{F}^{-1} a * v\|_\infty}{\|v\|_\infty} .$$

Assume now that $v \in \mathscr{C}_0^\infty$ and v vanishes for $x > 0$. For $\eta > 0$ we then have

$$\mathscr{F}\big(e^{\eta x} \mathscr{F}^{-1}(\omega_\alpha \hat{v})\big) = \omega_\alpha(\xi - i\eta)\, \hat{v}(\xi - i\eta) . \tag{13}$$

By our assumptions on v the right hand side is analytic for $\eta > 0$, and since $\hat{v}(\xi - i\eta)$ is bounded there it follows by the Paley–Wiener theorem that $e^{\eta x} \mathscr{F}^{-1}(\omega_\alpha \hat{v})$ also vanishes for $x > 0$. Hence

$$\|e^{\eta x} \mathscr{F}^{-1}(\omega_\alpha \hat{v})\|_\infty \leqq \|\mathscr{F}^{-1}(\omega_\alpha \hat{v})\|_\infty = \|v\|_{\infty,\alpha}^* . \tag{14}$$

Introducing the operator defined by

$$E_\eta(t) v = e^{\eta x} E(t)(e^{-\eta x} v) = e^{-\eta \varrho t} E(t) v ,$$

we have for $\eta > 0$,

$$\begin{aligned} \|\chi_{\delta-\varrho t} E_k^n v\|_\infty &= \|\chi_{\delta-\varrho t}(E_k^n - E(t)) v\|_\infty \\ &\leqq e^{\eta \varrho t - \eta \delta} \|(E_{k,\eta}^n - E_\eta(t))(e^{\eta x} v)\|_\infty . \end{aligned}$$

Taking Fourier transforms we get

$$\mathscr{F}\big((E^n_{k,\eta}-E_\eta(t))(e^{\eta x}v)\big)$$
$$= e^{-i\varrho\xi t-\varrho\eta t}\left(a_\varrho(h\xi-i\eta h)^n-1\right)\hat{v}(\xi-i\eta)$$
$$= e^{-i\varrho\xi t-\varrho\eta t}\,\frac{a_\varrho(h(\xi-i\eta))^n-1}{\omega_\alpha(\xi-i\eta)}\cdot\omega_\alpha(\xi-i\eta)\hat{v}(\xi-i\eta)\,.$$

Hence, by (13) and (14)

$$\|\chi_{\delta-\varrho t}E^n_k v\|_\infty \leqq e^{-\eta\delta}M\left(\frac{a_\varrho(h(\,.\,-i\eta))^n-1}{\omega_\alpha(\,.\,-i\eta)}\right)\|v\|^*_{\infty,\alpha}\,. \tag{15}$$

Let $h_r=h^{1-1/r}$ and let

$$\sigma_{\alpha,h,n,\eta}(\xi) = \frac{a_\varrho(h^{1/r}\xi-i\eta h)^n-1}{\omega_\alpha(\xi-i\eta h_r)}\,.$$

With $\zeta=\xi-i\eta h_r$ we have by a simple computation that, $\eta h\leqq\eta_0$, $D_\xi=\partial/\partial\xi$,

$$|a_\varrho(h^{1/r}\zeta)^n-1| \leqq Ct\exp\big(Cn(\eta h)^\tau\big)\min\{1,|\zeta|^r\}\,,$$
$$|D_\xi(a_\varrho(h^{1/r}\zeta)^n-1)| \leqq Ct\exp\big(Cn(\eta h)^\tau\big)|\zeta|^{r-1}\,.$$

Choosing an even function $\varphi\in\mathscr{C}_0^\infty$ with support in $\{y;\ \frac{1}{2}<|y|<2\}$ such that

$$\sum_{j=-\infty}^{\infty}\varphi(2^{-j}y) = 1,\quad y\neq 0\,,$$

(cf. Lemma 2.8 in [2]), and letting

$$\varphi_{j,\eta}(\xi) = \varphi(2^{-j}|\zeta|),\quad j=1,2,\dots\,,$$
$$\varphi_{0,\eta}(\xi) = 1-\sum_{1}^{\infty}\varphi(2^{-j}|\zeta|)\,,$$
$$\Phi_{J,\eta}(\xi) = \sum_{j=0}^{J}\varphi_{j,\eta}(\xi)\,,$$

we have for $0\leqq\alpha<r-\frac{1}{2}$ that

$$\|\varphi_{j,\eta}\,\sigma_{\alpha,h,n,\eta}\|_2 \leqq Ct\exp\big(Cn(\eta h)^\tau\big)\,2^{j(\frac{1}{2}-\alpha)},\quad j\geqq 0\,,$$
$$\|D_\xi(\varphi_{j,\eta}\,\sigma_{\alpha,h,n,\eta})\|_2 \leqq Ct\exp\big(Cn(\eta h)^\tau\big)\,2^{j(r-\frac{1}{2}-\alpha)},\quad j\geqq 0\,.$$

Actually, except for $j=0$ in the second inequality, these estimates hold for $0\leqq\alpha\leqq r$. Lemma 2.5 in [2] now yields

$$M(\varphi_{j,\eta}\sigma_{\alpha,h,n,\eta}) \leqq Ct\exp\left(Cn(\eta h)^\tau\right) 2^{j(\frac{1}{2}r-\alpha)}, \quad j \geqq 0 .$$

Adding these estimates we get for $\frac{1}{2}r < \alpha < r-\frac{1}{2}$,

$$M(\Phi_{J,\eta}\sigma_{\alpha,h,n,\eta}) \leqq Ct\exp\left(Cn(\eta h)^\tau\right),$$

and letting J tend to ∞ we obtain

$$M(\sigma_{\alpha,h,n,\eta}) \leqq Ct\exp\left(Cn(\eta h)^\tau\right).$$

Using (15) this implies

$$\|\chi_{\delta-\varrho t}E_k^n v\|_\infty \leqq Ct\exp\left(-\eta\delta+Cn(\eta h)^\tau\right) h^{\alpha(1-1/r)} \|v\|_{\infty,\alpha}^* .$$

The result now follows by Lemma 2.

We shall need the following version of van der Corput's lemma.

LEMMA 5. *Let $u \in \mathscr{C}_0^{\,1}$ and let $\tilde\psi \in \mathscr{C}^2$ be real and satisfy $|\tilde\psi''| \geqq \delta > 0$ in an interval containing the support of u. Then*

$$\|\mathscr{F}(\exp(i\tilde\psi)u)\|_\infty \leqq 8\delta^{-\frac{1}{2}}\|u'\|_1 .$$

PROOF. See Lemma 2.4 in [2].

In our next lemma we shall estimate the coefficients in $E_{k,\eta}^n$,

$$E_{k,\eta}^n v(x) = \sum_{j=-\infty}^{\infty} a_{nj}(\eta)\, v(x-jh) . \tag{16}$$

LEMMA 6. *With $a_{nj}(\eta)$ defined by (16) there is a positive constant C such that for $h\eta \leqq \eta_0$,*

$$|a_{nj}(\eta)| \leqq Cn^{-1/r}\exp\left(-\eta\varrho t+Cn(h\eta)^\tau\right).$$

PROOF. Since $a_{nj}(\eta)$ are the Fourier coefficients of $a(\xi-ih\eta)^n$ it is sufficient to prove that for any real y, and $0<\eta\leqq\eta_0$,

$$\left|\int_{-\pi}^{\pi} e^{-i\xi y}\, a_\varrho(\xi-i\eta)^n\, d\xi\right| \leqq Cn^{-1/r}\exp(Cn\eta^\tau) . \tag{17}$$

By (4) there is an $\varepsilon_0>0$ such that for $|\xi|\leqq\varepsilon_0$ and $0\leqq\eta\leqq\varepsilon_0|\xi|$,

$$|D_\xi^{\,2}\,\mathrm{Im}\,\psi(\xi-i\eta)| \geqq c|\xi|^{r-2}, \quad c>0 .$$

Further, by Lemma 1 for $0<\eta\leqq\eta_0$ and $|\xi|\leqq\pi$ we have with $\tau = s(s-r+1)^{-1}$,

(18) $$\operatorname{Re}\psi(\xi - i\eta) \leqq -c\xi^s + C\eta^\tau ,$$

and also

(19) $$|D_\xi \operatorname{Re}\psi(\xi - i\eta)| \leqq C(|\xi|^{s-1} + \eta|\xi|^{r-2} + \eta^{r-1}) .$$

Let for small positive ε, $h_\varepsilon \in \mathscr{C}_0^\infty$ be a function with support in $\frac{1}{2}\varepsilon \leqq |\xi| \leqq \varepsilon_0$ which satisfies the following:

(i) $h_\varepsilon = 1$ in $\varepsilon \leqq |\xi| \leqq \frac{1}{2}\varepsilon_0$,
(ii) $|h_\varepsilon| \leqq 1$,
(iii) $|h_\varepsilon'| \leqq C\varepsilon^{-1}$ in $|\xi| \leqq \varepsilon$ and $|h_\varepsilon'| \leqq C$ in $\frac{1}{2}\ \varepsilon_0 \leqq |\xi| \leqq \varepsilon_0$ for some constant C.

If we set $\chi_n(\xi,\eta) = h_\varepsilon(\xi)\exp(n \operatorname{Re}\psi(\xi - i\eta))$ we have by (18) and (19),

$$\|D_\xi\chi_n(\cdot,\eta)\|_1 \leqq C\int [n(|\xi|^{s-1} + \eta|\xi|^{r-2} + \eta^{r-1}) + |h_\varepsilon'|]\exp(-cn\xi^s + Cn\eta^\tau)\,d\xi ,$$

and hence after a simple computation,

$$\|D_\xi\chi_n(\cdot,\eta)\|_1 \leqq C\exp(Cn\eta^\tau) .$$

It follows by Lemma 5 that since $|\xi| \geqq \frac{1}{2}\varepsilon$ in the support of $\chi_n(\xi,\eta)$, we have for $\eta \leqq \frac{1}{2}\varepsilon_0\varepsilon$,

(20) $$\begin{aligned}\left|\int_{-\pi}^{\pi} e^{-i\xi y}\, h_\varepsilon(\xi)\, a_\varrho(\xi - i\eta)^n\,d\xi\right| &= \left|\int \exp(-i\xi y + ni\operatorname{Im}\psi(\xi - i\eta))\,\chi_n(\xi,\eta)\,d\xi\right| \\ &\leqq C\varepsilon^{-\frac{1}{2}(r-2)}\,n^{-\frac{1}{2}}\exp(Cn\eta^\tau) .\end{aligned}$$

On the other hand

(21) $$\left|\int_{-\pi}^{\pi} e^{-i\xi y}\,(1 - h_\varepsilon(\xi))\,a_\varrho(\xi - i\eta)^n\,d\xi\right| \leqq (C\varepsilon + \exp(-cn\varepsilon_0{}^s))\exp(Cn\eta^\tau) .$$

Choosing $\varepsilon = n^{-1/r}$ we may conclude from (20) and (21) that (17) holds for $\eta \leqq \frac{1}{2}\varepsilon_0 n^{-1/r}$. For $\frac{1}{2}\varepsilon_0 n^{-1/r} \leqq \eta \leqq \eta_0$ we have $n\eta^\tau \geqq cn^{1-\tau/r}$ and since $\tau < r$ we obtain

$$\int_{-\pi}^{\pi} |a_\varrho(\xi - i\eta)^n|\,d\xi \leqq C\exp(Cn\eta^\tau) \leqq Cn^{-1/r}\exp(Cn\eta^\tau) ,$$

which completes the proof of the lemma.

LEMMA 7. *There are positive constants c and C such that if* $v \in L_\infty$ *has support in* $[-\varepsilon, 0]$ *and* $\delta_t \leqq 1$, $t = nk$,

$$\|\chi_{\delta-\varrho t} E_k^n v\|_\infty \leqq C t^{-1/r} \varepsilon\, h^{-(1-1/r)} \exp(-cn\delta_t^\varkappa)\, \|v\|_\infty .$$

PROOF. We have by Lemma 6 for $\eta \leqq \eta_0$,

$$\|E_{k,\eta}^n(e^{\eta x} v)\|_\infty \leqq \sum_{x-jh\in[-\varepsilon,0]} |a_{nj}(\eta)|\, \|e^{\eta x} v\|_\infty$$
$$\leqq C t^{-1/r} \varepsilon\, h^{-(1-1/r)} \exp(-n\varrho t + Cn(h\eta)^\tau)\, \|v\|_\infty ,$$

so that

$$\|\chi_{\delta-\varrho t} E_k^n v\|_\infty \leqq C t^{-1/r} \varepsilon h^{-(1-1/r)} \exp(-\eta\delta + Cn(\eta h)^\tau)\, \|v\|_\infty .$$

The result now follows as above.

We can now state and prove the above mentioned improvement of Theorem 2. We shall denote by $\mathscr{C}_-^\alpha$ the set of functions v which are in $\mathscr{C}^\alpha$ for $x \leqq 0$ and vanish for $x > 0$ (notice that they may have a discontinuity at $x = 0$). We set with α as above,

$$\|v\|_{\mathscr{C}_-^\alpha} = \|v\|_\infty + \begin{cases} \sup_{x<0} |D^j v(x)|, & \alpha_0 = 0 , \\ \sup_{x,y<0} |D^j v(x) - D^j v(y)|\, |x-y|^{-\alpha_0}, & 0 < \alpha_0 < 1 . \end{cases}$$

THEOREM 3. *For any* $\alpha > \frac{1}{2} r$ *and* $T > 0$ *there are positive constants c and C such that for* $v \in \mathscr{C}_-^\alpha$ *and* $\delta_t \leqq 1$, $t \leqq T$, *we have with* $\varkappa = s/(r-1)$,

$$\|\chi_{\delta-\varrho t} E_k^n v\|_\infty \leqq C \exp(-cn\delta_t^\varkappa)\, \|v\|_{\mathscr{C}_-^\alpha} .$$

PROOF. We may clearly assume that $r < s$ and $\frac{1}{2} r < \alpha < r - \frac{1}{2}$. Let $\varepsilon > 0$ and let $v_\varepsilon \in \mathscr{C}^\alpha$ vanish for $x > 0$, coincide with v for $x \leqq -\varepsilon$ and satisfy an inequality of the form

$$\|v_\varepsilon\|_{\mathscr{C}^\alpha} \leqq C \varepsilon^{-\alpha} \|v\|_{\mathscr{C}_-^\alpha} .$$

We then obtain by Lemma 4,

$$\|\chi_{\delta-\varrho t} E_k^n v_\varepsilon\|_\infty \leqq C t \varepsilon^{-\alpha} h^{\alpha(1-1/r)} \exp(-cn\delta_t^\varkappa)\, \|v\|_{\mathscr{C}_-^\alpha} ,$$

and since $v - v_\varepsilon$ has its support in $[-\varepsilon, 0]$, by Lemma 6,

$$\|\chi_{\delta-\varrho t} E_k^n (v - v_\varepsilon)\|_\infty \leqq C t^{-1/r} \varepsilon h^{-(1-1/r)} \exp(-cn\delta_t^\varkappa)\, \|v\|_\infty .$$

The result now follows at once by the triangle inequality from these last two inequalities if we choose $\varepsilon = t^{1/r} h^{1-1/r}$.

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