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PONTRYAGIN'S MAXIMUM PRINCIPLE AND A MINIMAX PROBLEM

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1. Introduction.

The main purpose of this paper is to derive a counterpart of the Pontryagin maximum principle, valid for certain minimax problems. Our problem is to minimize the functional

$$H(x) = \sup_t F(t,x,dx/dt)$$

under given boundary conditions. The admissible functions are absolutely continuous vector functions. Such problems have been treated by D. S. Carter and the author. Carter [4] has given a thorough treatment of the case

$$F \equiv \|A(t)(dx/dt+B(t)x+c(t))\| ,$$

where $A(t)$ and $B(t)$ are matrix functions and $c(t)$ is a vector function. The author has treated the case x a scalar for fairly general nonlinear functions F, (cf. [1], [2], [3]).

We first state a partial result as a theorem, and in proving it we will derive our version of the maximum principle, which is given in Theorem 2. The principle is applicable only to minimizing functions which satisfy a certain condition, and this fact is illustrated by an example. After the maximum principle, we prove a theorem which shows a connection with the papers [1], [2], [3]. Finally, we derive a theorem about existence of solutions of the minimax problem.

2. Statement of the problem and Theorem 1.

Let us state the problem in detail. By D we denote a region in R^{n+1}, and points in D are written (t,x), where x has n components. By (T_0,X_0) and (T_1,X_1) we denote two points in D with $T_0<T_1$. The class $\mathscr{F}$ of *admissible functions* consists of all vector functions $x=x(t)$ with graph in D, defined for $T_0 \leqq t \leqq T_1$, absolutely continuous there and satisfying $x(T_0)=X_0$, $x(T_1)=X_1$.

For $x(t) \in \mathscr{F}$, we define the functional $H(x)$ by

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$$H(x) = \sup_{t\in E} F(t,x(t),dx(t)/dt)\ .$$

Here, $F(t,x,z)$ is a given function of $(2n+1)$ variables, defined on $D \times \mathsf{R}^n$, and E is the subset of $[T_0,T_1]$ where $\dot{x}(t)(=dx/dt)$ exists. Sometimes, it is convenient to consider the functional

$$\operatorname{ess\,sup}_{t\in E} F(t,x(t),dx(t)/dt)$$

and it will be proved in a lemma that $\operatorname{ess\,sup}_{t\in E} F(\cdot,\cdot,\cdot) = \sup_{t\in E} F(\cdot,\cdot,\cdot)$ for any $x(t) \in \mathscr{F}$ among those functions F which we are interested in, i.e. satisfying conditions 1)–3) below.

Further, we write $M_0 = \inf_{x\in\mathscr{F}} H(x)$ and it will be seen below that $M_0 > -\infty$. We impose the following conditions on $F(t,x,z)$:

1) $F(t,x,z) \in C^1(D \times \mathsf{R}^n)$

2) For any fixed $(t,x) \in D$, the function $\mu(z) \equiv F(t,x,z)$ is strictly convex in z. Further, there exists a mapping $\omega : D \to \mathsf{R}^n$ such that $F(t,x,\omega(t,x)) = \min_{z\in\mathsf{R}^n} F(t,x,z)$ for all $(t,x) \in D$. Also, $\omega \in C^1(D)$.

3) $\lim_{\|z\|\to\infty} F(t,x,z) = +\infty$, uniformly if (t,x) is restricted to a compact subset of D. (Compare the end of section 6.)

It is clear that $H(x) \geqq \max\{F(T_0,X_0,\omega(T_0,X_0)), F(T_1,X_1,\omega(T_1,X_1))\}$ for any admissible x, which shows that $M_0 > -\infty$.

THEOREM 1. *Suppose that $\bar{x}_0(t)$ is a solution of the minimization problem, and that F satisfies the conditions* 1), 2) *and* 3) *above. Suppose further that $F\big(t,\bar{x}_0(t),\omega(t,\bar{x}_0(t))\big) < M_0$ for $T_0 \leqq t \leqq T_1$. Then*

$$\bar{x}_0(t) \in C^1[T_0,T_1]$$

and $F(t,\bar{x}_0(t),d\bar{x}_0(t)/dt) = M_0$ for $T_0 \leqq t \leqq T_1$.

3. Proof of Theorem 1.

The proof consists of three parts:

1) The identification of a class of admissible functions (lying in a neighbourhood of $\bar{x}_0(t)$) with a class of solutions of a certain control system $dx/dt = f(t,x,u(t))$, where $u(t)$ is a measurable control vector ranging over a fixed control region $U \subset \mathsf{R}^n$.

2) The development of a variational method suited for our problem. This particular method is often used in proving the Pontryagin maximum principle, like for instance in [8, pp. 75–108], and [7, pp. 246–256]. It is based on so-called "needle-shaped" variations of the control function $u(t)$.

3) The variational method is applied to the minimizing function $\bar{x}_0(t)$. It turns out that, unless $\bar{x}_0(t)$ satisfies the maximum principle, it

is possible to construct a "better" admissible function $\bar{x}_1$, i.e. satisfying $H(\bar{x}_1) < M_0$, which is clearly impossible.

1) *Construction of the control system.*

Since $F(t,x,\omega(t,x)) < M_0$ along the curve $\bar{x}_0(t)$, we may introduce a region $D_1 \subset D$, containing the curve $x = \bar{x}_0(t)$, $T_0 \leqq t \leqq T_1$, and such that

$$F(t,x,\omega(t,x)) < M_0 \quad \text{on } D_1 .$$

Further, for $(t,x) \in D_1$ we put

$$C(t,x) = \{z \in \mathsf{R}^n \mid F(t,x,z) \leqq M_0\} .$$

Thus $C(t,x)$ is a closed, convex set in R^n and $\omega(t,x)$ is an interior point of $C(t,x)$. The closed unit sphere U in R^n will serve as the control region. Finally, we put $\Omega_u = D_1 \times U$ and

$$\Omega_z = \{(t,x,z) \mid (t,x) \in D_1;\ z \in C(t,x)\} .$$

Hence Ω_u and Ω_z are subsets of R^{2n+1} and we will now construct a topological mapping between these sets.

First, we define the mapping f: $\Omega_u \to \mathsf{R}^n$. We define $f(t,x,0) = \omega(t,x)$. Consider then (t,x,u) where $u \neq 0$. There is a unique $\alpha > 0$ such that

$$F(t,x,\omega(t,x) + \alpha u/\|u\|) = M_0 .$$

This is clear from our assumptions 2) and 3) regarding F, and from the fact that $F(t,x,\omega(t,x)) < M_0$. We put

$$f(t,x,u) = \omega(t,x) + \alpha u .$$

Geometrically, this means that for fixed (t,x), each radius in the sphere U is mapped on a parallel "radius" in $C(t,x)$ through $\omega(t,x)$ such that the endpoint where $\|u\| = 1$ is mapped on the endpoint $z \in \partial C(t,x)$ (where $F(t,x,z) = M_0$), and the origin is mapped on $\omega(t,x)$. Further, the correspondence within each pair of "radii" is linear.

Clearly, for fixed (t,x), this gives a one-to-one mapping of U onto $C(t,x)$. Therefore, if we define

$$\Phi(t,x,u) = (t,x,f(t,x,u)) ,$$

then Φ gives a one-to-one mapping of Ω_u onto Ω_z. Next we have to show that $f(t,x,u)$ and all derivatives $\partial f_i(t,x,u)/\partial x_k$ are continuous for $(t,x,u) \in \Omega_u$. We start with f and let $(t_n,x_n,u_n) \to (t_0,x_0,u_0)$.

A) $u_0 = 0$. Since ω is continuous and $f(t,x,0) = \omega(t,x)$, we may assume that $u_n \neq 0$ for $n = 1,2,3,\ldots$. We have

$$\|f(t_n,x_n,u_n)-f(t_0,x_0,0)\| = \|\omega(t_n,x_n)+\alpha_n u_n-\omega(t_0,x_0)\| \leqq \|\omega(t_n,x_n)-\omega(t_0,x_0)\|+\alpha_n\|u_n\| ,$$

where

$$F(t_n,x_n,\omega(t_n,x_n)+\alpha_n u_n/\|u_n\|) = M_0 .$$

Now the sequence $\{(t_n,x_n)\}_1^\infty$ belongs to some compact subset of D and it follows from condition 3) on F that $\alpha_n < K < \infty$. Thus $f(t_n,x_n,u_n) \to f(t_0,x_0,0)$.

B) $u_0 \neq 0$. We must study more closely the quantity α as a function of (t,x,u). We have

$$F(t,x,\omega(t,x)+\alpha u/\|u\|) = M_0 .$$

Put

$$G(t,x,u,\alpha) \equiv F(t,x,\omega(t,x)+\alpha u/\|u\|)-M_0 .$$

Clearly, G_t, G_{x_k}, G_{u_i}, G_α all exist and are continuous functions of (t,x,u,α) provided that $(t,x)\in D_1$ and $0 \neq u \in \mathsf{R}^n$ (no condition on α). The restriction $\|u\| \leqq 1$ can be left aside here. Further,

$$\frac{\partial G}{\partial \alpha} = \sum_{\nu=1}^{n} \frac{\partial F}{\partial z_\nu}\,(t,x,\omega(t,x)+\alpha u/\|u\|)\,\frac{u_\nu}{\|u\|} = \left(\operatorname{grad}_z F(t,x,\omega(t,x)+\alpha u/\|u\|),\ u/\|u\|\right) ,$$

and it follows from condition 2) on F that $\partial G/\partial\alpha > 0$ for $\alpha > 0$. Hence the relation $G(\cdots)=0$ will define α implicitly as a function $\alpha=\mu(t,x,u)$ in a neighbourhood V of (t_0,x_0,u_0), and $\mu \in C^1(V)$. It follows that

$$f(t,x,u) \equiv \omega(t,x)+\mu(t,x,u)u \in C^1(V) .$$

We have proved that $f(t,x,u)$ is continuous for $(t,x)\in D_1$ and, in fact, any $u \in \mathsf{R}^n$. We must also consider $\partial f_i/\partial x_k(t,x,u)$. We have

$$\partial f_i(t,x,0)/\partial x_k = \partial\omega_i(t,x)/\partial x_k$$

and

$$\partial f_i(t,x,u)/\partial x_k = \partial\omega_i(t,x)/\partial x_k + u_i\,\partial\mu(t,x,u)/\partial x_k$$

for $u \neq 0$. In order to show that $\partial f_i/\partial x_k$ is continuous, we need only prove continuity at a point $(t_0,x_0,0)$, since ω and μ belong to C^1, as is seen above. Now, in order to prove that $\partial f_i/\partial x_k$ is continuous at $(t_0,x_0,0)$, it is clearly sufficient to verify that $\partial\mu/\partial x_k$ is bounded for

$$\|(t,x,u)-(t_0,x_0,0)\| < \delta, \quad u \neq 0 .$$

We have

$$\frac{\partial\mu}{\partial x_k} = -\frac{G_{x_k}(t,x,u,\mu(t,x,u))}{G_\alpha(t,x,u,\mu(t,x,u))}.$$

Consider a sequence $(t_n, x_n, u_n) \to (t_0, x_0, 0)$ such that $u_n \neq 0$ for all n. We have seen above that $\mu_n = \mu(t_n, x_n, u_n)$ forms a bounded sequence. Further,

$$\frac{\partial G}{\partial x_k}(t,x,u,\mu(t,x,u)) = \frac{\partial F}{\partial x_k}\left(t,x,\omega(t,x)+\mu(t,x,u)\frac{u}{\|u\|}\right) + \\ + \sum_{i=1}^{n} \frac{\partial F}{\partial z_i}\left(t,x,\omega(t,x)+\mu(t,x,u)\frac{u}{\|u\|}\right)\frac{\partial \omega_i(t,x)}{\partial x_k}$$

and

$$\frac{\partial G(t,x,u,\mu(t,x,u))}{\partial \alpha} = \left(\mathrm{grad}_z F\left(t,x,\omega(t,x)+\mu(t,x,u)\frac{u}{\|u\|}\right), \frac{u}{\|u\|}\right).$$

Now, since $F, \omega \in C^1$, all partial derivatives F_{x_k} and F_{z_i} as well as $(\omega_i)_{x_k}$ are bounded on compact sets. Moreover, $(t_n, x_n) \to (t_0, x_0) \in D_1$, and $(\omega(t_n, x_n) + \mu_n u_n/\|u_n\|)$ is a bounded quantity. Therefore $G_{x_k}(t_n, x_n, \ldots)$ is clearly bounded. It remains to show that

$$\left(\mathrm{grad}_z F(t_n, x_n, \omega(t_n, x_n) + \mu_n u_n/\|u_n\|), u_n/\|u_n\|\right) > k > 0 .$$

If this is not true, there must be a subsequence for which

$$(\mathrm{grad}_z F(t_n, x_n, \ldots), u_n/\|u_n\|) \to 0 .$$

Further, we may select subsequences for which

$$\mu_n \to \mu_0 \geqq 0 \quad \text{and} \quad u_n/\|u_n\| \to u_0 \neq 0 .$$

We may assume that all this holds even without selection. It then follows from continuity that

$$F(t_0, x_0, \omega(t_0, x_0) + \mu_0 u_0) = M_0 .$$

This means that $\mu_0 > 0$ (remember that $(t_0, x_0) \in D_1$). From continuity we also get

$$\left(\mathrm{grad}_z F(t_0, x_0, \omega(t_0, x_0) + \mu_0 u_0), u_0\right) = 0 .$$

But since $\mu_0 > 0$ and $u_0 \neq 0$ this certainly contradicts the assumption 2) concerning $F(t,x,z)$.

We have thus proved that f and all derivatives $\partial f_i/\partial x_k$ are continuous functions of (t,x,u) in Ω_u. We also know that $\Phi(t,x,u) = (t,x,f(t,x,u))$ gives a one-to-one mapping of Ω_u onto Ω_z. Hence, there is an inverse mapping $\Phi^{-1} : \Omega_z \to \Omega_u$, and it follows from a simple selection argument using the continuity of Φ and the compactness of U that Φ^{-1} is continuous.

Now consider an absolutely continuous vector function $x(t)$, defined for $t_0' \leqq t \leqq t_0''$, with graph in D_1 and such that

$$\sup_t F(t,x(t),\dot{x}(t)) \leqq M_0 .$$

We may then define $u(t)$ by

$$\Phi^{-1}(t,x(t),\dot{x}(t)) = (t,x(t),u(t))$$

whenever $\dot{x}(t)$ exists, that is, a.e. Since Φ^{-1} is continuous, it is clear that the vector function $u(t)$ is Lebesgue measurable. Further, we have $\dot{x}(t)=f(t,x(t),u(t))$ a.e., and $u(t) \in U$.

Conversely, if $(t',x') \in D_1$ and $u(t)$ is defined in a neighbourhood of $t=t'$, measurable and with values in U, then the vector differential equation $dx/dt=f(t,x,u(t))$ under the initial condition $x(t')=x'$ has a unique absolutely continuous solution $x_u(t)$ in a neighbourhood of $t=t'$. (See [8, p. 78] or [10, pp. 291 and 298]. Here, the continuity of $f(t,x,u)$ and $(\partial f_i/\partial x_k)(t,x,u)$ is used). Also,

$$F(t,x_u(t),\dot{x}_u(t)) \leqq M_0 \quad \text{a.e.},$$

and according to our lemma this means that

$$\sup_t F(t,x_u(t),\dot{x}_u(t)) \leqq M_0 .$$

We may thus identify an absolutely continuous vector function $x(t)$ in D_1 satisfying $F(t,x(t),\dot{x}(t)) \leqq M_0$, with the solution of the control system $dx/dt=f(t,x,u(t))$, for some measurable control function $u(t)$. Further, any minimizing function for the minimax problem with graph in D_1 corresponds to a measurable control function $u(t)$ with values in U and defined a.e. on $[T_0,T_1]$. In the opposite direction, however, an arbitrary measurable control $u(t)$ with values in U and defined a.e. on $[T_0,T_1]$ need *not* correspond to a minimizing function, since it need not steer the system $\dot{x}=f(t,x,u(t))$ from the left endpoint (T_0,X_0) to the right endpoint (T_1,X_1). This difficulty, caused by the boundary conditions, will be taken care of by the variational method.

2) *The variational method.*

We shall use the technique developed in [8, chap. 2], and simplified in [7, chap. 4]. It is based on "needle-shaped" variations of the control u. We will follow the presentation in [7, pp. 247–251] (also most of the notations agree). We found above that $f(t,x,u)$ and $(\partial f_i/\partial x_k)(t,x,u)$ are continuous, and this agrees with the assumptions in [7]. There is one difference: our system is not autonomous, but this has no importance here.

Let $\bar{x}(t)$ be a trajectory and $\bar{u}(t)$ the corresponding control function over the interval $T_0 \leqq t \leqq T_1$. We consider an "elementary perturbation" of $\bar{u}(t)$ at some time t_1 and estimate the change in $\bar{x}(t)$, thus defining the "tangent vector" $v_{\pi_1}(t_1)$ at $\bar{x}(t_1)$. We define the "tangent perturbation cone" K_t and we consider finite combinations of elementary perturbations. The results in [7, pp. 248–250] are carried over with trivial changes.

3) *Variation of the minimizing function* $\bar{x}_0(t)$.

Let $\bar{x}_0(t)$ be the minimizing function in our theorem. We apply our variation method to $\bar{x}_0(t)$ over the interval $T_0 \leqq t \leqq T_1$, and we consider the tangent perturbation cone K_{T_1}. If K_{T_1} is not the whole tangent space at $\bar{x}_0(T_1)$, then K_{T_1} is contained in a halfspace bounded by a hyperplane through the origin. (See [10, p. 319, Lemma 39 C.1]). We want to show that K_{T_1} is contained in a halfspace. Suppose then that K_{T_1} is the whole tangent space at $\bar{x}_0(T_1)$. It is then possible to choose perturbation vectors $v_0, v_1, \ldots, v_n$ corresponding to data complexes $\pi_0, \ldots, \pi_n$, such that:

a) $(v_0, v_1, \ldots, v_n)$ is an n-simplex

b) 0 is an interior point of the convex hull of the points $v_0, \ldots, v_n$

c) the corresponding perturbation times are all distinct.

This is seen by arguing in the same way as in [7, Lemma 1 on p. 251]. Let G be the closed convex hull of the points $v_0, v_1, \ldots, v_n$. Then any point $x \in G$ is described by its barycentric coordinates

$$\mu_0, \mu_1, \ldots, \mu_n : x = \sum_{i=0}^{n} \mu_i v_i .$$

Also, these coordinates are continuous functions of $x, \mu_i = \mu_i(x)$.

Each perturbation vector v_i corresponds to a data complex

$$\pi_i = (t_{1,i}, \ldots, t_{s_i,i}; l_{1,i}, \ldots, l_{s_i,i}; u_{1,i}, \ldots, u_{s_i,i}) .$$

Let $\lambda_0, \ldots, \lambda_n$ be non-negative numbers such that $\sum \lambda_i = 1$. Consider the "composite data complex" $\pi = (\lambda_0 \pi_0, \lambda_1 \pi_1, \ldots, \lambda_n \pi_n)$, where multiplying π_i by λ_i just means that the numbers $l_{k,i}$ are multiplied by λ_i. It is fairly evident what is meant by the perturbation (π, ε). Now, for $\varepsilon > 0$ small enough, we have "the basic perturbation formula" (see [7, p. 250]):

$$\bar{x}_\pi(T_1, \varepsilon) = \bar{x}_0(T_1) + \varepsilon \sum_{i=0}^{n} \lambda_i v_i(T_1) + o(\varepsilon) ,$$

where $\lim_{\varepsilon \to +0} [o(\varepsilon)/\varepsilon] = 0$ *uniformly in* λ_i (as long as $0 \leqq \lambda_i \leqq 1$). Note also that the data complexes π_i and the vectors v_i are fixed all the time.

For $x \in G$, we form the composite data complex

$$\pi = \{\mu_0(x)\pi_0, \mu_1(x)\pi_1, \ldots, \mu_n(x)\pi_n\} = \pi(x)$$

and we consider the mapping

$$\varphi(x,\varepsilon) = \varepsilon^{-1}(\bar{x}_\pi(T_1,\varepsilon) - \bar{x}_0(T_1)) .$$

It follows from the perturbation formula that

$$\varphi(x,\varepsilon) = \sum_{i=0}^{n} \mu_i(x)v_i + \varepsilon^{-1}o(\varepsilon) = x + \varepsilon^{-1}o(\varepsilon) .$$

Clearly, for fixed $\varepsilon > 0$, $\varphi(x,\varepsilon)$ is a continuous mapping from G to R^n. Further

$$\lim_{\varepsilon \to +0}(\varphi(x,\varepsilon) - x) = 0$$

uniformly for $x \in G$. (Note that $0 \leqq \mu_k(x) \leqq 1$, for $k = 0,1,\ldots,n$, as required for uniformity.) Put $\delta = \min_{x \in \partial G} \|x\|$.

We can then fix $\varepsilon_1 > 0$ such that

$$\|\varphi(x,\varepsilon_1) - x\| < \tfrac{1}{3}\delta \quad \text{for } x \in G .$$

We shall now perform a final perturbation of the control functions. Let $u_{\pi(x)}(t)$ be the control, obtained from the unperturbed control $\bar{u}_0(t)$ by perturbing according to $\pi(x)$ and ε_1. Put

$$\tilde{u}_{\xi,x}(t) = \xi u_{\pi(x)}(t) ,$$

where ξ is a parameter, and $0 < \xi < 1$. It is seen from standard estimates that

$$\lim_{\xi \to 1-0} \|\bar{x}(T_1, \tilde{u}_{\xi,x}) - \bar{x}(T_1, u_{\pi(x)})\| = 0$$

uniformly for $x \in G$. Put

$$\psi(x,\xi) = \varepsilon_1^{-1}(\bar{x}(T_1,\tilde{u}_{\xi,x}) - \bar{x}_0(T_1)) .$$

Thus

$$\lim_{\xi \to 1-0} \|\psi(x,\xi) - \varphi(x,\varepsilon_1)\| = 0 ,$$

uniformly for $x \in G$. Choose ξ_1, $0 < \xi_1 < 1$, such that

$$\|\psi(x,\xi_1) - \varphi(x,\varepsilon_1)\| < \tfrac{1}{3}\delta \quad \text{for } x \in G .$$

Then $\psi(x,\xi_1)$ is a continuous mapping from G to R^n and if $z \in \partial G$, then

$$\|\psi(x,\xi_1) - x\| < \tfrac{2}{3}\delta \leqq \tfrac{2}{3}\|x\| .$$

Further, 0 is an interior point of G. Then degree$(\psi(\cdot,\xi_1), G, 0) = +1$, which implies that there is a point $x^* \in G$ such that $\psi(x^*,\xi_1) = 0$. (See [5, p. 32].) To see that degree$(\psi(\cdot,\xi_1), G, 0) = +1$, one may argue as follows: perform a homotopy of the mapping $\psi(x) = \psi(x,\xi_1)$ into the identity mapping I in such a way that the image of x moves along the straight segment from $\psi(x)$ to x. For $x \in \partial G$ we have

$$\|x\| > 0 \quad \text{and} \quad \|\psi(x) - x\| \leqq \tfrac{2}{3}\|x\| .$$

This implies that a point on ∂G is not mapped on the origin during the homotopy, and so

$$\text{degree}\,(\psi(\cdot, \xi_1), G, 0) = \text{degree}\,(I, G, 0) = +1$$

(cf. [5]). Thus $\bar{x}(T_1, \xi_1 u_{\pi(x^*)}) = \bar{x}_0(T_1)$, and this means that the trajectory $\bar{x}(t, \xi_1 u_{\pi(x^*)})$ is admissible for our minimax problem. Further,

$$\operatorname{ess\,sup}_t \|\xi_1 u_{\pi(x^*)}(t)\| \leqq \xi_1 < 1 .$$

It follows from this, and from the way we constructed the mapping between Ω_u and Ω_z (in part 1 of the proof) that $H(\bar{x}(t, \xi_1 u_{\pi(x^*)})) < M_0$. But this contradicts the definition of M_0, and the contradiction shows that K_{T_1} is not the whole tangent space at $\bar{x}_0(T_1)$.

As we have mentioned before, this means that K_{T_1} is contained in a half-space.

Let this halfspace be bounded by the hyperplane $y \cdot N = 0$ and let $y \cdot N \leqq 0$ for $y \in K_{T_1}$. Let $\bar{u}_0(t)$ be the control function corresponding to $\bar{x}_0(t)$. Consider the differential system (A) which is the adjoint of the variational system:

$$\text{(A)} \qquad \frac{dy_i}{dt} = -\sum_{k=1}^{n} \frac{\partial f_k}{\partial x_i}(t, \bar{x}_0(t), \bar{u}_0(t))\, y_k, \quad i = 1, 2, \ldots, n$$

Let $\psi(t)$ be the solution of this system which takes the value N for $t = T_1$. Now it follows in the usual way (see [7, the proof of Theorem 3, pp. 254–255]) that

$$(*) \qquad \sum_{\nu=1}^{n} \psi_\nu(t) f_\nu(t, \bar{x}_0(t), \bar{u}_0(t)) = \max_{u \in U} \{\sum_{\nu=1}^{n} \psi_\nu(t) f_\nu(t, \bar{x}_0(t), u)\}$$

for almost all $t \in [T_0, T_1]$, namely on the Lebesgue set for $f(t, \bar{x}_0(t), \bar{u}_0(t))$.

Consider the problem of finding $\max\{(\psi, f) \mid f \in C(t, x)\}$, where (t, x) is any point in D_1, and ψ a non-zero vector in R^n. It follows from conditions 2) and 3) on F that $C(t, x)$ is a compact and strictly convex set in R^n. Therefore the scalar product (ψ, f) takes its maximum at some well-defined point

$$f = w(t, x, \psi) \in \partial C(t, x) .$$

Thus, the relation (*) implies that

$$d\bar{x}_0/dt = w(t, \bar{x}_0(t), \psi(t)) \text{ a.e. on } [T_0, T_1] .$$

To be able to conclude that $\bar{x}_0(t) \in C^1$, we must verify that $w(t, x, \psi)$ is a continuous function of (t, x, ψ), provided that $(t, x) \in D_1$ and $\psi \neq 0$. Suppose that $(t_n,x_n,\psi_n) \to (t_0,x_0,\psi_0)$. The sequence $w(t_n,x_n,\psi_n)$, $n=1,2,3,\ldots$, is certainly bounded and we may assume that it is convergent:

$$w(t_n,x_n,\psi_n) = w_n \to w_0 .$$

It suffices to prove that $w_0=w(t_0,x_0,\psi_0)$. Let $z \in \mathsf{R}^n$ and $F(t_0,x_0,z)<M_0$. Then $F(t_n,x_n,z)<M_0$ for $n \geqq n_1$, and thus $z \in C(t_n,x_n)$. From the definition of w_n we find $(z,\psi_n) \leqq (w_n,\psi_n)$ for $n \geqq n_1$. A passage to the limit gives $(z,\psi_0) \leqq (w_0,\psi_0)$. Owing to the convexity of $\mu(y)=F(t_0,x_0,y)$, $C(t_0,x_0)$ is the closure of

$$\{z \mid z \in \mathsf{R}^n,\ F(t_0,x_0,z)<M_0\}$$

and thus $(y,\psi_0) \leqq (w_0,\psi_0)$ for any $y \in C(t_0,x_0)$. Further, $F(t_n,x_n,w_n)=M_0$ for $n=1,2,3,\ldots$, which gives $F(t_0,x_0,w_0)=M_0$, that is, $w_0 \in C(t_0,x_0)$. Consequently, $w_0=w(t_0,x_0,\psi_0)$, and $w(t,x,\psi)$ is shown to be continuous. (Cf. [6, pp. 411–413].)

Now $dx_0(t)/dt=w(t,\bar{x}_0(t),\psi(t))$ a.e., and the right member is continuous. Since $\bar{x}_0(t)$ is absolutely continuous, we have $dx_0(t)/dt=w(\cdot,\cdot,\cdot)$ for *all* $t \in [T_0,T_1]$. It follows that $\bar{x}_0(t) \in C^1[T_0,T_1]$ and that $F(t,\bar{x}_0(t),d\bar{x}_0(t)/dt)=M_0$ on the same interval. This completes the proof of Theorem 1.

It is clear that we have proved much more than stated in that theorem. For this reason, and also in order to bring out the similarity with the Pontryagin principle, we collect the results below.

THEOREM 2. THE MAXIMUM PRINCIPLE. *Let the assumptions in Theorem* 1 *be satisfied. Then there exists a nonzero vector function*

$$\psi(t) = (\psi_1(t),\ldots,\psi_n(t))$$

which is absolutely continuous over the interval $T_0 \leqq t \leqq T_1$ *such that:*

(a) $$\frac{d\psi_i(t)}{dt} = -\sum_{k=1}^{n} \frac{\partial f_k}{\partial x_i}(t,\bar{x}_0(t),\bar{u}_0(t))\psi_k(t) \quad \text{a.e. for } i=1.2,\ldots,n$$

(b) $$\frac{d\bar{x}_0(t)}{dt} = w(t,\bar{x}_0(t),\psi(t)) \quad \textit{for } T_0 \leqq t \leqq T_1 .$$

Here, $w(t,x,\psi)$ *is the* (*uniquely determined*) *vector* z *in* $C(t,x)$ *which maximizes* (z,ψ). *Further,* $w(t,x,\psi)$ *is a continuous function of* (t,x,ψ) *for* $(t,x) \in D_1$ *and* $\psi \neq 0$.

We also have

$$\bar{u}_0(t) = \frac{w(t,\bar{x}_0(t),\psi(t))-\omega(t,\bar{x}_0(t))}{\|w(t,\bar{x}_0(t),\psi(t))-\omega(t,\bar{x}_0(t))\|} \quad \text{a.e.}$$

which shows that $\bar{u}_0(t)$ *is* a.e. *equal to a continuous functions. Thus, after* *adjusting* $\bar{u}_0(t)$ *on a set of measure zero, the relation under* (a) *holds for* $T_0 \leqq t \leqq T_1$. *We find that both* $\bar{x}_0(t)$ *and* $\psi(t)$ *belong to* $C^1[T_0, T_1]$. *Here,* $\bar{u}_0(t)$ *is the control which corresponds to* $\bar{x}_0(t)$.

REMARK. The condition $F(t, \bar{x}_0(t), \omega(\ldots)) < M_0$ cannot be omitted, as is shown by the following simple example:

Choose $n = 1$ and

$$F(t,x,\dot{x}) \equiv 1 - t^2 + (\dot{x})^2 ,$$

$T_0 = -1$, $T_1 = 1$, $X_0 = X_1 = 0$. It is obvious that $M_0 = 1$. In this case, an admissible function is minimizing if and only if

$$1 - t^2 + (\dot{x})^2 \leqq 1 \text{ a.e.} ,$$

that is, $|\dot{x}| \leqq |t|$ a.e. Thus, the assertions of the theorem need not hold here.

4. A lemma on the definition of $H(x)$.

We have defined the functional $H(x)$ as $\sup_{t \in E} F(t, x(t), \dot{x}(t))$ and here E is the set where $\dot{x}(t)$ exists. In part 1) of the proof of Theorem 1 we found that

$$\operatorname{ess\,sup} F(t, x_u, \dot{x}_u) \leqq M_0$$

and we wished to conclude that $H(x_u) \leqq M_0$. Also, one might ask whether or not E is to include endpoints of the interval if the corresponding onesided derivatives exist. These questions are answered by the following result.

LEMMA. *Let* $F(t,x,z)$ *satisfy conditions* 1, 2 *and* 3 (*listed just before Theorem* 1), *and let* $x(t)$ *be absolutely continuous on an interval* I. *Denote by* E *the subset of* I *where* $\dot{x}(t)$ *exists, including endpoints of* I *if the appropriate one-sided derivatives exist. Then*

$$\sup_{t \in E} F(t, x(t), \dot{x}(t)) = \operatorname{ess\,sup}_{t \in E} F(t, x(t), \dot{x}(t))$$

(*in the sense that if one member is finite, then so is the other and they are equal*).

PROOF. Let $t' \in E$, and put $M = F(t', x(t'), \dot{x}(t'))$. Suppose that there are positive numbers η and δ such that

$$F(t, x(t), \dot{x}(t)) \leqq M - \eta$$

for almost all $t \in [t', t' + \delta]$. Let E_1 be the subset of $[t', t' + \delta]$ where $F(t, x(t), \dot{x}(t)) \leqq M - \eta$. It is clear from condition 3) on F that

$\sup_{t\in E_1}\|\dot{x}(t)\|<\infty$. By uniform continuity of F we are thus allowed to assume that

$$F(t',x(t'),\dot{x}(t)) \leqq M-\tfrac{1}{2}\eta \quad \text{for } t\in E_1$$

(possibly after decreasing δ). Hence $\dot{x}(t)\in C(t',x(t'),M-\frac{1}{2}\eta)$ for $t\in E_1$ (here, $C(t',\ldots)$ is defined in analogy with $C(t,x)$ in the proof of Theorem 1). Now $C(t',x(t'),M-\frac{1}{2}\eta)$ is convex and closed. It follows by considering the supporting planes that each difference quotient

$$\varepsilon^{-1}(x(t'+\varepsilon)-x(t')) = \varepsilon^{-1}\int_{t'}^{t'+\varepsilon}\dot{x}(t)\,dt$$

(for $\varepsilon\leqq\delta$) belongs to $C(t',\ldots)$ and also that the same holds for any limit of such quotients. We find that

$$\dot{x}(t')\in C(t',x(t'),\,M-\tfrac{1}{2}\eta)\,.$$

But this contradicts that, by definition, $M=F(t',x(t'),\dot{x}(t'))$. This proves the lemma.

Remark. We found above that $z\in C(t',x(t'),M-\frac{1}{2}\eta)$ where z is *any limit* of difference quotients $\varepsilon^{-1}(x(t'+\varepsilon)-x(t'))$ for $\varepsilon\to+0$. It is thus clear that we have

$$\sup_t\bigl(\sup\{F(t,x(t),z)\mid z\in D(t)\}\bigr) = \operatorname{ess\,sup}_{t\in E}F(t,x(t),\dot{x}(t))\,,$$

where $D(t)$ denotes the set of derived vectors at t, that is, the set of all limits of difference quotients $\varepsilon^{-1}(x(t+\varepsilon)-x(t))$ for $\varepsilon\to 0$.

5. A connection with absolutely minimizing functions. An existence theorem.

The so-called absolutely minimizing functions were introduced in [1, p. 45]. In words, $\bar{x}(t)$ is absolutely minimizing if it is a solution of the minimax problem, not only on the given basic interval but also on each sub-interval, the boundary values then being given by $\bar{x}(t)$ itself.

Theorem 3. *The function $\bar{x}_0(t)$, considered in Theorems* 1 *and* 2, *is absolutely minimizing on $T_0\leqq t\leqq T_1$.*

Proof. This will also be established by means of our control-theoretic set-up. Suppose that $\bar{x}_0(t)$ is not minimizing on the interval $[\alpha,\beta]\subset[T_0,T_1]$. Then there must exist an absolutely continuous function $\bar{x}_1(t)$ on $[\alpha,\beta]$ such that $\bar{x}_1(\alpha) = \bar{x}_0(\alpha)$, $\bar{x}_1(\beta)=\bar{x}_0(\beta)$, and

$$\begin{aligned} M_1 &= \operatorname{ess\,sup}_{\alpha \leqq t \leqq \beta} F(t, \bar{x}_1(t), \dot{\bar{x}}_1(t)) \\ &< \operatorname{ess\,sup}_{\alpha \leqq t \leqq \beta} F(t, \bar{x}_0(t), \dot{\bar{x}}_0(t)) = M_0 . \end{aligned}$$

We shall construct an admissible function $\bar{y}(t)$ for the basic minimax problem, satisfying $H(\bar{y}) < M_0$, which will give a contradiction. Clearly, $\|\dot{\bar{x}}_1(t)\| \leqq K < \infty$. Further, $F(t,x,z)$ is uniformly continuous on compact subsets of $D \times \mathsf{R}^n$. Therefore, there must be a $\delta > 0$ such that

$$F(t, \bar{x}_1(t) + a + b(t-\alpha), \dot{\bar{x}}_1(t) + b) \leqq \tfrac{1}{2}(M_0 + M_1) < M_0$$

if the constant vectors a, b satisfy $\|a\| < \delta$, $\|b\| < \delta$, and if t is a point in $[\alpha, \beta]$ where $\dot{\bar{x}}_1$ exists. (Note that $(t, \bar{x}_1(t))$ might not belong to D_1, but certainly to D.) Suppose that $T_0 < \alpha < \beta < T_1$. Now $\bar{u}_0(t)$ is the control corresponding to $\bar{x}_0(t)$. Let ξ, $0 < \xi \leqq 1$, be a parameter and consider the system $dx/dt = f(t, x, \xi \bar{u}_0(t))$. We formulate two initial-value problems for this system:

I) $x(T_0) = X_0$; a solution is sought over $[T_0, \alpha]$

II) $x(T_1) = X_1$, a solution is sought over $[\beta, T_1]$

(that is, a "backward" solution).

For $\xi = 1$, these problems have the solution $\bar{x}_0(t)$. Further, for $\xi < 1$ and close enough to 1, both problems have solutions, $y_0(t)$ defined for $T_0 \leqq t \leqq \alpha$, and $y_1(t)$ defined for $\beta \leqq t \leqq T_1$. This follows from standard estimates since $f(t,x,u)$ is uniformly continuous in (t,x,u) and Lipschitzian in x, for $\|u\| \leqq 1$ and for (t,x) belonging to some tubular neighbourhood of the curve $x = \bar{x}_0(t)$, $T_0 \leqq t \leqq T_1$. It also follows that the differences $\|\bar{x}_0(\alpha) - y_0(\alpha)\|$ and $\|\bar{x}_0(\beta) - y_1(\beta)\|$ can be made arbitrarily small by choosing ξ close enough to 1. Consider a function

$$\begin{aligned} y(t) &= y_0(t) && \text{for } T_0 \leqq t \leqq \alpha , \\ &= \bar{x}_1(t) + a + b(t-\alpha) && \text{for } \alpha < t < \beta , \\ &= y_1(t) && \text{for } \beta \leqq t \leqq T_1 . \end{aligned}$$

In order to make $y(t)$ continuous at $t = \alpha$ and $t = \beta$, we choose

$$\begin{aligned} a &= y_0(\alpha) - \bar{x}_1(\alpha) = y_0(\alpha) - \bar{x}_0(\alpha) , \\ b &= (\beta - \alpha)^{-1}(y_1(\beta) - \bar{x}_1(\beta) + \bar{x}_1(\alpha) - y_0(\alpha)) \\ &= (\beta - \alpha)^{-1}(y_1(\beta) - \bar{x}_0(\beta) + \bar{x}_0(\alpha) - y_0(\alpha)) . \end{aligned}$$

Now fix a value $\xi = \xi_1 < 1$ so close to 1 that the solutions $y_0(t)$ and $y_1(t)$ exist over $[T_0, \alpha]$ and $[\beta, T_1]$, respectively, *and* so close to 1 that $\|a\| < \delta$, $\|b\| < \delta$. Then $y(x)$ is admissible for our minimax problem over $[T_0, T_1]$. Further, since $\|a\| < \delta$ and $\|b\| < \delta$, we have

$$\operatorname{ess\,sup}_{\alpha < t < \beta} F(t, y(t), \dot{y}(t)) \leqq \tfrac{1}{2}(M_0 + M_1) < M_0 .$$

We may assume that $\bar{u}_0(t)$ is continuous; see Theorem 2. For $T_0 \leqq t < \alpha$ and $\beta < t \leqq T_1$ we thus have

$$\dot{y}(t) = f(t, y(t), \xi_1 \bar{u}_0(t))$$

and the right member is continuous for $T_0 \leqq t \leqq T_1$. Since $0 < \xi_1 < 1$ and $\|\bar{u}_0(t)\| = 1$, this obviously implies that

$$\sup\{F(t, y(t), \dot{y}(t)) \mid t \in [T_0, \alpha) \cup (\beta, T_1]\} < M_0 .$$

Consequently, we have $H(y) < M_0$, which contradicts the definition of M_0. The cases $T_0 = \alpha < \beta < T_1$ and $T_0 < \alpha < \beta = T_1$ are treated in analogous manner. This completes the proof.

We conclude the discussion of the minimax problem with an existence theorem. It is shown by an example in [2, p. 410] that a minimizing function need not exist. It turns out that in this example, every minimizing sequence is unbounded. In the converse direction we have the next theorem. Note that there are cases where a minimizing function exists, but none which satisfies $F(t, x(t), \omega(t, x(t))) < M_0$ for $T_0 \leqq t \leqq T_1$, as can be seen from the example after Theorem 2.

THEOREM 4. *If there exists a minimizing sequence of functions which is contained in a compact subset of D, then there is a minimizing function. (As before, $F(t,x,z)$ is assumed to satisfy conditions* 1), 2) *and* 3).)

PROOF. Let $x_n(t)$, $n = 1, 2, 3, \ldots$, be a minimizing sequence in $\mathscr{F}$ such that $(t, x_n(t)) \in E$, and such that

$$\sup_{t,n} F(t, x_n(t), \dot{x}_n(t)) < \infty .$$

Here, E is a compact subset of D. It follows that

$$\sup_{t,n} \|\dot{x}_n(t)\| < \infty .$$

Hence, these functions satisfy a uniform Lipschitz condition and a uniformly convergent subsequence can be selected. Let $\{x_n(t)\}_1^\infty$ be the subsequence and $x_0(t)$ the limit function. Thus, $x_0(t)$ satisfies the same Lipschitz condition, and $(t, x_0(t)) \in E \subset D$. Clearly, $x_0(t)$ is admissible. It only remains to prove that

$$H(x_0) \leqq M_0 + \varepsilon \quad \text{for every } \varepsilon > 0 ,$$

and the arguments for this will be very similar to those in the lemma. Let t_0 be an arbitrary point where $\dot{x}_0$ exists. Now

$$\sup_t F(t, x_n(t), \dot{x}_n(t)) < M_0 + \tfrac{1}{2}\varepsilon \quad \text{if } n \geqq N_1 .$$

Further,

$$\sup_{n,t}\|\dot{x}_n(t)\| = K < \infty .$$

Choose a closed spherical neighbourhood U_1 of $(t_0,x_0(t_0))$, with radius r, such that $U_1 \subset D$. Then $F(t,x,z)$ is uniformly continuous on

$$U_1 \times \{z \mid \|z\| \leqq K\} .$$

Thus, there is a $\delta \leqq r$ such that

$$(t_1,x_1) \in U_1,\ (t_2,x_2) \in U_1,\ \|z\| \leqq K, \text{ and } \|(t_1,x_1)-(t_2,x_2)\| \leqq \delta$$

implies that

$$|F(t_1,x_1,z) - F(t_2,x_2,z)| < \tfrac{1}{2}\varepsilon .$$

Let U be a spherical neighbourhood of $(t_0,x_0(t_0))$ with radius δ. Because of the equicontinuity and convergence of $\{x_n(t)\}_1^\infty$ there are $\delta_1 > 0$ and $N_2 \geqq N_1$ such that $(t,x_n(t)) \in U$ if $n \geqq N_2$ and $|t-t_0| \leqq \delta_1$. Thus

$$|F(t_0,x_0(t_0),\dot{x}_n(t)) - F(t,x_n(t),\dot{x}_n(t))| < \tfrac{1}{2}\varepsilon$$

if $n \geqq N_2$, $|t-t_0| \leqq \delta_1$ and if $\dot{x}_n(t)$ exists. For such n and t we have

$$F(t_0,x_0(t_0),\dot{x}_n(t)) \leqq M_0 + \varepsilon .$$

The set

$$C' = \{z \mid z \in \mathrm{R}^n, F(t_0,x_0(t_0),z) \leqq M_0 + \varepsilon\}$$

is closed and convex. As in the lemma of Section 4, it follows that

$$(t-t_0)^{-1}(x_n(t)-x_n(t_0)) \in C' \quad \text{for } 0 < |t-t_0| \leqq \delta_1 .$$

Making n tend to infinity, we find that

$$(t-t_0)^{-1}(x_0(t)-x_0(t_0)) \in C' .$$

Finally, making t tend to t_0, we find that $\dot{x}_0(t_0) \in C'$, which means that $F(t_0,x_0(t_0),\dot{x}_0(t_0)) \leqq M_0 + \varepsilon$. Since t_0 was arbitrary, it follows that $H(x_0) \leqq M_0 + \varepsilon$. This completes the proof.

If $D \supset [T_0,T_1] \times \mathrm{R}^n$, that is, there are no restrictions on x, the existence of a bounded minimizing sequence will imply the existence of a minimizing function. Further, the existence of such a sequence can be secured by imposing conditions on F, notably on the growth of $F(t,x,z)$ as a function of z. Compare [2, pp. 411–414].

6. Comments.

The special case for $n=1$ of Theorem 1 is contained in the result proved in [3]. The method of proof used in [3] cannot be carried over without essential changes.

In Carter's paper [4], the situation is as follows: we are given a linear differential operator $\mathscr{L}y \equiv A(t)(\dot{y}+B(t)y+c(t))$, where $A(t)$ and $B(t)$ are matrix functions and $c(t)$ a vector function. The admissible functions are (essentially) the same as in this paper. For a vector function $x(t)$ we consider the norm

$$\|x\| = \max_{1\leq i\leq n}(\operatorname{ess\,sup}_{T_0\leq t\leq T_1}|x_i(t)|)\,.$$

The problem is to minimize $\|\mathscr{L}y\|$ over all admissible $y(t)$. The solution is based on the fact that the differential equation $\mathscr{L}y=g$ can be solved for y if the function $g(t)$ is given. Further, since we have *two* boundary conditions, the boundary value problem is over-determined which means that only *certain* functions $g(t)\in L^\infty$ correspond to admissible functions $y(t)$. The problem is then to determine this class of functions $g(t)$ and to minimize $\|g\|$ over it. Well-known properties of adjoint systems are used in the analysis. The result is that the solution is always unique (we call it y_0), and it satisfies $|((\mathscr{L}y_0)(t))_i|=M_0$ a.e. for $i=1,2,\ldots,n$. Here, M_0 is the minimum value of $\|\mathscr{L}y\|$. (Compare the corollary of Theorem 9 in [1].) Thus the situation is much simpler in this case than it is in the more general case, treated in this paper. Carter also considers the minimax problem for linear differential operators of higher order.

We also want to mention the thesis by F. M. Waltz [9]. The problem here is to steer a control system from a given initial state to a prescribed final state, over a given time interval, in such a way that the "peak amplitude control", $\sup_t\|u(t)\|$, is as small as possible. The problem is solved in the linear case, $\dot{x}=A(t)x+B(t)u$. The solution is too long to be reported here. Naturally, there are parallels with Carter's results. We think the method of the present paper could also have been used, even in the nonlinear case. The relation

$$\text{(b)} \qquad d\bar{x}_0/dt = w(t,\bar{x}_0(t),\psi(t))$$

in Theorem 2 in this paper may be viewed as a counterpart of relation (5.4) in [4] and of relation (3.12), p. 38 in [9].

Finally, a few words on the conditions 1), 2) and 3) of Sectiou 2 on F. These conditions are not independent; in fact a tedious but routine argument shows that 3) can be deduced from 1) and 2). However, as it is convenient here, we work directly with conditions 1), 2) and 3). Also, in specific cases, it is usually easy to see whether 3) is satisfied or not. Other reductions of the conditions are also possible, but we do not go into these details.

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