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ARCS OF LOCAL CYCLIC ORDER THREE

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1. Introduction.

Differentiability properties of arcs of cyclic order three are quite familiar having been studied, among others, by Lane and Scherk [3] and M. Marchaud [5]. Such arcs have a unique pencil of general tangent circles at each point, varying continuously with the point. The condition of cyclic order three is not quite sufficient to ensure unique general osculating circles at each point, but there does exist a family of general osculating circles. There is at least one general osculating circle for every point p of the arc and the circle is unique except at a countable number of points. It is also known [4] that for arcs of cyclic order three this family of general osculating circles has essentially the same nesting property as the family of osculating circles of an arc of differentiable strictly monotone curvature. In this latter case the property is a trivial result of the familiar "unwinding string" property of the evolute of such an arc. It is the primary purpose of this paper to prove that this nesting property is a necessary and sufficient condition that an arc have local cyclic order three, i.e. that every point has a neighbourhood of cyclic order three (§ 2, § 3). Counter examples are given (§ 4) to two plausible but false conjectures regarding arcs of local cyclic order three.

2. The nesting property.

In this discussion an arc will be a topological image in the conformal plane (or on a sphere) of a closed interval. Let small letters p, q, etc. be used to denote points of such an arc. Following Lane and Scherk [3], a circle C is called a general tangent circle to an arc A at point p if there exists a sequence of triples of distinct points (t_i, u_i, R_i) such that $C = \lim C(t_i, u_i, R_i)$ where $\{t_i\}$ and $\{u_i\}$ converge on A to p, $\{R_i\}$ converges and $C(t_i, u_i, R_i)$ denotes the unique circle through the indicated points. If, in addition, $\{R_i\} = \{v_i\}$ and also converges on A to p, then C is called a general osculating circle at p.

It will sometimes be convenient to give a non-null circle C an orientation. The remaining points of the conformal plane are then separated

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into sets C_* and C^* which denote respectively the sets to the left and right of C.

Let A_3 be an arc of cyclic order three with endpoints a and b. If u, v, w are distinct interior points of A_3 and if the circle $C(u,v,w)$ is oriented by the order of points u, v, w on A_3, it is an easy consequence of the known properties of arcs of cyclic order three [3] that for all u, v, w either (1) $a \in C^*(u,v,w)$ and $b \in C_*(u,v,w)$ or (2) $a \in C_*(u,v,w)$ and $b \in C^*(u,v,w)$. The arc A_3 will be called positive or negative according as case (1) or case (2) holds. In the limit these properties also hold for the oriented general osculating circles at interior points of A_3. Thus if C_p is a general osculating circle at an interior point p of A_3, then for a positive arc $a \in C_p{}^*$, $b \in C_{p*}$ while for a negative arc $a \in C_{p*}$, $b \in C_p{}^*$. The obvious extensions for general osculating circles at the endpoints also hold.

Definition. The general osculating circles to an arc A are said to have the nesting property if the following condition is satisfied: If q_1, q_2, q_3 are distinct points of A with q_2 separating q_1 and q_3 on A and if C_1, C_2, C_3 are any general osculating circles at q_1, q_2, q_3 respectively, then C_2 separates C_1 and C_3.

Definition. If every sufficiently small neighbourhood of a point p of arc A has cyclic order k for some finite k, then A is said to have *cyclic order k at p.*

Definition. If an arc A has cyclic order three at each point, it is said to have *local cyclic order three.*

The following fact has been established by Lane, Singh, and Scherk [4].

Lemma A. *Let p_1 and p_2 be distinct points of an arc A_3 of cyclic order three, and let C_1 and C_2 be general osculating circles at p_1 and p_2 respectively. Then $C_1 \cap C_2 = \emptyset$.*

By use of Lemma A it will be shown that the nesting property can be extended to arcs of local order three.

Theorem 1. *The general osculating circles to an arc of local cyclic order three have the nesting property.*

Proof. By the Heine–Borel Theorem, arc A is the union of a finite number of overlapping arcs, each of cyclic order three. If two such neighbourhood arcs overlap they are either both positive or both negative, whence it follows that all these neighborhood arcs of A are of the same kind. It will be sufficient to consider the case when they are all positive.

For a positive arc of cyclic order three, the definition together with Lemma A shows that if p precedes q on the arc and if C_p and C_q are general osculating circles at p and q respectively, then $C_{q*} \cup C_q \subset C_{p*}$. By an entirely routine argument this property can be extended to the whole arc A.

The proof of the theorem is now easily completed. Let q_1, q_2, q_3 be three distinct points in this order on A, and let C_1, C_2, C_3 be any general osculating circles at these points. By the property noted above since q_1 precedes q_2 and q_2 precedes q_3, we have $C_2 \cup C_{2*} \subset C_{1*}$ and $C_3 \cup C_{3*} \subset C_{2*}$. This first relation, by taking complements, is equivalent to $C_2{}^* \supset C_1 \cup C_1{}^*$. Thus $C_1 \subset C_2{}^*$ and $C_3 \subset C_{2*}$ so C_2 separates C_1 and C_3. This completes the proof.

3. Some properties of arcs.

In the following discussion it is to be established that the nesting property is not only a necessary condition for A to be of local cyclic order three (as is proved in Theorem 1) but that it is also sufficient.

LEMMA B. *If at point p of arc A the null circle is not a general osculating circle, then the general tangent circles at p form a unique pencil of the second kind with p as fundamental point.*

PROOF. Consider a sequence of circles $C(p,r_i,r_i')$ where $\{r_i\}$ and $\{r_i'\}$ converge on A to p. We may assume (by choosing a subsequence if necessary) that this sequence converges to a limit circle C which is, by definition, a general osculating circle at p. By hypothesis C is not a point circle. Let $P \neq p$ be any other point of C. There exists therefore a sequence $\{P_i\}$ converging to P such that $C(p,r_i,r_i')=C(p,r_i,P_i)$ so that $C=\lim C(p,r_i,P_i)$. Moreover, the angle between $C(p,r_i,P_i)$ and $C(p,r_i,P)$ approaches 0 whence it follows that $C=\lim C(p,r_i,P)$.

Let K now be any non-null general tangent circle to A at p. Then by definition $K=\lim C(q_i,q_i',Q_i)$ where $\{q_i\}$, $\{q_i'\}$ converge on A to p and $\{Q_i\}$ converges. Without loss of generality we may suppose sequence $\{Q_i\}$ chosen to converge to a point $Q \neq p$ on K. Since, as above, the angle between $C(q_i,q_i',Q_i)$ and $C(q_i,q_i',Q)$ approaches 0, we may conclude that $K=\lim C(q_i,q_i',Q)$. We may also assume, without loss of generality, that q_i, $q_i' \neq p$ for all i.

Consider next the sequence $C(p,q_i,q_i')$ which we may assume convergent to a limit circle C', and since C' is a general osculating circle it is non-null. If $R \neq p$ is a point of C', then, using again the argument above, it follows that $C' = \lim C(q_i, q_i', R)$ and also that $C' = \lim C(p, q_i, R)$. Since the angle between $C(q_i, q_i', R)$ and $C(q_i, q_i', Q)$ approaches 0, it follows that their limit circles C' and K are tangent at p.

Consider finally the sequence $C(p, q_i, r_i)$ which, as before, we may assume convergent to a non-null circle C''. If $S \neq p$ is a point of C'', the same argument as before shows that $C'' = \lim C(p, q_i, S)$ and $C'' = \lim C(p, r_i, S)$. Since the angle between $C(p, q_i, S)$ and $C(p, q_i, R)$ approaches 0, the limit circles C'' and C' are tangent at p. Similarly, since the angle between $C(p, r_i, S)$ and $C(p, r_i, P)$ approaches 0, circles C'' and C are tangent at p. This discussion shows circles K, C', C'', C to be mutually tangent at p. Since K was an arbitrary general tangent circle at p, it follows that all such circles belong to the pencil of circles tangent at p to the fixed circle C. Since it is trivial that all circles of this pencil are general tangent circles, this completes the proof of the lemma.

LEMMA C. *If the general osculating circles of arc A at point p do not contain every point of the plane, then there is a neighbourhood B of p on A such that if q, r, s, t are distinct points of B which occur in this order on A and which also lie on a circle C, then q, r, s, t are in cyclic order on C.*

PROOF. Suppose the lemma is false. Then there must be a sequence of quadruples of distinct points q_i, r_i, s_i, t_i which have the following properties: (a) each of the sequences $\{q_i\}$, $\{r_i\}$, $\{s_i\}$, $\{t_i\}$ converges on A to p; (b) for each i the points q_i, r_i, s_i, t_i belong to a circle C_i; (c) for each i the points q_i, r_i, s_i, t_i are in this order on A but are not in cyclic order on C_i.

Condition (c) means that the orientation induced on C_i by q_i, r_i, s_i is opposite to that induced by q_i, s_i, t_i. Consider the family of circles $C(q_i, u, v)$. If the pair of distinct points u, v vary continuously on A from r_i, s_i to s_i, t_i, the circle $C(q_i, u, v)$ varies continuously from C_i with one orientation to C_i with opposite orientation. In such a variation the circle $C(q_i, u, v)$ must pass through every point of the plane since the right and left sides of C_i have been interchanged. Thus if $z \neq p$ is an arbitrary point, there must exist points u_i, v_i between q_i and t_i on A such that $z \in C(q_i, u_i, v_i)$. As usual, we may assume this sequence of circles converges to a limit circle C. Clearly $z \in C$. Since, from condition (a), $\{u_i\}$, $\{v_i\}$, $\{q_i\}$ converge on A to p, C is a general osculating circle to A at p. Thus there is a general osculating circle at p through every point of the plane. But this contradicts the hypothesis on p and thus is impossible. Hence the lemma cannot be false and the proof is complete.

If at a point p of an arc A the general tangent circles form a unique family of the second kind, it is easy to show that arc A induces a uniquely determined orientation of the circles, so it is meaningful to speak of the oriented general tangent pencil. The following lemma uses this concept.

LEMMA D. *Let A be an arc whose general tangent circles at any point form a unique pencil of the second kind, and let K_0 be an oriented circle satisfying the following conditions:*

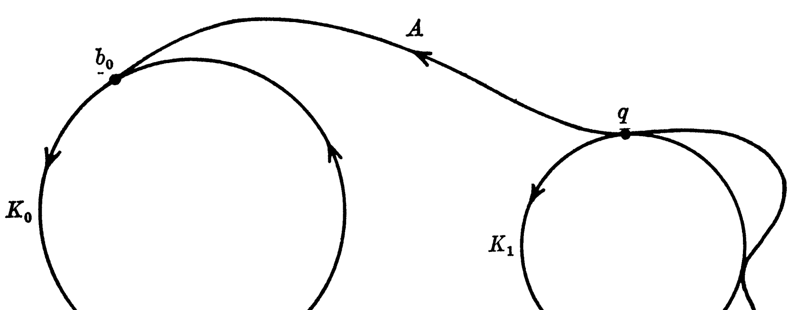
(a) *K_0 contains the endpoints a_0 and b_0 of A and does not meet A elsewhere;*

(b) *K_0 belongs to the oriented tangent pencils to A at a_0 and b_0.*

Then there is an interior point p of A and a general osculating circle C at p such that A never crosses C.

PROOF. We shall give a proof by contradiction. Thus we assume that every general osculating circle to A at an interior point has points of A on both sides of it. In particular, no null circle at an interior point can be a general osculating circle.

Since A meets K_0 only at a_0 and b_0, then $A \setminus \{a_0, b_0\}$ is a subset either of K_{0*} or $K_0{}^*$. For definiteness suppose that $A \setminus \{a_0, b_0\} \subset K_0{}^*$. Let S_0 denote the simple closed curve which is the union of A and the directed arc $b_0 a_0$ of K_0. It is readily seen that S_0 has a unique pencil of general tangent circles at every point. The only question might be for points a_0 and b_0 where it follows by hypothesis (b). Curve S_0 divides the plane into two parts. Let S_{0*} be the set containing K_{0*}. Thus S_{0*} is the region to the left of S_0.



Let q be the midpoint of the arc A, i.e., the point corresponding to the midpoint of the parameter interval. In the pencil τ of general tangent circles at q there must be some whose interiors are subsets of S_{0*}, for otherwise the null circle at q would be a general osculating circle. If a variable circle moves to the left through the pencil τ, let K_1 be the first such circle for which $K_1 \cup K_{1*} \subset S_0 \cup S_{0*}$. Clearly K_1 belongs to the oriented general tangent pencil at any point where it meets S_0. Moreover, K_1 can contain no point of arc $b_0 a_0$ of K_0, since a circle tangent to K_0 on this arc and contained in $S_0 \cup S_{0*}$ would necessarily be a subset of $K_0 \cup K_{0*}$ while K_1 contains the point q in $K_0{}^*$. Thus $K_1 \cap A$ contains only interior points of A.

Since A does not cross K_1, it follows from the contradiction assumption that K_1 is not a general osculating circle at q. Let Q_i be a sequence of circles of τ converging to K_1 from the right, i.e., such that $K_{1*} \subset Q_{i*}$. Then, by definition of K_1, each circle Q_i must contain a point u_i of A different from q. By choosing a subsequence if necessary we may assume that sequence $\{u_i\}$ converges to a point r of A. We may conclude that $r \neq q$ since if $r = q$, the circle K_1 would be a general osculating circle contrary to the assumption. Thus $K_1 \cap A$ contains the distinct points q, r but does not contain the entire arc qr since K_1 is not a general osculating circle. It is then possible to find points a_1, b_1 of the subarc qr of A such that $a_1, b_1 \in K_1$, but no other points of subarc $a_1 b_1$ belong to K_1. Thus circle K_1 and subarc $a_1 b_1$ of A satisfy the same hypotheses (a) and (b) as circle K_0 and arc A. The procedure above may then be iterated so that S_1 is the simple closed curve which is the union of subarc $a_1 b_1$ of A and arc $b_1 a_1$ of K_1 while S_{1*} is the region bounded by S_1 which contains K_{1*}. Note that by its definition $S_1 \cup S_{1*} \subset S_0 \cup S_{0*}$ and that the parameter length of $a_1 b_1$ does not exceed half the parameter length of A.

Proceeding inductively we may therefore define a sequence of circles K_i, with a corresponding sequence of arcs $a_i b_i$ on A and a resulting sequence of simple closed curves S_i. As noted above, the sequence of regions $S_i \cup S_{i*}$ is a decreasing nested sequence, the subarcs $a_i b_i$ of A are also a decreasing nested sequence whose parameter lengths approach 0, and each K_i is tangent to A at the distinct points a_i, b_i. By choosing a subsequence if necessary, we may assume that the arcs $a_i b_i$ converge to a point p of A and that the circles K_i converge to a limit circle C. Point p belongs to all the arcs $a_i b_i$ and is therefore an interior point of A since none of these arcs after the first contains a_0 or b_0. Circle C is clearly a general osculating circle at p since the points of tangency a_i, b_i converge to p. Moreover, by the nesting property of the regions $S_i \cup S_{i*}$, circle C is a subset of $S_0 \cup S_{0*}$ and hence A never crosses C. This contradicts the intial assumption that A crosses every general osculating circle at an interior point. Thus the contradiction proof of Lemma D is complete.

It may be noted that for the special case when arc A has continuous curvature this lemma is a familiar result. It is, for example, an immediate consequence of Lemma 4.1 and Corollary 2.1.1 of reference [2].

It is now possible to prove the following result.

THEOREM 2. *Let A be an arc satisfying the following conditions:*

(a) *At each interior point A crosses each general osculating circle;*

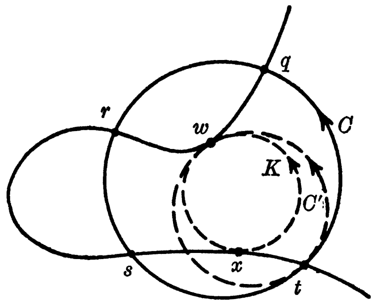
(b) *At neither endpoint does the set of general osculating circles cover the entire plane.*

Then A is of local cyclic order three.

PROOF. The proof will be by contradiction. Assume therefore that there exists a point of A for which the cyclic order is greater than three. Then any neighbourhood of p on A contains at least four concyclic points. If p is an endpoint, then by (b) the general osculating circles do not cover the entire plane. Suppose that p is an interior point of A. Since, by (a), arc A crosses any general osculating circle at an interior point, there is no null osculating circle at such a point. It follows from Lemma B that the general tangent circles at any interior point form a pencil of the second kind. In particular this holds at p. The general osculating circles at p are a subset of this pencil. Hence the only way the set of general osculating circles at p could cover the plane would be for the entire pencil, except perhaps the null circle, to be general osculating circles. But the null circle would be a limit of general osculating circles and hence a general osculating circle itself. Since this is false, it follows that the general osculating circles at p cannot cover the plane. Thus in any case, the hypothesis of Lemma C is satisfied at p. By Lemma C, therefore, there is a neighbourhood of p within which the order of any set of concyclic points on A agrees with their cyclic order on circle C. Let four such common points be selected q, r, s, t. It is clear that the selection can be made in such a way that subarc qr of A meets C only at q and r. For definiteness suppose this arc is contained in $C \cup C_*$ where the orientation of C is induced by the cyclic order of q, r, s, t. Note also that C may be selected so that all points of arc qt are interior points of A.

It was established above by the use of Lemma B that the general tangent circles at any interior point of A form a unique pencil. This holds therefore for all points of qt. As noted just before Lemma D, these pencils have a unique orientation induced by A.

Consider the pencil of circles tangent to C at t and given the orientation induced by C. Let a circle of this pencil move continuously through C_* from C to the null circle at t. In this motion there is a last position, C', for which the circle contains a point of qr. Let w be such a point. Since C', by definition, has no points of qr to its left, it is a member



of the pencil of general tangent circles at w. Moreover C' clearly has the orientation induced by A at w, so C' belongs to the oriented general tangent pencil at w. Now let a circle of this tangent pencil at w move continuously through C_*' from position C' to the null circle at w. Since all such circles meet qr only at w, and since $t \in C'$, there is a last position K for which the circle has any point other than w in common with wt. Let x be the first point of $A \cap K$ following w on A. Then, as before, K is a member of the oriented tangent pencil at x. The arc wx and the circle K satisfy the conditions of Lemma D. By Lemma D there is a point of wx at which some general osculating circle fails to cross A. This is impossible since, by (a), arc A crosses every general osculating circle at an interior point. This contradiction establishes the theorem.

It is now a trivial matter to verify the following converse of Theorem 1.

THEOREM 3. *If the general osculating circles of an arc A have the nesting property, then A is of local cyclic order three.*

PROOF. If the general osculating circles have the nesting property, then two points on opposite sides of p on A are separated by any general osculating circle at p, so condition (a) of Theorem 2 holds. Moreover, at any point, endpoint or otherwise, a general osculating circle can never contain any other point of A. Hence the general osculating circles at a point can never cover the entire plane, so condition (b) also holds. Theorem 3 then follows from Theorem 2.

For the special case of arcs of continuous curvature the following, fairly obvious, result follows readily.

THEOREM 4. *If A is an arc of continuous curvature, a necessary and sufficient condition that the arc be of local cyclic order three is that the curvature be strictly monotone.*

PROOF. For an arc of continuous curvature it is well known that the general osculating circles are precisely the ordinary circles of curvature.

Let A be of local cyclic order three. It follows from Theorem 1 that A crosses every general osculating circle, i.e. every circle of curvature. But it is known [2, Cor. 2.1.1] that at any extremum of the curvature the arc does not cross its circle of curvature. Hence the curvature has no extremum and must be monotone. Moreover, it must be strictly monotone since otherwise the arc would contain circular arcs where A would not have local cyclic order three.

Conversely, let A have strictly monotone curvature. By [2, Cor. 2.1.2] at each interior point A crosses each circle of curvature. Hence condition (a) of Theorem 2 is satisfied. But at all points including end-points, A has a unique circle of curvature i.e. a unique general osculating circle. Thus condition (b) of Theorem 2 holds also and the desired result follows from Theorem 2.

4. Counterexamples.

It is a natural question to ask whether the awkward appearing condition (b) of Theorem 2 could be deleted, as it appeared to play a very minor role in the proof. The answer is in the negative. Consider the following arc:

$$\begin{aligned} x(t) &= t\cos(\pi/t) \\ y(t) &= t\sin(\pi/t) \end{aligned} \quad \text{for } 0<t\leqq 1, \quad \begin{aligned} x(0)&=0\,, \\ y(0)&=0\,. \end{aligned}$$

This is merely the curve which in polar coordinates would be $r=\pi/\theta$ with the origin adjoined as an endpoint. It is readily verified that for $t>0$ it has continuous monotone decreasing curvature and hence at each interior point is of cyclic order three by Theorem 4. Hence it crosses each general osculating circle at an interior point and therefore satisfies condition (a) of Theorem 2. But is is not of local cyclic order three because of trouble at $t=0$. As t approaches 0 the arc spirals infinitely often about the origin. Hence every non-null circle through the origin is met by the arc in an infinite sequence of points converging to the origin. Thus every circle whatever through the origin is a general osculating circle so condition (b) is not satisfied. But since the arc has unbounded cyclic order at the origin the conclusion of Theorem 2 is false. Thus it is not possible to delete hypothesis (b) in Theorem 2.

Lane and Scherk [3] in discussing conformal differentiability introduce the concepts of tangent circles and osculating circles to an arc A at a point p as follows. If P is any point different from p, then C is called *the tangent circle at p through P* if $C=\lim C(p,q_i,P)$ where $\{q_i\}$ approaches p on A and the relation is to hold independent of sequence $\{q_i\}$. If this limit exists for some P, then it exists for all P and the tangent circles at A form a pencil τ of the second kind with fundamental point p. Circle K is called *the osculating circle at p* provided $K=\lim C(\tau,q_i)$ where, as before, $\{q_i\}$ is an arbitrary sequence converging to p on A and where $C(\tau,q_i)$ denotes the circle of pencil τ which contains q_i. When the tangent circles and osculating circle exist at p, then arc A is said to be conformally differentiable at p. If A is conformally differentiable at every point, there are therefore sets of tangent circles and osculating circles which are subsets of the general tangent circles and general osculating circles. For conformally differentiable arcs, it is a natural question to ask if Theorem 3 would be true if the nesting property were required only for the osculating circles, rather than for the (possibly) larger set of general osculating circles. In other words if an arc is conformally differentiable and if the osculating circles have the nesting property, is it of local cyclic order three? The answer again is in the negative. The following arc provides a counterexample:

$$x = |t|, \quad y = t^3, \qquad -\tfrac{1}{3} \leqq t \leqq \tfrac{1}{3}. \tag{1}$$

To verify the properties, consider first the arc $y=x^3$ with $-\frac{1}{3}\leqq x\leqq\frac{1}{3}$. On this range it is easily verified that the arc has continuous and strictly increasing curvature. It is therefore, by Theorem 4, of local cyclic order three and its general osculating circles, which are its circles of curvature, have the nesting property. For $x=0$, the general osculating circle is the line $y=0$. The arc given by (1) is obtained by leaving the part of $y=x^3$ in the upper half plane alone but reflecting the lower half in the y-axis. The osculating circles for (1) are precisely the images of those for the corresponding points of $y=x^3$. The only question might be for $t=0$, but it is readily verified that the required limit for $t=0$ exists and is the line $y=0$ whether one takes the limit for positive or negative t. The nesting property for the osculating circles of (1) follows from the known nesting property for $y=x^3$. But arc (1) has a cusp at $t=0$ and hence a cyclic order at least four at this point. Hence arc (1) does not have local cyclic order three. It may be noted that for $t=0$, because of the cusp the point circle is a general osculating circle, so the general osculating circles of (1) do not have the nesting property.

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