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$R(X)$ AS A DIRICHLET ALGEBRA AND REPRESENTATION OF ORTHOGONAL MEASURES BY DIFFERENTIALS

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1. Introduction.

Let X be a compact subset of the complex plane C, and let $R(X)$ denote the uniform closure on X of the rational functions with poles off X. Let $A(X)$ be the continuous functions on X which are analytic in the interior X° of X. In this paper we will always assume that $\mathsf{C}\setminus X^\circ$ is connected. We say that $R(X)$ is a Dirichlet algebra if $\mathrm{Re}(R(X))$, the real parts of the functions in $R(X)$, are uniformly dense on the boundary $\mathrm{b}X$ of X in the real continuous functions on $\mathrm{b}X$. This occurs if and only if there are no non-zero real measures on $\mathrm{b}X$ orthogonal to $R(X)$. If $R(X)$ is a Dirichlet algebra, then $\mathsf{C}\setminus X^\circ$ is connected [6, section 4]. For information on Dirichlet algebras, see [17], [9] and [6].

In this paper we treat a problem, raised by Bishop in [2] and [3]: When can every measure μ on $\mathrm{b}X$ which is orthogonal to $R(X)$ be represented by its analytic differential $(2\pi i)^{-1}\hat{\mu}(z)\,dz$, where

$$\hat{\mu}(z) = \int (\zeta - z)^{-1}\,d\mu(\zeta) \quad \text{for } z \in X^\circ\,?$$

(See definitions below.) In section 2 we give a necessary and sufficient condition for this to be true, theorem 2.2, and then we use this result to prove that if $R(X)$ is a Dirichlet algebra, then every orthogonal measure μ on $\mathrm{b}X$ is represented by its differential $(2\pi i)^{-1}\hat{\mu}(z)\,dz$ (theorem 2.4). It is an interesting question whether the converse of this is also true.

In section 3 we prove three results on when $R(X)$ is a Dirichlet algebra. The first states, roughly speaking, that if $R(X)$ "locally" is a Dirichlet algebra, then $R(X)$ is a Dirichlet algebra. Using the same technique we then prove that if $R(bX) = C(\mathrm{b}X)$ and every bounded analytic function on X° can be approximated pointwise by a bounded sequence in $R(X)$, then $R(X)$ is a Dirichlet algebra. The same proof, without claiming $R(\mathrm{b}X) = C(\mathrm{b}X)$, works for $A(X)$. The third result states that if X is an

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intersection of compact sets X_n such that $R(X_n)$ is a Dirichlet algebra, then $R(X)$ is a Dirichlet algebra. These results are contained in recent work done by Gamelin, Garnett and Davie (see [7], [10] and [11]) but our proofs are entirely different.

At the end of section 3 we combine these results with theorem 2.4 to obtain a generalization of a result of Bishop in [3].

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2. Representation of orthogonal measures by differentials.

2.1. DEFINITIONS. (See [2] and [3].) A sequence $\Gamma=\{\Gamma_n\}$ of compact subsets of X° is said to converge to bX provided:

(i) Each Γ_n is the union of a finite number of disjoint piecewise smooth simple closed curves lying in X°, no two of which belong to the same component of X°, and

(ii) If S is any compact subset of X°, then for all n sufficiently large, S will lie in the union of the bounded components of $\mathbf{C}\setminus\Gamma_n$.

If μ is a measure on bX and $dw=f(z)dz$ is an analytic differential in X°, we say that dw represents μ (with respect to Γ) if there exists a sequence $\Gamma=\{\Gamma_n\}$ converging to bX such that for all continuous functions h on X, we have

$$\int_{\mathrm{b}X} h(t)\,d\mu(t) = \lim_n \int_{\Gamma_n} h(z)f(z)\,dz\,.$$

Let $\{a_n\}_{n=1}^\infty$ and $\{r_n\}_{n=1}^\infty$ be fixed sequences of non-negative numbers such that

(i) $a_1=0,\ r_1=1$,
(ii) $a_n\uparrow a<\infty$,
(iii) $r_n\downarrow 0$,
(iv) $a_{n+1}-r_{n+1}>a_n+r_n, \quad n=1,2,\ldots$.

Put $\Delta_n=\{z;\ |z-a_n|<r_n\}$. Let $V_1, V_2,\ldots$ denote the components of X°, and let m denote the number of such components (which may be countably infinite). Then we define $K=K_X$ to be the closure of $\bigcup_{n=1}^m \Delta_n$. We let Φ denote a fixed conformal mapping from X° onto K° and let $\Psi\colon K^\circ\to X^\circ$ be its inverse. By a theorem of Fatou (see [15]) Ψ has

non-tangetial boundary values Ψ^* a.e. on bK with respect to the measure $d\theta = \sum_n 2^{-n} d\theta_n$, where $d\theta_n$ denotes the normalized Lebesgue measure on bΔ_n.

If ν is a measure on bK orthogonal to $R(K)$, that is, $\nu \in R(K)^\perp$, then by the F. and M. Riesz theorem $\nu \ll d\theta$, and so we can define $\Psi^*(\nu)$ as the measure on bX whose value on the Borel set E is $\nu(\Psi^{*-1}(E))$.

With these notations we get the following characterisation of measures μ on bX that is represented by its differential:

2.2. THEOREM. *Let μ be a measure on* bX. *The following are equivalent:*
(i) *μ is represented by its analytic differential* $(2\pi i)^{-1}\hat{\mu}(z)\,dz$.
(ii) *There exists a measure $\nu \in R(K)^\perp$ carried on* bK *with* $\mu = \Psi^*(\nu)$.

PROOF. If there exists a sequence $\Gamma = \{\Gamma_n\}$ convering to bX such that $(2\pi i)^{-1}\hat{\mu}(z)\,dz$ represents μ with respect to Γ, then we can find a subsequence $\{\Gamma_{n_k}\}$ such that $\delta = \{\delta_k\} = \{\Phi(\Gamma_{n_k})\}$ converges to bK, and the differential

$$\Phi\big((2\pi i)^{-1}\hat{\mu}(z)\,dz\big) = (2\pi i)^{-1}\,\hat{\mu}\big(\Psi(z)\big)\,\Psi'(z)\,dz$$

represents a measure ν on bK with respect to δ. The proof of this is word for word the same as the proof of theorem 2 in [2]. As remarked on page 283 in [2] we then have

$$\hat{\mu}\big(\Psi(z)\big)\,\Psi'(z) = \hat{\nu}(z) \quad \text{for } z \in K^\circ\,.$$

Using Cauchys theorem we get $\nu \in R(K)^\perp$.

Let $\sigma = \Psi^*(\nu)$. We will prove that $\sigma = \mu$. If $z \in K^\circ$ we have

$$\hat{\sigma}\big(\Psi(z)\big) = \int \frac{d\sigma(s)}{s - \Psi(z)} = \int \frac{d\nu(t)}{\Psi^*(t) - \Psi(z)}$$

and

$$\hat{\mu}\big(\Psi(z)\big) = \frac{1}{\Psi'(z)}\,\hat{\nu}(z) = \frac{1}{\Psi'(z)} \int \frac{d\nu(t)}{t - z}\,.$$

Now since

$$G(t) = \frac{1}{\Psi(t) - \Psi(z)} - \frac{1}{\Psi'(z)}\,\frac{1}{t - z}$$

is bounded and analytic in K°, we have $\int G^*(t)\,d\nu(t) = 0$, where $G^*(t)$ denotes the non-tangential boundary values of G, such that $\hat{\mu}(w) = \hat{\sigma}(w)$ for all $w \in X^\circ$. Since both μ and σ are orthogonal to $A(X)$, lemma 1.1 in [7] gives $\sigma = \mu$.

Suppose $\mu = \Psi^*(\nu)$, where $\nu \in R(K)^\perp$ is a measure on bK. Then, essentially by the F. and M. Riesz theorem (see [2, p. 283]), there exists a sequence $\delta=\{\delta_n\}$ converging to bK such that $(2\pi i)^{-1}\hat{\nu}(z)dz$ represents ν with respect to δ. Again, by the proof of theorem 2 in [2], there exists a subsequence $\{\delta_{n_k}\}$ such that $\Gamma=\{\Gamma_k\}=\{\Psi(\delta_{n_k})\}$ converges to bX and the differential

$$\Psi((2\pi i)^{-1}\hat{\nu}(z)dz) = (2\pi i)^{-1}\hat{\nu}(\Phi(z))\,\Phi'(z)dz$$

represents a measure σ on bX with respect to Γ. By the remark p. 283 in [2], we have $\hat{\sigma}(z)=\hat{\nu}(\Phi(z)\Phi'(z)$, so it is enough to prove that $\sigma=\mu$. Since both σ and μ are orthogonal to $A(X)$, it is enough to prove, by lemma 1.1 in [7], that $\hat{\mu}(w)=\hat{\sigma}(w)$ for all $w\in X^\circ$. Letting $z=\Phi(w)$ we have

$$\hat{\sigma}(w) = \Phi'(w)\,\hat{\nu}(\Phi(w)) = \frac{1}{\Psi'(z)}\,\hat{\nu}(z) = \frac{1}{\Psi'(z)}\int\frac{d\nu(t)}{t-z}$$

and

$$\hat{\mu}(w) = \int\frac{d\mu(s)}{s-w} = \int\frac{d\nu(t)}{\Psi^*(t)-\Psi(z)}\,.$$

Hence, since $G(t)$ is bounded and analytic in K°, the result follows.

If $z\in X^\circ$, let λ_z be the harmonic measure for z with respect to X°, and if $z\in$ bX, let λ_z be the pointmass at z. Choose points $z_n\in V_n$, $n=1,2,\ldots$, (the components of X°) and put $\lambda_n=\lambda_{z_n}$ and $\lambda=\sum_{n=1}^{m}2^{-n}\lambda_n$, where m is, as before, the number of components of X°. If $f\in L^1(\lambda)$ we let $\tilde{f}$ be the harmonic function defined on X° by $\tilde{f}(z)=\int f d\lambda_z$. $\tilde{f}$ is called the harmonic extention of f (see [14, ch. 8] for properties of harmonic measures). If σ is a positive measure on bX with $\lambda\ll\sigma$ we define $H^\infty(\sigma)$ to be the weak* closure of $R(X)$ in $L^\infty(\sigma)$. We denote by $H^\infty(X^\circ)$ the set of bounded analytic functions on X°. The map S_σ: $H^\infty(\sigma)\to H^\infty(X^\circ)$ given by $S_\sigma(f)=\tilde{f}$ is a continuous homomorphism of the algebra $H^\infty(\sigma)$ into the algebra $H^\infty(X^\circ)$. (See [9, p. 226].) Moreover, we have:

2.3. Lemma. *If $R(X)$ is a Dirichlet algebra, then S_λ is an isometric isomorphism of $H^\infty(\lambda)$ onto $H^\infty(X^\circ)$.*

Proof. This follows from lemma 2.1 of [7] and theorem VIII.11.1 of [9].

2.4. Theorem. *Suppose $R(X)$ is a Dirichlet algebra. Then every measure $\mu\in R(X)^\perp$ carried on* bX *is represented by its differential* $(2\pi i)^{-1}\hat{\mu}(z)dz$.

Proof. Let μ be a measure on bX orthogonal to $R(X)$. By theorem

2.2 it is enough to prove that there exists a measure ν on bK orthogonal to $R(K)$ such that $\mu=\Psi^*(\nu)$. Since $\Phi \in H^\infty(X^\circ)$ there exists, by lemma 2.3, a function $\Phi^* \in H^\infty(\lambda)$ such that $\tilde{\Phi}^*=\Phi$. Let $z \in X^\circ$ and put $x=\Phi(z) \in K^\circ$. Let ϱ_x denote the harmonic measure (the Poisson measure) for x with respect to Δ_n, where $x \in \Delta_n$. Then it is easy to see that $\lambda_z=\Psi^*(\varrho_x)$. This gives

$$x = \Phi(z) = \int \Phi^*\, d\lambda_z = \int \Phi^* \circ \Psi^*\, d\varrho_x = \int t\, d\varrho_x(t)\,,$$

and hence $\Phi^*(\Psi^*(t))=t$ a.e. $d\theta$. But then $\Phi^*(\Psi^*(\varrho_x))=\varrho_x$ or

$$\Phi^*(\lambda_z) = \varrho_x\,.$$

This implies that

$$\int \Psi^*(\Phi^*(t))\, d\lambda_z(t) = \int \Psi^*(t)\, d\varrho_x(t) = \Psi(x) = z\,,$$

and so $\Psi^*(\Phi^*(t))=t$ a.e. λ. Since $R(X)$ is a Dirichlet algebra, we have $|\mu| \ll \lambda$, by the Wermer–Glicksberg theorem [17, Satz 3] and the fact that $R(X)$ has no non-zero completely singular orthogonal measures [18]. Hence $\Psi^*(\Phi^*(t))=t$ a.e. μ. If we define $\nu=\Phi^*(\mu)$, we have

$$\int t^k\, d\nu(t) = \int (\Phi^*)^k(\zeta)\, d\mu(\zeta) = 0\,,$$

since $\mu \ll \lambda$ and $\Phi^* \in H^\infty(\lambda)$. Therefore ν is orthogonal to the polynomials, and so $\nu \in R(K)^\perp$ since $\mathbf{C} \setminus K$ is connected. Since ν is carried on bK and $\mu=\Psi^*(\Phi^*(\mu))=\Psi^*(\nu)$, the result follows.

3. Conditions under which $R(X)$ is a Dirichlet algebra.

We now turn to the question: When is $R(X)$ a Dirichlet algebra? We will not try to give a complete discussion of this problem, but concentrate on three special results. We need the following lemma about splitting of orthogonal measures, due to Bishop (see [1, lemma 6]):

3.1. LEMMA. *Let F be compact and $\mu \in R(F)^\perp$. Then for almost all $x_0 \in \mathbf{R}$ with respect to Lebesgue measure there exists a measure β on $L=\{z \in F;\ \mathrm{Re}\, z=x_0\}$ such that*

$$\int_{F_1} h\, d\mu = -\int_{F_2} h\, d\mu = \int_L h\, d\beta \quad \textit{for all } h \in R(F)\,,$$

where $F_1=\{z \in F;\ \mathrm{Re}\, z \leqq x_0\}$, $F_2=\{z \in F;\ x_0 \leqq \mathrm{Re}\, z\}$. Further $|\mu|(L)=0$, and if we define

$$\mu_1 = \mu/F_1 - \beta, \quad \mu_2 = \mu/F_2 + \beta ,$$

then $\mu = \mu_1 + \mu_2$ *and* $\mu_i \in R(F_i)^{\perp}$, $i = 1, 2$.

We do not give the proof of lemma 3.1, since the proof of lemma 6 in [1] applies with only minor changes.

If S is a compact plane set, F is a closed subset of S, and B is a closed subspace of $C(S)$ including constants and separating points on S, we say that F is an interpolation set for B if $B/F = C(F)$. We say that F is a peak set for B if there exists a function $f \in B$ such that $f = 1$ on F, $|f| < 1$ off F. A point $x \in S$ is called a peak point for B if $\{x\}$ is a peak set for B. If F is both an interpolation set and a peak set for B, we call F a peak interpolation set for B. We mention the following result, known as the (Bishop–) Rudin–Carleson theorem: If B is an algebra, then F is a peak interpolation set for B if and only if every measure orthogonal to B vanishes on F. (See [4], [8, p. 284] and [13, p. 429].)

3.2. Lemma. *Let S be a compact set and let F be a closed subset of* bS. *Then F is a peak interpolation set for $R(S)/\mathrm{b}S$ if and only if F is a peak interpolation set for $R(S)$.*

Proof. Suppose F is a peak interpolation set for $R(S)/\mathrm{b}S$. Choose $f \in R(S)$ such that $f = 1$ on F and $|f| < 1$ on $\mathrm{b}S \setminus F$. Suppose there exists $x_0 \in S^\circ$ such that $|f(x_0)| = 1$. By the maximum modulus principle we then have $f = 1$ on the closure of the component V of S° which contains x_0. Therefore $\mathrm{b}V \subset F$. But since $R(S)/F = C(F)$ and $h(z) = (z - x_0)^{-1} \in C(F)$ there exists $g \in R(S)$ such that $g(z) = (z - x_0)^{-1}$ or $g(z)\ (z - x_0) = 1$ on $\mathrm{b}V$ and so

$$g(z)\ (z - x_0) = 1 \quad \text{for all } z \in \overline{V} ,$$

which is impossible for $z = x_0$. This contradiction proves that F is a peak set for $R(S)$, and we are done.

3.3. Theorem. *Suppose that for all $x \in X$ there exists an open neighbourhood W of x such that $R(\overline{W} \cap X)$ is a Dirichlet algebra. Then $R(X)$ is a Dirichlet algebra.*

Proof. Choose open sets $W_1, \ldots, W_n$ such that $R(\overline{W}_i \cap X)$ is a Dirichlet algebra, $1 \leqq i \leqq n$, and $X \subset \bigcup_{i=1}^n W_i$. Choose $\delta > 0$ such that for every closed rectangle R of diameter less than δ there exists W_i with $R \cap X \subset W_i$. Let μ be a measure on bX which is orthogonal to $R(X)$. Using lemma 3.1 a finite number of times we can write

$$\mu = \sum_{i=1}^{N} \mu_i, \quad X = \bigcup_{i=1}^{N} X_i ,$$

where $X_i = X \cap R_i$, R_i is a closed rectangle of diameter less that δ, and μ_i is a measure on

$$\mathrm{b}X_i = (R_i \cap \mathrm{b}X) \cup (X \cap \mathrm{b}R_i)$$

orthogonal to $R(X_i)$. Moreover, lemma 3.1 tells us that

$$\mu_i / R_i \cap \mathrm{b}X = \mu / R_i \cap \mathrm{b}X .$$

Let i be fixed, $1 \leqq i \leqq N$. Choose $W = W_j$ such that $X_i \subset W$ and put $Y = Y_i = \overline{W} \cap X$. For $x \in Y^\circ$ we let λ_x denote the harmonic measure for x with respect to Y°. Let $S_1, S_2, \ldots$ be the components of Y°, choose $y_i \in S_i$ for all i, and define $\lambda = \sum_i 2^{-i} \lambda_{y_i}$. Let $K = K_X$, and Φ, Ψ, Ψ^* be as in 2.1.

The idea of the proof is to construct a function $\Phi^* \in L^\infty(\mu)$ which satisfies:

(i)	$\Phi^*(t) \in \mathrm{b}K$	for a.a. t with respect to μ ,
(ii)	$\int (\Phi^*)^k d\mu = 0$	for $k = 0, 1, 2, \ldots$,
(iii)	$\Psi^*(\Phi^*(t)) = t$	a.e. μ .

When (i), (ii) and (iii) are established, it is easy to prove that if μ is real, μ must be the zero measure. For if μ is real, so is the measure $\nu = \Phi^*(\mu)$. ν is carried on $\mathrm{b}K$, by (i), and it is orthogonal to the polynomials, by (ii). It is well known and easy to see that $R(K)$ is a Dirichlet algebra. Hence $\nu = 0$, and (iii) gives that

$$\mu = \Psi^*(\Phi^*(\mu)) = \Psi^*(\nu) = 0 .$$

To construct such a function Φ^* we first prove this local result: There exists a function $\Phi_Y{}^* \in H^\infty(\lambda)$ such that

(I) $\Phi_Y{}^*(t) \in \mathrm{b}K$ for a.a. t on $\mathrm{b}X_i \cap \mathrm{b}Y$ with respect to μ_i

(II) $$\int_{\mathrm{b}X_i \cap \mathrm{b}Y} (\Phi_Y{}^*)^k d\mu_i = -\int_{\mathrm{b}X_i \cap Y^\circ} \Phi^k \, d\mu, \quad k = 0, 1, 2, \ldots ,$$

(III) $\Psi^*(\Phi_Y{}^*(t)) = t$ a.e. μ_i on $\mathrm{b}X_i \cap \mathrm{b}Y$.

When (I), (II) and (III) are established, the construction of Φ^* goes as follows: For each i between 1 and N we construct one such function $\Phi_Y{}^* = \Phi^*_{Y_i}$, and since μ has no mass on $\bigcup_{i=1}^N \mathrm{b}R_i$, by lemma 3.1, we can define $\Phi^* \in L^\infty(\mu)$ by

$$\Phi^*(t) = \Phi^*_{Y_i}(t) \quad \text{for} \quad t \in R_i \cap \mathrm{b}X = \mathrm{b}X_i \cap \mathrm{b}Y_i .$$

Then of course $\Phi^*(t) \in \mathrm{b}K$ a.e. μ, and since $\mu = \sum_{i=1}^{N} \mu_i$, (II) gives that

$$\int_{\mathrm{b}X} (\Phi^*)^k \, d\mu = \sum_i \int_{R_i \cap \mathrm{b}X} (\Phi^*)^k \, d\mu_i$$

$$= \sum_i \int_{\mathrm{b}X_i \cap \mathrm{b}Y_i} (\Phi^*_{Y_i})^k \, d\mu_i$$

$$= -\sum_i \int_{\mathrm{b}X_i \cap X^\circ} \Phi^k \, d\mu_i = -\int_{X^\circ} \Phi^k \, d\mu = 0, \quad k = 0,1,2,\ldots .$$

Here we have used the fact that $\mathrm{b}X_i \cap Y^\circ = \mathrm{b}X_i \cap X^\circ$ and that μ is a measure on $\mathrm{b}X$. Moreover, since (III) is valid, it is easy to see that Φ^* also satisfies (iii).

It remains to prove the existence of $\Phi_Y{}^*$.

Since $\Phi / Y^\circ \in H^\infty(Y^\circ)$ and $R(Y)$ is a Dirichlet algebra, there exists by lemma 2.3 a function $\Phi_Y{}^* \in H^\infty(\lambda)$ such that

$$\tilde{\Phi}_Y{}^*(z) = \Phi(z) \quad \text{for all } z \in Y^\circ .$$

Since λ_z is a multiplicative measure with respect to $H^\infty(\lambda)$, we get

$$\int (\Phi_Y{}^*)^k \, d\lambda_z = \Phi^k(z) \quad \text{for all } z \in Y^\circ .$$

Let $E \subset \mathrm{b}X_i \cap \mathrm{b}Y$ be closed and suppose $\lambda(E) = 0$. Since $R(Y)$ is a Dirichlet algebra, all measures on $\mathrm{b}Y$ orthogonal to $R(Y)$ vanishes on E, by the Wermer–Glicksberg theorem ([17]) and the fact that $R(Y)$ has no nonzero completely singular orthogonal measures. Hence by the Rudin–Carleson theorem E is a peak interpolation set for $R(Y)/\mathrm{b}Y$ and by lemma 3.2 E is a peak interpolation set for $R(Y)$. Again by the Rudin–Carleson theorem we get that $|\mu_i|(E) = 0$, since

$$\mu_i \in R(X_i)^\perp \subset R(Y)^\perp .$$

Hence

(1) $$\mu_i / bX_i \cap \mathrm{b}Y \ll \lambda / \mathrm{b}X_i \cap \mathrm{b}Y .$$

Define the measure σ_i on $\mathrm{b}Y$ to be the sweep of μ_i to $\mathrm{b}Y$, that is

$$\int f \, d\sigma_i = \int \tilde{f} \, d\mu_i \quad \text{for } f \in C(bY) ,$$

where as before $\tilde{f}$ denotes the harmonic extension of f in Y°, and we set $\tilde{f}(x) = f(x)$ for $x \in \mathrm{b}Y$. Then $\sigma_i \in R(Y)^\perp$, $\sigma_i \ll \lambda$ and therefore

$$(2) \qquad \int (\Phi_Y{}^*)^k \, d\sigma_i = 0, \quad k=0,1,2,\ldots,$$

since $\Phi_Y{}^* \in H^\infty(\lambda)$. Choose a sequence $\{f_n\}_{n=1}^\infty \subset C(\mathrm{b}Y)$ such that $\|f_n\|_\infty \leqq \|\Phi_Y{}^*\|_\infty$ and $f_n(t) \to \Phi_Y{}^*(t)$ a.e. λ. Then by the dominated convergence

$$(f_n{}^k)^\sim(z) \to \Phi^k(z) \text{ for } z \in Y^\circ,$$

$k=0,1,2,\ldots$, and (1) and (2) gives

$$\begin{aligned} 0 = \int_{\mathrm{b}Y} (\Phi_Y{}^*)^k \, d\sigma_i &= \lim_n \int_{\mathrm{b}Y} f_n{}^k \, d\sigma_i \\ &= \lim_n \int_{\mathrm{b}X_i} (f_n{}^k)^\sim \, d\mu_i \\ &= \lim_n \int_{\mathrm{b}X_i \cap Y^\circ} (f_n{}^k)^\sim \, d\mu_i + \lim_n \int_{\mathrm{b}X_i \cap \mathrm{b}Y} f_n{}^k \, d\mu_i \\ &= \int_{\mathrm{b}X_i \cap Y^\circ} \Phi^k \, d\mu_i + \int_{\mathrm{b}X_i \cap \mathrm{b}Y} (\Phi_Y{}^*)^k \, d\mu_i, \quad k=0,1,2,\ldots, \end{aligned}$$

which is (II).

To verify (I) and (III) define $\Psi_0 \colon K \to X$ by

$$\begin{aligned} \Psi_0(t) &= \Psi^*(t) \quad \text{for } t \in \mathrm{b}K, \\ &= \Psi(t) \quad \text{for } t \in K^\circ. \end{aligned}$$

We want to show that

$$\Psi_0(\Phi_Y{}^*(t)) = t \quad \text{a.e. } \lambda.$$

Let $G=\Phi(Y^\circ)$. We assert that

$$\Phi_Y{}^*(t) \in \bar{G} \quad \text{for a.a. } t \text{ with respect to } \lambda.$$

Suppose $a \notin \bar{G}$. Then $(\Phi/Y^\circ - a)^{-1} \in H^\infty(Y^\circ)$, and by lemma 2.3 there exists $\beta^* \in H^\infty(\lambda)$ such that

$$\tilde{\beta}^* = (\Phi/Y^\circ - a)^{-1} \quad \text{and} \quad \|\beta^*\|_\infty = \|(\Phi(Y^\circ - a)^{-1}\|.$$

Since $\Phi_Y{}^* - a \in H^\infty(\lambda)$ we then have

$$\int \beta^*(\Phi_Y{}^* - a) \, d\lambda_z = (\Phi(z)-a)^{-1}(\Phi(z)-a) = 1$$

for all $z \in Y^\circ$. Hence by injectivity $\beta^* = (\Phi_Y{}^* - a)^{-1}$ and so

$$\|(\Phi_Y{}^* - a)^{-1}\|_\infty = \|(\Phi/Y^\circ - a)^{-1}\|.$$

Since this is valid for all $a \notin \bar{G}$, we must have

$$\Phi_Y^*(t) \in \bar{G} \subset K \quad \text{for a.a. } t \text{ with respect to } \lambda .$$

Now choose $z \in Y^\circ$ and put $x = \Phi(z) \in K^\circ$. Since

$$\int (\Phi_Y^*)^k \, d\lambda_z = \Phi^k(z) = x^k, \quad k = 0, 1, 2, \ldots ,$$

the measure $\beta_x = \Phi_Y^*(\lambda_z)$ is a representing measure on $\bar{G} \subset K$ for x with respect to the polynomials. Choose k such that $x \in \Delta_k$. Since the function which is 1 on $\bar{\Delta}_k$ and 0 on $K \setminus \bar{\Delta}_k$ belongs to $R(K) = P(K)$ (this follows, for instance, by lemma 3.1), β_x must be a measure on $\bar{\Delta}_k$. Let ϱ_x be the Poisson measure for x with respect to Δ_k. Then $\varrho_x - \beta_x \in R(\bar{\Delta}_k)^\perp$.

Let F be a closed subset of $b\Delta_k$ such that $d\theta_k(F) = 0$. Then, by the F. and M. Riesz theorem, lemma 3.2 and the Rudin–Carleson theorem we get that $|\varrho_x - \beta_x|(F) = 0$. Since $\varrho_x(F) = 0$, $\beta_x(F) = 0$ and so $\beta_x / b\Delta_k \ll d\theta_k$.

Now let $\{P_n\}$ be a sequence of polynomials which is uniformly bounded on $\bar{\Delta}_k$ and converges to Ψ^* a.e. $d\theta_k$ on $b\Delta_k$ [17, lemma 5]. Then $P_n(z) \to \Psi(z)$ for all $z \in \Delta_k$ and we have

$$\begin{aligned} \int \Psi_0(\Phi_Y^*(t)) \, d\lambda_z(t) &= \int \Psi_0(t) \, d\beta_x(t) \\ &= \lim_n \int P_n(t) \, d\beta_x(t) \\ &= \lim_n P_n(x) = \Psi(x) = z = \int t \, d\lambda_z(t) . \end{aligned}$$

Since this is valid for all $z \in Y^\circ$, lemma 2.3 gives that

$$\Psi_0(\Phi_Y^*(t)) = t \text{ a.e. } \lambda .$$

Using the fact that $\Psi_0(K^\circ) = X^\circ$, it follows that

$$\Psi_0(\Phi_Y^*(t)) = \Psi^*(\Phi_Y^*(t)) = t \quad \text{a.e. } \lambda \text{ on } \mathrm{b}X_i \cap \mathrm{b}Y = R_i \cap \mathrm{b}X ,$$

and since

$$\mu_i / \mathrm{b}X_i \cap \mathrm{b}Y \ll \lambda / \mathrm{b}X_i \cap \mathrm{b}Y$$

by (1), we have (I) and (III).

This completes the proof.

3.4. Corollary. *Suppose that the diameters of the components of* $\mathbf{C} \setminus X$ *are bounded away from zero. Then* $R(X)$ *is a Dirichlet algebra.*

PROOF. This is an immediate consequence of theorem 3.3 and Walsh' theorem, which states that $R(Y)$ is a Dirichlet algebra whenever $\mathbf{C} \setminus Y$ is connected [5, lemma 3].

We say that $R(X)$ is pointwise boundedly dense (p.b.d.) in $H^\infty(X^\circ)$ if every function in $H^\infty(X^\circ)$ can be approximated pointwise in X° by a bounded sequence of functions in $R(X)$. From lemma 1.2 in [7] and theorem VIII.11.1 in [9] we have the following lemma (λ is as defined before lemma 2.3).

3.5. LEMMA. *Suppose $R(\mathrm{b}X) = C(\mathrm{b}X)$ and $R(X)$ is* p.b.d. *in $H^\infty(X^\circ)$. Then the map $f \to \tilde{f}$ is an isometric isomorphism from $H^\infty(\lambda + |\mu|)$ onto $H^\infty(X^\circ)$, for every measure μ on* bX *orthogonal to $R(X)$.*

When we have this fact, an argument similar to the one in the proof of theorem 3.3 gives the next result.

3.6. THEOREM. *Suppose $R(\mathrm{b}X) = C(\mathrm{b}X)$ and $R(X)$ is* p.b.d. *in $H^\infty(X^\circ)$. Then $R(X)$ is a Dirichlet algebra.*

PROOF. We use the same notation as in theorem 3.3. Let μ be a measure on bX orthogonal to $R(X)$. Then by lemma 3.5 there exists $\Phi^* \in H^\infty(\lambda + |\mu|)$ such that $\tilde{\Phi}^* = \Phi$, and using the same technique as in the proof of theorem 3.3 we get that

$$\Phi^*(t) \in \overline{\Phi(X^\circ)} = K \quad \text{a.e. } \lambda + |\mu| .$$

Choose $z \in X^\circ$ and let $x = \Phi(z)$. Then if ϱ_x denotes the poissonmeasure for x, we have that $\Psi^*(\varrho_x)$ is a representing measure for x with respect to the functions which are continuous on X and harmonic on X°. Hence by theorem 5.3 in [12], $\Psi^*(\varrho_x) = \lambda_z$. This gives that

$$x = \Phi(z) = \int \Phi^* \, d\lambda_z = \int \Phi^* \circ \Psi^* \, d\varrho_x \quad \text{for all } x \in K^\circ ,$$

and so

$$\Phi^*(\Psi^*(t)) = t \quad \text{a.e. } d\theta .$$

Therefore

$$\varrho_x = \Phi^*(\Psi^*(\varrho_x)) = \Phi^*(\lambda_z) \quad \text{for all } x \in K^\circ . \tag{1}$$

If Ψ_0 is defined as in the proof of theorem 3.3, then $\Psi_0 \circ \Phi^* \in L^\infty(\lambda)$, and for $z \in X^\circ$ we have

$$\int \Psi_0 \circ \Phi^* \, d\lambda_z = \int \Psi_0 \, d\varrho_x = \Psi(x) = z$$

Hence $\int \Psi_0 \circ \Phi^* \circ \Psi^* \, d\varrho_x = \Psi(x)$ and therefore

$$\Psi_0 \circ \Phi^* \circ \Psi^*(t) = \Psi^*(t) \quad \text{a.e. } d\theta,$$

that is,

$$\Psi_0 \circ \Phi^*(t) = t \quad \text{a.e. } \lambda . \tag{2}$$

Choose a sequence $\{P_n\}$ of polynomials such that $\|P_n\|_K \leqq M < \infty$ and

$$P_n(z) \to \Psi_0(z) \quad \text{for all } z \in K^\circ ,$$
$$P_n(t) \to \Psi_0(t) \quad \text{a.e. } d\theta \text{ on } \mathrm{b}K .$$

Then $\{P_n \circ \Phi^*\}$ is a bounded sequence of functions in $H^\infty(\lambda + |\mu|)$. Therefore there exists a function $F \in H^\infty(\lambda + |\mu|)$ and a subnet $\{P_i \circ \Phi^*\}$ converging to F in the weak* topology in $L^\infty(\lambda + |\mu|)$. Using (1) we get for $z \in X^\circ$:

$$\tilde{F}(z) = \int F \, d\lambda_z = \lim_i \int P_i \circ \Phi^* \, d\lambda_z$$
$$= \lim_i \int P_i \, d\varrho_x = \int \Psi^* \, d\varrho_x = \Psi(x) = z .$$

Hence by lemma 3.4

$$F(t) = t \quad \text{a.e. } \lambda + |\mu| . \tag{3}$$

Let E denote the convex hull of $\{P_i \circ \Phi^*\}$. Then since

$$\int g \, P_i \circ \Phi^* \, d(\lambda + |\mu|) \to \int g \, F \, d(\lambda + |\mu|)$$

for all $g \in L^2(\lambda + |\mu|)$ and the unit ball in $L^\infty(\lambda + |\mu|)$ is metrizable, there exists a sequence $\{Q_n\}$ of polynomials such that

$$Q_n \circ \Phi^* \in E \quad \text{and} \quad Q_n \circ \Phi^* \to F$$

in $L^2(\lambda + |\mu|)$. We can assume that

$$Q_n \circ \Phi^*(t) \to F(t) \quad \text{a.e. } \lambda + |\mu| ,$$

and since $\|P_m \circ \Phi^*\| \leqq M$ for all m,

$$\|Q_n \circ \Phi^*\| \leqq M \quad \text{for all } n .$$

Since

$$F(t) = \Psi_0(\Phi^*(t)) \quad \text{a.e. } \lambda$$

by (2) and (3) this gives that, if $\varrho_k = \Phi^*(\lambda_k)$, then

$$\begin{aligned}\Sigma_k 2^{-k}\int|\Psi_0-Q_n|\,d\varrho_k &= \Sigma_k 2^{-k}\int|\Psi_0\circ\Phi^*-Q_n\circ\Phi^*|\,d\lambda_k \\ &= \Sigma_k 2^{-k}\int|F-Q_n\circ\Phi^*|\,d\lambda_k \\ &= \int|F-Q_n\circ\Phi^*|\,d\lambda \to 0\,.\end{aligned}$$

By passing to a subsequence, we may therefore assume that

$$Q_n(t)\to\Psi_0(t)\quad \text{a.e. } d\theta \text{ on } \mathrm{b}K\,.$$

Hence

$$Q_n(x)\to\Psi_0(x)\quad \text{for all } x\in K^\circ\,.$$

Define $G(t)=\lim_n Q_n(t)$ for those $t\in K$ such that $\lim_n Q_n(t)$ exists. Put

$$T = \{t\in \mathrm{b}X;\ \lim_n Q_n(\Phi^*(t))=F(t)=t\}\,.$$

Then $(\lambda+|\mu|)(\mathrm{b}X\setminus T)=0$ so that

$$G(\Phi^*(t)) = t\quad \text{a.e. } \lambda+|\mu|\,.$$

Since $G(K^\circ)\subset X^\circ$, it follows that

$$\Phi^*(t)\in \mathrm{b}K\quad \text{a.e. } \lambda+|\mu|\,.$$

Hence if we define $\nu=\Phi^*(\mu)$, then ν is a measure on bK. Since $\Phi^*\in H^\infty(\lambda+|\mu|)$,

$$\int(\Phi^*)^k\,d\mu = 0\quad \text{for } k = 0,1,2,\ldots\,,$$

and ν is orthogonal to the polynomials.

Now suppose μ is a real measure. Then ν is real, and so $\nu=0$. Hence $\mu=G(\Phi^*(\mu))=G(\nu)=0$, and the proof is complete.

If Y is a compact plane set we define $H(Y)$ to be the space of real-valued functions harmonic in a neighbourhood of Y, and $\bar{H}(Y)$ to be the uniform closure of $H(Y)$ on Y. To get the other main result of this section, we need the following two lemmas from potential theory, which we state without proof. The first is due to Davie [6, lemma 2.1] and the other is due to Carleson [5, lemma 1] and Davie [6, lemma 1.5].

3.7. Lemma. *Let Y be a compact set and suppose x_0 is a peak point for $\bar{H}(Y)$. Let f be a superharmonic function defined in a neighbourhood of x_0. Then*

$$f(x_0) = \liminf_{x\to x_0} f(x),\quad x\in \mathbf{C}\setminus Y\,.$$

3.8. Lemma. *Let Y and E be subsets of $\mathbf{C}$ such that E has zero one-dimensional Hausdorff outer measure and $Y \cup E$ is connected. Let μ be a finite real measure with compact support such that*

$$P_{|\mu|}(z) = \int \log|\zeta - z|^{-1}\, d|\mu|(\zeta) < \infty$$

for all $z \in Y$. Let f be a real continuous function on an open set U and define

$$g(z) = P_\mu(z) - f(z) ,$$

where

$$P_\mu(z) = \int \log|\zeta - z|^{-1}\, d\mu(\zeta) ,$$

for those $z \in U$ which satisfies $P_{|\mu|}(z) < \infty$. Suppose $g(z)$ is a constant α for all $z \in Y \cap U$. Let $z_0 \in \overline{Y} \cap U$ and suppose $P_{|\mu|}(z_0) < \infty$.

Then $g(z_0) = \alpha$.

We can now prove

3.9. Theorem. *Suppose $X, X_1, X_2, \ldots$ are compact sets such that*

(i) $X_{n+1} \subset X_n$ *for* $n = 1, 2, \ldots,$
(ii) $X = \bigcap_{n=1}^{\infty} X_n,$
(iii) $R(X_n)$ *is a Dirichlet algebra for* $n = 1, 2, \ldots .$

Then $R(X)$ is a Dirichlet algebra.

Proof. First we want to prove that if $x_0 \in \mathrm{b}X$ then the pointmass at x, λ_x, is the only representing measure for x on $\mathrm{b}X$ with respect to $\overline{H}(X)$. To prove this, it is useful to establish the following:

(*) If $x_0 \in X$ and μ is a (positive) representing measure for x_0 with respect to $\overline{H}(X)$, then

$$P_\mu(z) = \log|z - x_0|^{-1} \quad \text{for all } z \in \mathrm{b}X \setminus \{x_0\} .$$

Proof of (*): Since $t \to \log|t - z|^{-1} \in H(x)$ for all $z \in \mathbf{C} \setminus X$, we have that

$$P_\mu(z) = \log|z - x_0|^{-1} \quad \text{for all } z \in \mathbf{C} \setminus X ,$$

and especially for $z \in \mathbf{C} \setminus X_n$, for all n. Since $R(X_n)$ is a Dirichlet algebra, it is easy to see that each point in $\mathrm{b}X_n$ is a peak point for $\overline{H}(X_n)$. Since $P_\mu(z)$ is superharmonic in $\mathbf{C}$ [16, II.23], lemma 3.7 gives that

$$P_\mu(z) = \log|z - x_0|^{-1} \quad \text{for } z \in \mathrm{b}X_n, \quad n = 1, 2, \ldots .$$

Hence

$$P_\mu(z) = \log|z-x_0|^{-1} \quad \text{for } z \in Y$$

where $Y = \bigcup_{n=1}^{\infty}(\mathbf{C} \setminus X_n{}^\circ)$. Since $R(X_n)$ is a Dirichlet algebra, $\mathbf{C} \setminus X_n{}^\circ$ is connected for all n, and so Y is connected. Let E be the empty set, $U = \mathbf{C} \setminus \{x_0\}$ and define

$$f(z) = \log|z-x_0|^{-1} \quad \text{for } z \in U .$$

Then $P_\mu(z) < \infty$ on Y and

$$g(z) = P_\mu(z) - f(z) = 0 \quad \text{on } Y \cap U .$$

Let $w \in \mathrm{b}X \setminus \{z_0\}$. By lower semicontinuity

$$P_\mu(w) \leqq \liminf_{z \to w} P_\mu(z) \leqq \log|w-x_0|^{-1} < \infty ,$$

and since $w \in \mathrm{b}X \setminus \{x_0\} \subset Y \cap U$, we can apply lemma 3.8 and get that $g(w) = 0$, which proves (*).

Let $x_0 \in \mathrm{b}X$ and choose two representing measures μ_1 and μ_2 for x_0 on $\mathrm{b}X$ with respect to $\bar{H}(X)$. Let $\mu = \mu_1 - \mu_2$. Using (*) we get that

$$P_\mu(z) = 0 \quad \text{for all } z \in \mathrm{b}X \setminus \{x_0\} .$$

Moreover, if $a \in X^\circ$, then (*) gives that

$$P_{\lambda_a}(z) = \log|z-a|^{-1} \quad \text{for all } z \in \mathrm{b}X ,$$

where λ_a as before denotes the harmonic measure for a with respect to X°. Since $\lambda_a(\{x_0\}) = 0$ (see [12, lemma 5.2]), $P_\mu(z) = 0$ a.e. λ_a. Since

$$\int \left(\int \log|t-z|^{-1} \, d\lambda_a(z) \right) d\,|\mu|(t) = \int \log|t-a|^{-1} \, d|\mu|(t) < \infty ,$$

the Fubini theorem gives

$$\begin{aligned} 0 = \int P_\mu(z)\, d\lambda_a(z) &= \int \left(\int \log|t-z|^{-1} \, d\mu(t) \right) d\lambda_a(z) \\ &= \int \left(\int \log|t-z|^{-1} \, d\lambda_a(z) \right) d\mu(t) \\ &= \int \log|t-a|^{-1} \, d\mu(t) = P_\mu(a) . \end{aligned}$$

Hence $P_\mu(a) = 0$ for all $a \in X^\circ$. Since we also have $P_\mu(z) = 0$ for $z \in (\mathbf{C} \setminus X) \cup (\mathrm{b}X \setminus \{x_0\})$, it is well known that μ must be the zero measure [5, lemma 2]. Therefore $\mu_1 = \mu_2$, and so λ_{x_0} is the only representing measure for x_0 on $\mathrm{b}X$ with respect to $\bar{H}(X)$.

For $x \in bX$, $n=1,2,\ldots$, define λ_x^n as follows: If $x \in X_n^\circ$ we let λ_x^n denote the harmonic measure for x with respect to X_n° and if $x \in bX_n$ we let λ_x^n denote the pointmass at x. Then $\|\lambda_x^n\|=1$ for all n, and so by the theorem of Alaoglu the sequence $\{\lambda_x^n\}_{n=1}^\infty$ has at least one clusterpoint in the weak* topology on measures. Let σ be one such point. Since the unit ball of $C(X_1)^*$ is metrizable, we can find a subsequence $\{\lambda_x^{n_k}\}_{k=1}^\infty$ which converges weak* to σ. Since $\lambda_x^{n_k}$ is a measure on bX_{n_k} it is easy to see that σ must be a measure on bX. Moreover, if f is harmonic in a neighbourhood U of X then $X_n \subset U$ for n big enough, so that

$$\int f\,d\sigma = \lim_k \int f\,d\lambda_x^{n_k} = f(x)\,.$$

Taking uniform limits, we get that σ is a representing measure for x on bX with respect to $\bar{H}(X)$. Hence $\sigma=\lambda_x$, and this proves that λ_x^n converges to λ_x in the weak* topology.

Now suppose $\mu \in R(X)^\perp$ is real and carried on bX. Let μ_n be the sweep of μ to bX_n defined as follows:

$$\int f\,d\mu_n = \int \tilde{f}^{(n)}\,d\mu \quad \text{for } f \in C(bX_n)\,,$$

where

$$\tilde{f}^{(n)}(z) = \int f\,d\lambda_z^n \quad \text{for } z \in X_n\,.$$

Then it is easy to see that $\mu_n \in R(X_n)^\perp$. Since $R(X_n)$ is a Dirichlet algebra, $\mu_n=0$. But if $f \in C(X_1)$, then

$$f(x) = \int f\,d\lambda_x = \lim_n \int f\,d\lambda_x^n = \lim_n \tilde{f}^{(n)}(x)$$

for all $x \in bX$, and since

$$|\tilde{f}^{(n)}(x)| \leqq \|f\| \quad \text{for all } x \in bX\,,$$

the dominated convergence gives

$$\int f\,d\mu = \lim_n \int \tilde{f}^{(n)}\,d\mu = \lim_n \int f\,d\mu_n = 0\,,$$

and so μ must be the zero measure.

Let U_0 (unbounded), $U_1, U_2,\ldots$ be the components of $\mathbf{C} \setminus X$. In [3] Bishop proved that if $bX=bU_0$, then every measure μ on bX orthogonal to $R(X)$ is represented by its differential $(2\pi i)^{-1}\hat{\mu}(z)\,dz$. As a consequence of theorem 3.9 and corollary 3.4 we get the following, which by theorem 2.4 generalizes the following result of Bishop.

3.8. COROLLARY. *Suppose that*

$$(**) \qquad bU_j \cap \bigcup_{i=0}^{j-1} bU_i \text{ is non-empty for } j=1,2,\ldots .$$

Then $R(X)$ is a Dirichlet algebra.

PROOF. Let $X_n = X \cup \bigcup_{i=n+1}^{\infty} U_i$, $n=1,2,\ldots$. Then

$$C \setminus X_n = \bigcup_{i=0}^{n} U_i \quad \text{and} \quad C \setminus X_n^{\circ} = \bigcup_{i=0}^{n} \bar{U}_i ,$$

which is connected for all n by (**). Therefore by corollary 3.4, $R(X_n)$ is a Dirichlet algebra for all n, and since $X = \bigcap_{n=1}^{\infty} X_n$, $R(X)$ is a Dirichlet algebra by theorem 3.9.

REFERENCES

1. E. Bishop, *Subalgebras of functions on a Riemann surface*, Pacific J. Math. 8 (1958), 29–50.
2. E. Bishop, *The structure of certain measures*, Duke Math. J. 25 (1958), 283–290.
3. E. Bishop, *Boundary measures of analytic differentials*, Duke Math. J. 27 (1960), 331–340.
4. E. Bishop, *A general Rudin–Carleson theorem*, Proc. Amer. Math. Soc. 13 (1962), 140–143.
5. L. Carleson, *Mergelyans theorem on uniform approximation*, Math. Scand. 15 (1964), 167–175.
6. A. M. Davie, *Real annihilating measures for $R(K)$*, J. Functional Analysis 6 (1970), 357–386.
7. A. M. Davie, *Bounded approximation and Dirichlet sets*, J. Functional Analysis 6 (1970), 460–467.
8. T. W. Gamelin, *Restrictions of subspaces of $C(X)$*, Trans. Amer. Math. Soc. 112 (1964), 278–286.
9. T. W. Gamelin, *Uniform Algebras*, Prentice–Hall Inc., Englewood Cliffs, N.J., 1969.
10. T. W. Gamelin and J. Garnett, *Constructive techniques in rational approximation*, Trans. Amer. Math. Soc. 143 (1969), 187–200.
11. T. W. Gamelin and J. Garnett, *Pointwise bounded approximation and Dirichlet algebras*, J. Functional Analysis 8 (1971), 360–404.
12. T. W. Gamelin and H. Rossi, *Jensen measures*, in *Function algebras* (Proc. Int. Symp. Function Algebras, Tulane Univ., 1965, ed. Frank T. Birtel), Scott, Foresman and Co., Chicago, 1966, 15–35.
13. I. Glicksberg, *Measures orthogonal to algebras and sets of antisymmetry*, Trans. Amer. Math. Soc. 105 (1962), 415–435.
14. L. L. Helms, *Introduction to potential theory*, Wiley-Interscience, New York · London · Sydney · Toronto, 1969.
15. K. Hoffman, *Banach spaces of analytic functions*, Prentice–Hall International · London, 1965.
16. M. Tsuji, *Potential theory in modern function theory*, Maruzen Co., Tokyo, 1959.
17. J. Wermer, *Seminar über Funktionen-Algebren* (Lecture Notes in Mathematics 1), Springer-Verlag. Berlin · Göttingen · Heidelberg, 1964.
18. D. Wilken, *Lebesgue measure for parts of $R(X)$*, Proc. Amer. Math. Soc. 18 (1967), 508–512.

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