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ON THE FINE STRUCTURE OF SPECTRA

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1. Introduction.

In this paper we shall consider the fine structure of the spectra of bounded linear operators on complex Banach spaces. We shall use the notion of the state of an operator to give a subclassification of its spectrum.

A linear operator A with domain and range in a normed linear space X, is classified I, II or III according as its range, $\mathscr{R}(A)$, is all of X; is not all of X, but is dense in X; or is not dense in X. In addition A is classified 1, 2 or 3 according as A^{-1} exists and is continuous; exists, but is not continuous; or does not exist. The *state* of an operator is the combination of its Roman and Arabic numerical classifications and is denoted by the Roman numeral with the Arabic numeral as a subscript [2, p. 94], [3, p. 235].

For a specific operator A on a complex Banach space we partition the complex plane into subsets corresponding to the states of the operator $A_\lambda = \lambda - A$. For example, the subset consisting of those λ for which the state of the operator A_λ is II_3 will be denoted by $\mathrm{II}_3(A)$. Thus the resolvent set, $\varrho(A)$, of the operator A consists of the union of $\mathrm{I}_1(A)$ and $\mathrm{II}_1(A)$ and its spectrum, $\sigma(A)$, consists of the union of the remaining seven subsets [2, p. 109], [3, p. 264]. We shall call these seven subsets *spectral subsets* and the sets $\mathrm{I}_1(A)$ and $\mathrm{II}_1(A)$ *resolvent subsets.*

The partitioning of the spectrum of an operator A into its unique family of nonvoid spectral subsets will be called the *spectral decomposition* of A. This unique family will be called the *fine structure* of the spectrum of A, and the set of states corresponding to the elements in the fine structure will be called the *type* of the fine structure.

Our purpose is to determine which types of fine structures are possible for bounded linear operators on complex Banach spaces. For such operators the subsets $\mathrm{I}_2(A)$ and $\mathrm{II}_1(A)$ are void [3, p. 236]; hence the state I_2 plays no rôle in the consideration of the types of fine structures for these operators.

Our main result is that the only types of fine structure that can never

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occur for a bounded linear operator on a complex Banach space are $\{\mathrm{I}_3\}$, $\{\mathrm{III}_1\}$, and $\{\mathrm{I}_3, \mathrm{III}_1\}$.

We shall also consider the special case of a compact linear operator A on a complex Banach space. The spectrum of such an operator can consist of no more than two nonvoid spectral subsets, and, if there are two, one must be $\mathrm{III}_3(A)$ [3, p. 281]. In Section 2 we prove that for a compact operator A the spectral subsets $\mathrm{I}_3(A)$ and $\mathrm{III}_1(A)$ are empty. Thus, a priori, there are seven conceivable types of fine structure for a compact operator. In Section 3 we show that each of these types can occur.

2. Principal theorems.

Our first objective is to prove that for any linear operator A on a complex normed linear space X the spectral subset $\mathrm{III}_1(A)$ is open. This is a consequence of the following theorem.

THEOREM 2.1. *Let A be a linear operator on a normed linear space X into a normed linear space Y. If A is classified* III *and is the strong limit of a sequence of linear operators on X into Y which are not classified* III, *then A is not classified* 1.

PROOF. Since A is classified III there exists a positive number ε and a $y \in Y$ such that $\|Ax-y\| > \varepsilon$ for every $x \in X$. Let A be the strong limit of the sequence $\{A_k\}$, where A_k is not classified III for any k. Then there exists a sequence $\{x_k\}$ of elements of X, such that $\|A_k x_k - y\| < \frac{1}{2}\varepsilon$ for each k. It follows that

$$\varepsilon < \|Ax_k - y\| \leqq \|A - A_k\| \, \|x_k\| + \tfrac{1}{2}\varepsilon .$$

Thus

$$2\|x_k\| \geqq \varepsilon \|A - A_k\|^{-1}$$

and $\|x_k\| \to \infty$ as $k \to \infty$. Hence, with $y_k = x_k \|x_k\|^{-1}$,

$$\|Ay_k\| \leqq \|A - A_k\| + \|A_k y_k\| \leqq \|A - A_k\| + (\|y\| + \tfrac{1}{2}\varepsilon)\|x_k\|^{-1} .$$

Therefore $\inf_{\|x\|=1} \|Ax\| = 0$ and the theorem is proved.

COROLLARY 2.2. *Let A be a linear operator with domain and range in a complex normed linear space X. Then $\mathrm{III}_1(A)$ is an open set. In particular if A is compact then $\mathrm{III}_1(A)$ is empty.*

PROOF. From Theorem 2.1 it follows that every λ in $\mathrm{III}_1(A)$ is at a positive distance from the open set $\varrho(A) = \mathrm{I}_1(A) \cup \mathrm{II}_1(A)$. Moreover, the set of all λ such that A_λ has classification 1 is open [3, p. 256] and comprise the set $\varrho(A) \cup \mathrm{III}_1(A)$. Hence $\mathrm{III}_1(A)$ is open. The final assertion follows since the spectrum of a compact operator is at most countable.

REMARK. Corollary 2.2 has been proved otherwise by Gindler and Taylor, cf. [1, Theorems 3.2 and 4.2].

COROLLARY 2.3. *Let A be a bounded linear operator on a complex Banach space. Then $\mathrm{I}_3(A)$ is an open set. In particular if A is compact then $\mathrm{I}_3(A)$ is empty.*

Corollary 2.3 follows from Corollary 2.2 since under the hypothesis $\mathrm{I}_3(A) = \mathrm{III}_1(A')$, where A' is the conjugate operator of A [2, p. 100], [3, p. 237].

The next corollary is a consequence of the preceding two corollaries and the fact that the spectrum of a bounded linear operator on a complex Banach space is a nonvoid compact set. We use the notation $\partial\sigma(A)$ to designate the boundary of the spectrum $\sigma(A)$. Also, for convenience, we shall refer to the set $\mathrm{II}_2(A) \cup \mathrm{II}_3(A) \cup \mathrm{III}_2(A) \cup \mathrm{III}_3(A)$ as the *frame* of the spectrum and denote it by $F\sigma(A)$.

COROLLARY 2.4. *If A is a bounded linear operator on a complex Banach space then the frame, $F\sigma(A)$, is a nonvoid compact subset of the spectrum $\sigma(A)$, containing the boundary $\partial\sigma(A)$.*

Next we state a theorem concerning the relationship of the fine structure of the spectrum of a linear operator A which is completely reduced by a pair of closed subspaces and the fine structure of the spectra of the restrictions of A to these subspaces.

THEOREM 2.5. *Suppose A is a linear operator on a Banach space X and that A is completely reduced by a pair of closed subspaces X_1 and X_2. Let A_k be the restriction of A to X_k, $k = 1, 2$.*

Then, if λ belongs to the resolvent or spectral subsets $L_\alpha(A_1)$ and $M_\beta(A_2)$, it follows that λ belongs to the resolvent or spectral subset $N_\gamma(A)$, where $N = \max(L, M)$ and $\gamma = \max(\alpha, \beta)$.

The proof of this theorem is omitted since the results are easily deducible from existing theorems [3, p. 270]. A conclusion implicit in Theorem 2.5 is that $\sigma(A) = \sigma(A_1) \cup \sigma(A_2)$.

We find an immediate application of Theorem 2.5 as an aid in the development of a method of constructing certain operators which will be useful in the ensuing portions of this paper. The construction involves a *mixing* technique, whereby given two operators A and B an operator C is produced which satisfies the hypotheses of Theorem 2.5, and the restrictions of C to X_1 and X_2 have the fine spectral structures of A and B, respectively. We let A and B be bounded linear operators on the sequence space l_p, $1 \leqq p \leqq \infty$, with infinite matrix representations (a_{ij}) and (b_{ij}), respectively. The operator C, which we shall denote by $[A,B]$, is then defined by the infinite matrix (c_{ij}), where

$$c_{ij} = \begin{cases} 0 & \text{if } i+j \text{ is odd ,} \\ a_{\frac{1}{2}(i+1),\frac{1}{2}(j+1)} & \text{if } i \text{ and } j \text{ are both odd ,} \\ b_{\frac{1}{2}i,\frac{1}{2}j} & \text{if } i \text{ and } j \text{ are both even .} \end{cases}$$

Completely analogous techniques can be used to mix any finite number of operators.

Our next theorems deal with subclasses of the algebra of bounded linear operators on a sequence space. For a given sequence space l_p, where $1 \leqq p \leqq \infty$, we denote this algebra by $[l_p]$.

Let $\{a_k\}_0^\infty$ be a sequence of nonzero complex numbers, such that

$$\lim_{k\to\infty} |a_{k-1}a_k^{-1}| = R < \infty ,$$

and let $x = \{\xi_k\}_1^\infty$ be an element of l_p. Define the operator A by

$$Ax = \{a_{k-1}a_k^{-1}\xi_{k+1}\}_1^\infty .$$

Then $A \in [l_p]$, for $1 \leqq p \leqq \infty$. We will call this operator the *weighted shift operator* generated by the sequence $\{a_k\}_0^\infty$. The weighted shift operator generated by the sequence $\{1\}_0^\infty$ will in the sequel be denoted by T.

If $R > 0$, then $A \in [l_p]$ is classified I. It is convenient to introduce the right inverse $\tilde{A} \in [l_p]$, uniquely determined by

$$I = A\tilde{A} = \tilde{A}A + P , \tag{2.6}$$

where I is the identity operator and P is the projection of l_p onto the subspace generated by the first fundamental basis vector e_1.

One of the properties of a weighted shift operator A is that its spectral radius, $|\sigma(A)|$, equals R, independently of p. To see this, we observe, that

$$\|A^n\| = \sup_k |a_k/a_{k+n}| ,$$

and therefore

$$|\sigma(A)| = \lim_{n\to\infty} \|A^n\|^{1/n} = R .$$

Analogously, if $R > 0$, then

$$\|\tilde{A}^n\| = \sup_k |a_k{}^{-1}/a_{k+n}^{-1}| ,$$

and

$$|\sigma(\tilde{A})| = R^{-1} = |\sigma(A)|^{-1} .$$

A second property of such an operator is that λ belongs to its point spectrum, $P\sigma(A)$, if and only if $\{\lambda^{k-1}a_{k-1}\}_1^\infty \in l_p$. In particular, if $|\lambda| < R$, then $\lambda \in P\sigma(A)$ for any $p \geqq 1$. For further use we note that if $\lambda \in P\sigma(A)$ and $y \in l_p$, then there exists a unique vector x in the null space of A_λ, denoted by $\mathcal{N}(A_\lambda)$, such that $Px = Py$.

It is now apparent that the spectrum, $\sigma(A)$, of a weighted shift operator A is the closed disc, centered at the origin, with radius R.

A further property of a weighted shift operator A is that $\mathrm{I}_3(A)$ is the interior of the spectrum of A, i.e. $\mathrm{Int}\,\sigma(A) = \mathrm{I}_3(A)$. To see this we need only observe that, for $|\lambda| < R$, the operator $\tilde{A}(\lambda\tilde{A} - \mathrm{I})^{-1} \in [l_p]$ is a right inverse of A_λ. The property now follows from Corollary 2.3 and the fact that $P\sigma(A) \supset \mathrm{Int}\,\sigma(A)$.

These several properties of a weighted shift operator imply the first conclusion of the following theorem.

Theorem 2.7. *Let A be the weighted shift operator generated by the sequence $\{a_k\}_0^\infty$ where*

$$R = \lim_{k\to\infty} |a_{k-1}a_k{}^{-1}| .$$

Then $\sigma(A)$ is the closed disc, centered at the origin, with radius R, and $\mathrm{Int}\,\sigma(A) = \mathrm{I}_3(A)$ *for* $1 \leqq p \leqq \infty$.

If $1 \leqq p < \infty$, then $\partial\sigma(A)$ is $\mathrm{II}_3(A)$ or $\mathrm{II}_2(A)$ according as $\{a_{k-1}R^{k-1}\}_1^\infty$ belongs l_p or not.

If $p = \infty$, then $\partial\sigma(A)$ is $\mathrm{III}_3(A)$ or $\mathrm{III}_2(A)$ according as $\{a_{k-1}R^{k-1}\}_1^\infty$ belongs to l_∞ or not.

Proof. In view of Corollary 2.4 and the fact that $\lambda \in P\sigma(A)$ if and only if $\{\lambda^{k-1}a_{k-1}\}_1^\infty \in l_p$, it only remains to be proved that if $|\lambda| = R$, then A_λ is classified II or III according as p is finite or infinite. If p is finite and $|\lambda| = R$, direct computations show that each of the members of the fundamental basis in l_p is in the range of A_λ, which is therefore classified II. If $p = \infty$ and $|\lambda| = R > 0$, then A_λ is classified III, since the vector $y = \{\eta_k\}$,

$$\eta_k = \lambda^k R^{-k} \exp(i \arg a_{k-1})$$

has a neighborhood, which belongs to the complement of $\mathcal{R}(A_\lambda)$. To see this assume that $x=\{\xi_k\}_1^\infty \in l_\infty$ and $z=A_\lambda x=\{\zeta_k\}_1^\infty \in l_\infty$ where $\zeta_k=\eta_k(1+\theta_k)$ with $|\theta_k| \leqq K < 1$. Direct computation yields

$$\xi_{k+1} = \lambda^k a_k \left\{\frac{\xi_1}{a_0} - \sum_{\nu=1}^{k} \frac{1+\theta_\nu}{|a_{\nu-1}|R^\nu}\right\}.$$

Hence, if the series $\sum |a_\nu|^{-1} R^{-\nu}$ converges, then

$$|a_k| R^k \sum_{\nu=k}^{\infty} |a_\nu|^{-1} R^{-\nu}$$

is bounded, while if the series diverges, then

$$|a_k| R^k \sum_{\nu=1}^{k} |a_\nu|^{-1} R^{-\nu}$$

is bounded, contradicting the fact that $\lim_{k\to\infty} |a_{k-1} a_k{}^{-1}| = R$. If $R=0$, then A is classified III, since the vector $\{1\}_1^\infty \in l_\infty$ has a neighborhood in the complement of $\mathscr{R}(A)$.

Corollary 2.8. *Let A be the weighted shift operator generated by the sequence $\{a_k\}_0^\infty$ and let $|\sigma(A)| > 0$. Then, if $1 \leqq p < \infty$, the spectrum of $\tilde{A}$ is the closed disc, centered at the origin, with radius $|\sigma(A)|^{-1}$. The interior of $\sigma(\tilde{A})$ is the set $\mathrm{III}_1(\tilde{A})$ and the boundary of $\sigma(\tilde{A})$ is the set $\mathrm{III}_2(\tilde{A})$ or the set $\mathrm{II}_2(\tilde{A})$ according as the vector $\{a_k{}^{-1}|\sigma(A)|^{-k}\}_1^\infty$ is an element of the conjugate space of l_p or not.*

Proof. It is easy to see that $P\sigma(\tilde{A})$ is empty. Moreover, if $\tilde{A} \in [l_p]$, $1 \leqq p < \infty$, then its conjugate operator $\tilde{A}' \in [l_q]$, $p^{-1}+q^{-1}=1$, is the weighted shift operator generated by the sequence $\{a_k{}^{-1}\}_0^\infty$. These facts together with the above theorem and the state diagram [2, p. 100], [3, p. 237] give the corollary.

The condition in Corollary 2.8 which determines the nature of $\partial\sigma(\tilde{A})$ can also be given in terms of geometric properties of the operator A. We shall say that $A \in [l_p]$ is of *minimal type* if given $y \in l_p$ and any $\lambda \in \partial\sigma(A)$, there exists a sequence of elements $x_n \in l_p$, such that $A_\lambda x_n \to 0$ as $n \to \infty$ and $Px_n = Py$. We shall say that $A \in [l_p]$ is of *maximal type* if $\lambda \in \partial\sigma(A)$ and $A_\lambda x_n \to 0$ as $n \to \infty$ implies that $Px_n \to 0$. With this terminology an alternative characterization of $\partial\sigma(\tilde{A})$ is given by the following theorem.

Theorem 2.9. *Let A be a weighted shift operator on l_p, $1 \leqq p < \infty$, and let $|\sigma(A)| > 0$. Then A is of minimal type if and only if $\partial\sigma(\tilde{A}) = \mathrm{II}_2(\tilde{A})$ and A is of maximal type if and only if $\partial\sigma(\tilde{A}) = \mathrm{III}_2(\tilde{A})$.*

PROOF. Suppose that A is of minimal type and $\lambda \in \partial\sigma(\tilde{A})$. Then $\lambda^{-1} \in \partial\sigma(A)$ and therefore $A_{\lambda^{-1}}$ is classified II. Thus given any $y \in l_p$ and $\varepsilon > 0$, there exists an $x_1 \in l_p$ such that $\|A_{\lambda^{-1}}x_1 - y\| < \frac{1}{2}\varepsilon$. Moreover, there exists an x_2 such that $\|A_{\lambda^{-1}}x_2\| < \frac{1}{2}\varepsilon$ and $Px_2 = Px_1$. Since, by (2.6), $\mathscr{R}(\tilde{A}) = \mathscr{R}(I-P)$, there is an x such that $-\lambda\tilde{A}x = x_1 - x_2$. Thus

$$\|\tilde{A}_\lambda x - y\| = \|-\lambda A_{\lambda^{-1}}\tilde{A}x - y\| = \|A_{\lambda^{-1}}x_1 - y - A_{\lambda^{-1}}x_2\| < \varepsilon .$$

From Corollary 2.8 it follows that $\tilde{A}_\lambda$ has state II_2 and therefore $\partial\sigma(\tilde{A}) = \mathrm{II}_2(\tilde{A})$.

To prove the converse assume that $\partial\sigma(\tilde{A}) = \mathrm{II}_2(\tilde{A})$, $\lambda \in \partial\sigma(A)$ and $y \in l_p$. Thus $\lambda^{-1} \in \partial\sigma(\tilde{A})$ and hence there exists a sequence $\{y_n\}_1^\infty$, where $y_n \in l_p$, such that

$$\tilde{A}_{\lambda^{-1}}y_n - Py \to 0 \quad \text{as } n \to \infty .$$

Since A has state I_3 we can choose x_n such that $Ax_n = y_n$ and $Px_n = Py$. Thus, by (2.6),

$$A_\lambda x_n = -\lambda(\tilde{A}_{\lambda^{-1}}Ax_n - Px_n) = -\lambda(\tilde{A}_{\lambda^{-1}}y_n - Py) ,$$

and we have constructed a sequence $\{x_n\}_1^\infty$ with $Px_n = Py$ such that

$$A_\lambda x_n \to 0 \quad \text{as } n \to \infty .$$

This proves that A is of minimal type, completing the first part of the proof. We have also, in essence, proved that $\partial\sigma(\tilde{A}) = \mathrm{III}_2(\tilde{A})$ if and only if $e_1 \notin \overline{\mathscr{R}(\tilde{A}_\lambda)}$ for any $\lambda \in \partial\sigma(\tilde{A})$. For assume that there exists a $\lambda \in \partial\sigma(\tilde{A})$ such that $e_1 \in \overline{\mathscr{R}(\tilde{A}_\lambda)}$. Then from the second part of the above proof it follows that given $y \in l_p$ there is a sequence $\{x_n\}_1^\infty$ such that $A_{\lambda^{-1}}x_n \to 0$ and $Px_n = Py$. But then, from the first part of the proof, it follows that $\tilde{A}_\lambda$ has the state II_2, which is a contradiction.

We shall use this last fact to prove the second part of the theorem. All that remains to be proved is that if $\partial\sigma(\tilde{A}) = \mathrm{III}_2(\tilde{A})$, then A is of maximal type. Therefore assume that $\lambda \in \partial\sigma(A)$. Then $\lambda^{-1} \in \partial\sigma(\tilde{A})$ and hence there exists an $\varepsilon > 0$ such that $\|\tilde{A}_{\lambda^{-1}}y - e_1\| > \varepsilon$ for all $y \in l_p$. Thus if $x \in l_p$ and $Px \neq 0$, we have

$$\|A_\lambda x\| = \|\lambda(\tilde{A}_{\lambda^{-1}}Ax - Px)\| \geqq \varepsilon|\lambda|\,\|Px\| .$$

The conclusion follows readily from this inequality.

It should be observed that a consequence of this theorem is that if A is a weighted shift operator on l_p, $1 \leqq p < \infty$, then A is either of maximal type or minimal type.

We now introduce a class of transformations which we call *mixed weighted shift* operators. If A and B are weighted shift operators such that $|\sigma(B)| > 0$, then the ordered pair (A, B) gives rise to a mixed weighted shift operator $M \in [l_p]$, $1 \leqq p \leqq \infty$, defined by $M = [A, \tilde{B}] - \tilde{T}P$.

Our next theorems will deal with the fine structure of the spectrum of a mixed weighted shift operator. Before stating these theorems we shall prove certain properties of such an operator. In these proofs we associate with the vector $y = \{\eta_k\}_{k=1}^{\infty}$ the pair of vectors $y_1 = \{\eta_{2k-1}\}_{k=1}^{\infty}$ and $y_2 = \{\eta_{2k}\}_{k=1}^{\infty}$. It is clear that y determines the pair y_1 and y_2 uniquely, and conversely. Note also that

$$\tfrac{1}{2}(\|y_1\| + \|y_2\|) \leqq \|y\| \leqq \|y_1\| + \|y_2\| .$$

With this notation $y = M_\lambda x$ if and only if $y_1 = A_\lambda x_1$ and $y_2 = \tilde{B}_\lambda x_2 + Px_1$. Using the identities (2.6) we also observe that $y_2 = \tilde{B}_\lambda x_2 + Px_1$ if and only if $(\lambda B - I)x_2 = By_2$ and $Px_1 + \lambda Px_2 = Py_2$.

One of the properties of a mixed weighted shift operator is that

$$P\sigma(M) = P\sigma(A) \cap (P\sigma(B))^{-1} ,$$

where we use the set notation $S^{-1} = \{\lambda^{-1}; \lambda \in S\}$. If we assume that $M_\lambda x = 0$ and $x \neq 0$ then $A_\lambda x_1 = 0$, $x_2 = \lambda B x_2$ and $Px_1 + \lambda Px_2 = 0$. If $x_2 = 0$, then $Px_1 = 0$ and therefore $x_1 = 0$ contradicting the assumption that $x \neq 0$. But $x_2 \neq 0$ implies $\lambda \neq 0$ and thus $(\lambda B - I)x_2 = 0$. Therefore $Px_2 \neq 0$; which implies that $Px_1 \neq 0$. Hence $A_\lambda x_1 = 0$, $x_1 \neq 0$ and $(\lambda B - I)x_2 = 0$, $x_2 \neq 0$. On the other hand if $\lambda \in P\sigma(A) \cap (P\sigma(B))^{-1}$ we can choose nonzero vectors x_1 and x_2 such that $A_\lambda x_1 = 0$, $(\lambda B - I)x_2 = 0$ and $Px_1 + \lambda Px_2 = 0$. Then the vector x determined by x_1 and x_2 is an eigenvector of M corresponding to the eigenvalue λ.

A second property of a mixed weighted shift operator is that M_λ is classified I i fand only if the mixed operator $[A_\lambda, \lambda B - I]$ has state I_3. Let us first assume that M_λ is classified I. Then $M_\lambda x = y$ implies that $A_\lambda x_1 = y_1$ and $(\lambda B - I)x_2 = By_2$. Since $|\sigma(B)| > 0$, the operator B is classified I and therefore the mixed operator $[A_\lambda, \lambda B - I]$ is classified I. Moreover, the state of $[A_\lambda, \lambda B - I]$ must be I_3 for if it were I_1, then $M_\lambda x = \lambda e_1$, would imply that $x_1 = \lambda A_\lambda^{-1} e_1 = e_1$ and $x_2 = (\lambda B - I)^{-1} 0 = 0$, which contradicts the requirement that $Px_1 + \lambda Px_2 = Py_2$. On the other hand, if $[A_\lambda, \lambda B - I]$ has the state I_3, then for arbitrary elements y_1 and y_2 we can determine x_1 and x_2 so that $A_\lambda x_1 = y_1$, $(\lambda B - I)x_2 = By_2$ and, since at least one of the operators A_λ and $\lambda B - I$ is classified 3, so that $Px_1 + \lambda Px_2 = Py_2$ holds. If x is determined by x_1 and x_2, and y is determined by y_1 and y_2, then $M_\lambda x = y$ and our proof is complete.

A direct consequence of the first two properties is that M_λ has the state I_1 if and only if $[A_\lambda, \lambda B - I]$ has the state I_3 and one of the operators A_λ and $\lambda B - I$ is not classified 3.

A third property of a mixed weighted shift operator M is that M_λ is classified III if $[A_\lambda, \lambda B - I]$ is classified III. This follows directly from the following inequality

$$\begin{aligned} 2\|M_\lambda x - y\| &\geqq \|A_\lambda x_1 - y_1\| + \|\tilde{B}_\lambda x_2 + Px_1 - y_2\| \\ &\geqq \|A_\lambda x_1 - y_1\| + \|B\|^{-1} \|B(\tilde{B}_\lambda x_2 + Px_1) - By_2\| \\ &= \|A_\lambda x_1 - y_1\| + \|B\|^{-1} \|(\lambda B - I)x_2 - By_2\| . \end{aligned}$$

A fourth property of a mixed weighted shift operator M is that M_λ has the state III_1 if $[A_\lambda, \lambda B - I]$ has the state I_1. This follows directly from the above inequality with $y = 0$.

A fifth property of a mixed weighted shift operator M is that M_λ is classified II if $[A_\lambda, \lambda B - I]$ has the state II_3. Assuming that the state of $[A_\lambda, \lambda B - I]$ is II_3, we see that given any vector y and positive number ε, there exist vectors x_1, x_2, z_1 and z_2 such that

$$\begin{aligned} A_\lambda x_1 &= y_1 + z_1, & \|z_1\| &< \tfrac{1}{2}\varepsilon , \\ (\lambda B - I)x_2 &= By_2 + z_2, & \|z_2\| &< \tfrac{1}{2}\varepsilon \|\tilde{B}\|^{-1} \end{aligned}$$

and

$$Px_1 + \lambda P x_2 = Py_2 .$$

The last two identities give $\tilde{B}_\lambda x_2 + Px_1 = y_2 + \tilde{B} z_2$, and hence

$$\|M_\lambda x - y\| \leqq \|A_\lambda x_1 - y_1\| + \|\tilde{B}_\lambda x_2 + Px_1 - y_2\| < \varepsilon .$$

Since M_λ cannot be classified I it must be classified II.

These several properties of a mixed weighted shift operator imply the following two theorems.

THEOREM 2.10. *Let $M \in [l_p]$ be the mixed weighted shift operator generated by the weighted shift operators A and B and let $|\sigma(A)|\,|\sigma(B)| > 1$. Then $\sigma(M) = \sigma(A) \cap (\sigma(B))^{-1}$ and $\mathrm{I}_3(M) = \mathrm{Int}\,\sigma(M)$. Moreover, if $1 \leqq p < \infty$, then*

$$\begin{aligned} \mathrm{II}_2(M) &= \mathrm{II}_2(A) \cup (\mathrm{II}_2(B))^{-1} , \\ \mathrm{II}_3(M) &= \mathrm{II}_3(A) \cup (\mathrm{II}_3(B))^{-1} , \end{aligned}$$

and, if $p = \infty$, then

$$\mathrm{III}_2(M) = \mathrm{III}_2(A) \cup (\mathrm{III}_2(B))^{-1}$$

and

$$\mathrm{III}_3(M) = \mathrm{III}_3(A) \cup (\mathrm{III}_3(B))^{-1} .$$

Note that Theorem 2.10 gives the complete fine structure of the spectrum of M.

THEOREM 2.11. *Let $M \in [l_p]$ be the mixed weighted shift operator generated by the weighted shift operators A and B and let $|\sigma(A)|\,|\sigma(B)| = 1$. Then $\sigma(M) = \sigma(A) \cap (\sigma(B))^{-1}$ and $\operatorname{Int}\sigma(M) = \emptyset$.*

If $1 \leqq p < \infty$ and $\mathrm{II}_2(A) \cap (\mathrm{II}_2(B))^{-1}$ is empty, then $\sigma(M)$ is the set $\mathrm{II}_2(M)$ or the set $\mathrm{II}_3(M)$, according as $\mathrm{II}_3(A) \cap (\mathrm{II}_3(B))^{-1}$ is empty or not.

If $p = \infty$, then $\sigma(M)$ is the set $\mathrm{III}_2(M)$ or the set $\mathrm{III}_3(M)$, according as $\mathrm{III}_3(A) \cap (\mathrm{III}_3(B))^{-1}$ is empty or not.

For the case $\sigma(M) = \mathrm{II}_2(A) \cap (\mathrm{II}_2(B))^{-1}$, which was omitted in the preceding theorem, knowledge of the fine structure of $\sigma(A)$ and $\sigma(B)$ is not sufficient to determine the fine structure of $\sigma(M)$. In this case the fine structure also depends on the types of the operators A and B.

THEOREM 2.12. *Let $M \in [l_p]$ be the mixed weighted shift operator generated by the weighted shift operators A and B and let $\sigma(M) = \mathrm{II}_2(A) \cap (\mathrm{II}_2(B))^{-1}$. Then $\sigma(M)$ is the set $\mathrm{III}_2(M)$ or the set $\mathrm{II}_2(M)$ according as both A and B are of maximal type or not.*

PROOF. First assume that both A and B are of maximal type, and let $\lambda \in \partial\sigma(A) = (\partial\sigma(B))^{-1}$. Now assume that $e_2 \in \overline{\mathscr{R}(M_\lambda)}$. Then there exists a sequence $\{x^{(n)}\}$ such that $\lim_{n\to\infty} M_\lambda x^{(n)} = e_2$. Since

$$2\|M_\lambda x - e_2\| \geqq \|A_\lambda x_1\| + \|\tilde{B}_\lambda x_2 - e_1 + Px_1\|\,,$$

it follows that $\lim_{n\to\infty} A_\lambda x_1^{(n)} = 0$ and therefore $\lim_{n\to\infty} Px_1^{(n)} = 0$. Another use of the inequality shows that $\lim_{n\to\infty} \tilde{B}_\lambda x_2^{(n)} = e_1$, contradicting the fact that $e_1 \notin \overline{\mathscr{R}(\tilde{B}_\lambda)}$. (Cf. proof of Theorem 2.9.)

Next we assume that A is of minimal type, that $\lambda \in \partial\sigma(A) = (\partial\sigma(B))^{-1}$, and that y is an arbitrary element of l_p. Then, given $\varepsilon > 0$, there exists a $z \in l_p$ such that

$$\|(\lambda B - I)z - y_2\| < \tfrac{1}{2}\varepsilon\,.$$

We now determine an x_2 such that $\tilde{B}x_2 = z - Pz$. Since A is of minimal type, there exists an $x_1 \in l_p$ such that

$$\|A_\lambda x_1 - y_1\| < \tfrac{1}{2}\varepsilon \quad \text{and} \quad Px_1 + Pz = 0\,.$$

Thus

$$\begin{aligned}\|M_\lambda x - y\| &\leqq \|A_\lambda x_1 - y_1\| + \|\tilde{B}_\lambda x_2 + Px_1 - y_2\| \\ &= \|A_\lambda x_1 - y_1\| + \|(\lambda B - \mathrm{I})(\tilde{B}x_2 - Px_1) - y_2\| \\ &= \|A_\lambda x_1 - y_1\| + \|(\lambda B - \mathrm{I})z - y_2\| < \varepsilon\,.\end{aligned}$$

Hence $\lambda \in \mathrm{II}_2(M)$ and therefore $\sigma(M) = \mathrm{II}_2(M)$. The case when B is of minimal type follows from the same inequality since in that case $\partial\sigma(\tilde{B}) = \mathrm{II}_2(\tilde{B})$.

Using the same techniques employed in the proofs of the preceding theorems one can prove the following theorem.

THEOREM 2.13. *Let* $M \in [l_p]$ *be the mixed weighted shift operator generated by the weighted shift operators* A *and* B *and let* $|\sigma(A)|\,|\sigma(B)| < 1$. *Then* $\mathrm{III}_1(M) = \varrho(A) \cap (\varrho(B))^{-1}$ *and* $\sigma(M) = \mathrm{III}_1(M)$. *Moreover, if* $1 \leqq p < \infty$, *then the set* $\partial\sigma(A)$ *is a subset of* $\mathrm{III}_2(M)$ *or* $\mathrm{II}_2(M)$ *according as* A *is of maximal type or not, and the set* $(\partial\sigma(B))^{-1}$ *is a subset of* $\mathrm{III}_2(M)$ *or* $\mathrm{II}_2(M)$ *according as* B *is of maximal type or not. If* $p = \infty$, *then* $\partial\sigma(A) \cup (\partial\sigma(B))^{-1} = \mathrm{III}_2(M)$.

3. Spectral decomposition.

In this section we prove the statements made in the Introduction concerning spectral decompositions for compact and bounded operators.

That six of the seven conceivable types of fine structure for compact operators do occur is an immediate consequence of the following theorems.

THEOREM 3.1. *A weighted shift operator* $A \in [l_p]$ *is compact if and only if* $|\sigma(A)| = 0$.

THEOREM 3.2. *If* $A \in [l_2]$ *is a weighted shift operator and* $|\sigma(A)| = 0$, *then the operators* A, A', $[A, A']$, $A\tilde{T}$, $[A\tilde{T}, A]$ *and* $[A\tilde{T}, A']$ *are all compact elements of* $[l_2]$ *with fine structures of types* $\{\mathrm{II}_3\}$, $\{\mathrm{III}_2\}$, $\{\mathrm{III}_3\}$, $\{\mathrm{II}_2, \mathrm{III}_3\}$, $\{\mathrm{II}_3, \mathrm{III}_3\}$ *and* $\{\mathrm{III}_2, \mathrm{III}_3\}$, *respectively.*

The necessity of the condition in Theorem 3.1 follows from Theorem 2.7 and the fact that the spectrum of a compact operator is at most countable. To prove the sufficiency of the condition one uses standard techniques to show that if S is a bounded subset of l_p, then the image of S under A is conditionally sequentially compact.

That all of the operators considered in Theorem 3.2 are compact is easily verified. The remaining conclusions of Theorem 3.2 follow from Theorems 2.5 and 2.7 and the state diagram once we have shown that the fine structure of $A\tilde{T}$ is $\{\mathrm{II}_2(A\tilde{T}), \mathrm{III}_3(A\tilde{T})\}$, which is an immediate consequence of the facts that

$$\mathscr{R}(A\tilde{T}) = \mathscr{R}(A), \quad \mathscr{R}(\tilde{T}) \cap \mathscr{N}(A) = \{0\}$$

and e_1 is an eigenvector of $A\tilde{T}$ corresponding to the eigenvalue $a_0 a_1^{-1}$.

We shall now exhibit a compact operator with fine structure of type $\{II_2\}$.

Let $A \in [l_p]$, $1 \leqq p < \infty$, be a compact weighted shift operator generated by a sequence $\{a_k\}_0^\infty$, such that $\{a_{k-1}a_k^{-1}\}_1^\infty \in l_p$, and let $P_A \in [l_p]$ be defined by $P_A e_1 = \{a_{k-1}a_k^{-1}\}_1^\infty$ and $P_A e_k = 0$ for $k > 1$. Then the operator $A - P_A$ is compact,

$$\mathscr{R}(A) \cap \mathscr{R}(P_A) = \{0\}, \quad \text{and} \quad \mathscr{R}(A - P_A) \supset \mathscr{R}(A).$$

These facts imply that the operator $A - P_A$ has the state II_2. To see this assume that $Ax = P_A x$. Then $P_A x = 0$ and therefore $Px = 0$. Hence $Ax = 0$ and $Px = 0$, which implies that $x = 0$. Since the frame of the spectrum of any compact operator on an infinite dimensional space contains the origin, the operator $A - P_A$ is classified 2. Moreover, since A has the state II_3, the range of $A - P_A$ is dense in l_p. Thus $A - P_A$ has the state II_2.

If we now impose the additional condition on the sequence $\{a_k\}_0^\infty$ that the entire function

$$f(z) = \sum_{k=0}^{\infty} a_k^{-1} z^k$$

has no zeros, then $P\sigma(A - P_A)$ is empty and $\sigma(A - P_A) = II_2(A - P_A)$. It is easy to see that sequences satisfying these conditions exist. For example, if $1 < p < \infty$, the sequence $\{n!\}$ will do.

We now consider spectral decompositions for bounded operators.

THEOREM 3.3. *The only types of fine structure that can never occur for a bounded linear operator on a complex Banach space are* $\{I_3\}$, $\{III_1\}$ *and* $\{I_3, III_1\}$.

PROOF. That the types of fine structure given in the theorem are impossible is a direct consequence of the fact that the frame of the spectrum of a bounded linear operator on a complex Banach space is nonvoid (Cor. 2.4.).

To prove that all other conceivable types of fine structure are possible we shall work with elements of the algebra $[l_p]$, where $1 \leqq p < \infty$. Let A_ν, $\nu = 1, 2, 3$, be weighted shift operators with spectral radii 1 and fine structures of types $\{I_3, II_3\}$, $\{I_3, II_2\}$ and $\{I_3, II_2\}$, respectively. Moreover let A_2 be of minimal type and A_3 of maximal type. Then, by Theorems 2.11 and 2.12, the mixed weighted shift operators M_ν, $\nu = 1, 2, 3$, generated by the pairs (A_ν, A_ν) have circular spectra, centered at the origin, of radii 1, and fine structures of types $\{II_3\}$, $\{II_2\}$ and $\{III_2\}$, respectively. By Theorem 2.5, the mixed operator $M_4=[M_1,M_3]$ has a circular spectrum, centered at the origin, of radius 1, and fine structure of type $\{III_3\}$. Finally, by Theorem 2.13, the mixed weighted shift operator M_5 generated by the ordered pair $(A_2, \frac{1}{2}A_2)$ has an annular spectrum, centered at the origin, with radii 1 and 2, and a fine structure of type $\{III_1, II_2\}$.

Operators with all possible types of fine structure can now be obtained by mixing multiples of the operators A_2 and M_ν, $\nu=1,2,3,4,5$. For example, the mixed operator $[A_2, 2M_1, \frac{3}{2}M_3, \frac{1}{2}M_4, M_5]$ has a fine structure of type $\{I_3, II_2, II_3, III_1, III_2, III_3\}$.

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