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ON MULTIPLICATIVE LEBESGUE INTEGRATION AND FAMILIES OF EVOLUTION OPERATORS

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Introduction.

Several authors working in various contexts have developed theories of multiplicative integration, involving limits of products rather than limits of sums. References to this work are given at the end of the paper. However we take an independent approach exploiting the connection between multiplicative integrals and evolution operators: this allows us to make full use of the theorems of ordinary (vector-valued) integration. We develop a theory of multiplicative Lebesgue integration in which there is a dominated convergence theorem and for which the notions of absolute continuity and bounded variation play their appropriate role. As a natural by-product we obtain a structure theorem for two parameter families of evolution operators analogous to the most elementary of the theorems which describe the structure of one-parameter semi-groups.

We begin with some notation, a definition and motivation.

Let X be a Banach space and $B(X)$ be the space of continuous linear transformations of X into itself. Let $\mathsf{R}_+^2 = \{(s,t) \in \mathsf{R}^2;\ s \leqq t\}$. A family of evolution operators is then defined to be a map $T(\cdot,\cdot)\colon \mathsf{R}_+^2 \to B(X)$ satisfying the requirements

$$(1) \qquad T(s,s) = I, \qquad T(r,t)\,T(s,r) = T(s,t) \quad \text{for } s \leqq r \leqq t\,.$$

When $T(s,t)$ depends only on $t-s$ we obtain a 1-parameter semi-group $S(t)=T(0,t)$ of operators in $B(X)$. The structure theorems for 1-parameter semi-groups all generalize the following (see [2, page 614]): $S(t)=\exp[At]$ for some A in $B(X)$ if and only if $S(t)$ is continuous in the norm topology of $B(X)$. The appropriate generalization of this result to 2-parameter families of evolution operators originally motivated this work.

In the case that X and $B(X)$ are simply the complex numbers it is rather easy to see:

(i) $T(s,t)=\exp[\int_s^t a(r)\,dr]$, where $a(r)$ is integrable over any finite interval, if and only if $T(\cdot,\cdot)$ is absolutely continuous in each of its variables.

(ii) $T(s,t)=\exp[\int_{[s,t)} a(\cdot)\,d\lambda]$, where λ is a positive Borel measure and

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$a(\cdot)$ is integrable with respect to λ over any finite interval, if and only if $T(\cdot,\cdot)$ is left continuous and of bounded variation in each of its variables.

In case (ii), $T(s,t)$ satisfies the following integral equation:

$$T(s,t) = I + \int_{[s,t)} b(\cdot)T(s,\cdot)\, d\lambda\,, \tag{2}$$

where

$$b(t) = \begin{cases} a(t), & \text{when } \lambda(\{t\})=0 \\ [\exp[a(t)\lambda(\{t\})]-1]\lambda(\{t\})^{-1}, & \text{otherwise}\,. \end{cases}$$

When $T(s,t)$ is absolutely continuous this equation is satisfied with $b \equiv a$ and λ the Lebesgue measure.

We wish to generalize these results. However, we should at once note the following difficulties:

(i) In general $\exp[A]\exp[B] \neq \exp[A+B]$ for A and B in $B(X)$. Thus given an integrable function $A(\cdot)\colon \mathsf{R}^1 \to B(X)$, the family of operators defined by $T(s,t)=\exp[\int_{[s,t)} A(\cdot)\,d\lambda]$ does not generally satisfy (1).

(ii) If A is an operator which cannot be expressed as an exponential, the family of evolution operators defined by

$$T(s,t) = \begin{cases} A, & \text{if } s \leqq 0 < t\,, \\ I, & \text{otherwise}\,, \end{cases}$$

does not have the structure suggested by the scalar case. (Even if A is invertible it may not be expressible as an exponential, see [4].)

These difficulties can be resolved by expressing the evolution operators as limits of ordered products rather than as the exponentials of integrals. Intuitively it seems appropriate to consider the evolution of a physical system as taking place through a time-ordered process of "bumps". Moreover the procedure to be used below is also suggested by the following identities valid in the scalar case (when $a(\cdot)$ is Riemann integrable), but not in general:

$$\exp\left[\int_s^t a(r)\,dr\right] = \exp \lim_{\max|t_n-t_{n-1}|\to 0} \textstyle\sum_n a(t_n)(t_n - t_{n-1})]$$

$$= \lim_{\max|t_n-t_{n-1}|\to 0} \textstyle\prod_n \exp[a(t_n)(t_n - t_{n-1})]\,.$$

Multiplicative Lebesgue integration.

We begin by introducing a definition.

DEFINITION. $A(\cdot)\colon \mathsf{R}^n \to B(X)$ is said to be *strongly measurable* if for each x in X $A(\cdot)x\colon \mathsf{R}^n \to X$ is approximable a.e. by a sequence of measurable simple functions. $A(\cdot)$ is said to be *strongly integrable* with respect to a measure λ if it is strongly measurable and $\|A(\cdot)\|$: $\mathsf{R}^n \to \mathsf{R}^1$ is both measurable and integrable with respect to λ. (When X is separable the measurability of $\|A(\cdot)\|$ follows from the strong measurability of $A(\cdot)$.)

Let λ be Borel measure on R^1 and suppose that $A(\cdot)$: $\mathsf{R}^1 \to B(X)$ is strongly integrable with respect to λ. We wish to define the multiplicative Lebesgue integral of A over a Borel set E to be denoted by $\bigcap_E \exp[A(\cdot)d\lambda]$. We define this notion first for a particular class of simple functions and then extend the definition.

DEFINITIONS. (i) A *time-ordered simple function* S: $\mathsf{R}^1 \to B(X)$ is a function which can be written in the form

$$S(t) = \sum_{i=1}^{N} \chi_{Ei}(t) A_i ,$$

where $\{A_i\} \subset B(X)$, E_i is Borel measurable, $E_i \cap E_j = \emptyset$ for $i \neq j$, and $\sup E_i \leqq \inf E_{i+1}$.

(ii) Let S be a time-ordered simple function, let λ be a Borel measure and E a Borel measurable set with $\lambda(E) < \infty$. Then we define

$$\bigcap_E \exp[S(\cdot)d\lambda] = \prod_i \exp[A_i \lambda(E \cap E_i)] ,$$

where the product is ordered by the requirement that $\exp[A_{i+1}\lambda(E \cap E_{i+1})]$ should lie to the left of $\exp[A_i \lambda(E \cap E_i)]$.

In order to extend this definition for an arbitrary strongly integrable function $A(\cdot)$: $\mathsf{R}^1 \to B(X)$, we have to prove:

(i) There exists a sequence $\{S_n(\cdot)\}$ of time ordered simple functions converging suitably to A.

(ii) The sequence of multiplicative integrals $\{\bigcap_E \exp[S_n(\cdot)d\lambda]\}$ converges to a limit, and that limit is independent of the approximating sequence.

It is in fact enough to prove these facts for sets E of the form $[s,t)$.

PROPOSITION 1. *Let* $A(\cdot)$: $\mathsf{R}^1 \to B(X)$ *be strongly integrable over any finite interval with respect to a Borel measure* λ. *Then there exists a sequence* $\{S_n(\cdot)\}$ *of time ordered simple functions such that for any finite interval* J

$$\sup_n \int_J \|S_n(\cdot)\| \, d\lambda < \infty$$

and for any x *in* X

$$\int_J \|[S_n(\cdot) - A(\cdot)]x\| \, d\lambda \to 0 .$$

PROOF. We denote the continuous and discrete parts of λ by λ_c and λ_d respectively. Let $\{t_j\}$ be the points at which $A(t_j)\lambda(\{t_j\}) \neq 0$. Let $R_n(\cdot)$ be simple functions defined by

$$R_n(t) = \begin{cases} A(t_i) & \text{when } t = t_i,\ i \leqq n\ , \\ 0 & \text{otherwise}\ . \end{cases}$$

Since for any finite interval J,

$$\Sigma\{\|A(t_i)\|\lambda(\{t_i\})\ ;\ t_i \in J\} < \infty\ ,$$

we have

$$\int_J \|R_n(\cdot) - A(\cdot)\|\ d\lambda_d \to 0 \quad \text{as } n \to \infty\ .$$

This proves both the assertions of the proposition for λ_d.

The approximation of the continuous part of A is a bit trickier. For each integer, $n > 0$, define a countable partition of the real line into intervals of λ_c-measure 2^{-n} (if $\lambda_c(\mathrm{R}^1)$ is finite we suppose it to be equal to 1). It is possible to obtain such a partition, since λ_c is continuous, positive and finite on finite intervals. We construct these partitions in such a way that as n increases we get refinements of the previous partitions. We then define

$$T_n(t) = \Sigma_i\, \chi_{n,i}(t)\, 2^n \int \chi_{n,i}(\cdot) A(\cdot)\ d\lambda_c\ ,$$

where $\chi_{n,i}$ is the characteristic function of the ith interval of the nth partition. Now clearly

$$\|[T_n(t) - A(t)]x\| \leqq \Sigma_i \chi_{n,i}(t)\, 2^n \int \chi_{n,i}(\cdot)\, \|[T_n(\cdot) - A(t)]x\|\ d\lambda_c\ .$$

Strong convergence to zero now follows from a differentiation theorem which is well-known in the case that λ is Lebesgue measure (see e.g. [2, page 217]), but which can also be readily proved for continuous Borel measures, as is done in Appendix 1 to this paper. When $A(\cdot)$ is norm bounded on finite intervals, $T_n(\cdot)$ clearly is likewise bounded and as a consequence of the bounded convergence theorem we get

$$(3) \qquad \int_J \|[T_n(\cdot) - A(\cdot)x\|\ d\lambda_c \ \to\ 0 \quad \text{as } n \to \infty\ .$$

When $A(\cdot)$ is not locally bounded we can approximate it arbitrarily closely by bounded functions $A'(\cdot)$ obtained by truncating $A(\cdot)$, in the sense that we can make $\int_J \|A(\cdot) - A'(\cdot)\| d\lambda_c$ arbitrarily small. We note that for intervals J having partition points as end points we have from the definition of $T_n(\cdot)$ and $T_n'(\cdot)$ that

$$\int_J \|T_n(\cdot)-T_n'(\cdot)\|\, d\lambda_c \leqq \int_J \|A(\cdot)-A'(\cdot)\|\, d\lambda_c .$$

Consequently,

$$\int_J \|[T_n(\cdot)-A(\cdot)]x\|\, d\lambda_c \leqq 2\|x\|\int_J \|A(\cdot)-A'(\cdot)\|\, d\lambda_c + \int_J \|[T_n'(\cdot)-A'(\cdot)]x\|\, d\lambda_c$$

from which (3) follows in general.

The proof is completed by putting $S_n(\cdot)=R_n(\cdot)+T_n(\cdot)$, and by verifying with the aid of easy estimates that

$$\int_J \|S_n(\cdot)\|\, d\lambda \leqq \int_J \|A(\cdot)\|\, d\lambda$$

for any interval J whose endpoints are partition points of S_n.

The question of the convergence of $\bigcap_{[s,t)}\exp[S_n(\cdot)d\lambda]$ is much more complicated. The connection between multiplicative integrals and evolution operators is established in the next proposition.

Proposition 2. *Let $S(\cdot)$ be a time-ordered simple function, and λ a Borel measure finite on finite intervals. Then*

$$T(s,t) = \bigcap_{[s,t)}\exp[S(\cdot)d\lambda]$$

is a family of evolution operators, and furthermore satisfies the integral equation

$$T(s,t)x = x + \int_{[s,t)} B(\cdot)T(s,\cdot)x\, d\lambda , \tag{4}$$

where

$$B(\tau) = \begin{cases} S(\tau), & \text{when } S(\tau)\lambda(\{\tau\})=0 , \\ [\exp[S(\tau)\lambda(\{\tau\})]-I]\lambda(\{\tau\})^{-1}, & \text{when } S(\tau)\lambda(\{\tau\})\neq 0 . \end{cases} \tag{5}$$

Proof. That $T(\cdot,\cdot)$ is a family of evolution operators is obvious from its definition. A little more care needs to be taken in verifying that the integral equation is satisfied.

It is enough to establish the integral equation for a simple function of the form $S(t)=A\chi_E(t)$; the generalization to an arbitrary simple function is easy (in fact, it is closely analogous to the argument at the end of this proof.) We can also suppose that E contains only a finite number of atoms of λ. For an arbitrary E can be approximated by sets E_n containing only a finite number of atoms in such a way that $\lambda(E-E_n)$ tends to 0 as n becomes infinite; one easily checks that the integral equation for $A\chi_{E_n}(\cdot)$ converges appropriately.

We can now follow two different procedures. For the first of these we notice that $\exp[A\lambda(E\cap[s,t))]$ is of bounded variation in t (in the classical sense with absolute values replaced by norms) and that the variation measure is dominated by λ. According to Appendix 2 we may, when X is a dual space of another Banach space, conclude that

$$\exp[A\lambda(E\cap[s,t))]x - x = \int_{[s,t)} B_s(\cdot)x\,d\lambda\,.$$

Letting t_i, $i=1,\ldots,n$, be the atoms in E, and $\lambda_i=\lambda(\{t_i\})$ we have $B_s(t_i)=[\exp[A\lambda_i]-I]\lambda_i^{-1}$. Otherwise we may (since E contains only a finite number of atoms) use the differentiation theorem of Appendix 1 to conclude that

$$B_s(\cdot) = A\chi_E(\cdot)\exp[A\lambda(E\cap[s,\cdot))] \quad \text{a.e.}$$

In order to avoid the extra assumption on X we can argue as follows. For the atoms we have

$$\exp[A\lambda(E\cap\{t_i\})]x - x = \int_{\{t_i\}} [\exp[A\lambda_i]-I]\lambda_i^{-1}x\,d\lambda\,. \tag{6}$$

If (s,t) is an interval such that $E\cap(s,t)$ contains no atoms of λ we define μ to be the continuous measure obtained by restricting λ to $E\cap(s,t)$. Then we compute:

$$\begin{aligned}\int_{(s,t)} A(\cdot)\chi_E(\cdot)\exp[A\lambda(E\cap(s,\cdot))]\,d\lambda &= \int_{(s,t)} A\exp[A\mu(s,\cdot)]\,d\mu\\ &= \sum_{n=0}^{\infty}\int_{(s,t)} A^{n+1}[\mu(s,\cdot)]^n n!^{-1}\,d\mu\\ &= \sum_{n=0}^{\infty}[A\mu(s,t)]^{n+1}(n+1)!^{-1}\,,\end{aligned}$$

where we have used the change of variables described in Appendix 1. Thus

$$\exp\big[A\lambda(E\cap(s,t))\big] - I = \int_{(s,t)} A(\cdot)\chi_E(\cdot)\exp\big[A\lambda(E\cap(s,\cdot))\big]\,d\lambda\,. \tag{7}$$

To obtain the integral equation for arbitrary s and t we write

$$[s,t) = [s,t_j)\cup\bigcup_{i=j}^{k-1}[\{t_i\}\cup(t_i,t_{i+1})]\cup\{t_k\}\cup(t_k,t)$$

and then use equations (6) and (7) for the atoms and the atom free intervals respectively. Explicitly, noting that $T(s,t)$ is left continuous in each variable, we get

$$[T(s,t)-I]x = [T(s,t)-T(s,t_k+)]x + \sum_{i=j}^{k-1} \{[T(s,t_{i+1}+)-T(s,t_{i+1})]x + \\ + [T(s,t_{i+1})-T(s,t_i+)]x + [T(s,t_i+)-T(s,t_i)]x\} + \\ + [T(s,t_j)-I]x .$$

We now apply (7) and (6) to the terms of the form

$$[T(s,t_{i+1})-T(s,t_i+)]x = [T(t_i+,t_{i+1})-I]T(s,t_i+)x$$

and

$$[T(s,t_i+)-T(s,t_i)]x = [T(t_i,t_i+)-I]T(s,t_i)x ,$$

respectively. This leads to the integral equation, and completes the proof of the proposition.

Our next two propositions will deal with evolution operators and the integral equations they satisfy; they will allow us to define the multiplicative integral in general. We now work somewhat within the framework of the paper by Hackman [3], who proved Proposition 3 below in the case when Lebesgue measure was involved. (We precede the propositions by some notational comments.)

Let $t^{(n)}$ denote points $(t_1,\ldots,t_n)$ of $R^{(n)}$ and $\lambda^{(n)}$ the n-fold product measure of λ. Given $B(\cdot) : \mathsf{R}^1 \to B(X)$, we define

$$B^{(n)}(t^{(n)}) = B(t_1)\, B(t_2) \ldots B(t_n) ,$$

and note that if B is strongly integrable with respect to λ, $B^{(n)}$ is strongly integrable with respect to $\lambda^{(n)}$ (which is proved in Hackman's paper).

THEOREM 3. *Suppose $B(\cdot) : \mathsf{R}^1 \to B(X)$ is strongly integrable over every finite interval with respect to a Borel measure λ. Then there exists uniquely a family $T(\cdot,\cdot)$: $\mathsf{R}_+{}^2 \to B(X)$ of evolution operators satisfying the integral equation*

$$T(s,t)x = x + \int_{[s,t)} B(\cdot)T(s,\cdot)x \, d\lambda$$

and locally bounded in norm in $\mathsf{R}_+{}^2$. Explicitly

$$T(s,t) = \begin{cases} \sum_{n=0}^{\infty} T^{(n)}(s,t)\left(1+B(s)\lambda(\{s\})\right), & \text{if } t>s , \\ I, & \text{if } t=s , \end{cases}$$

where

$$T^{(0)}(s,t) = I, \quad T^{(n)}(s,t) = \int\limits_{s<\tau_n<\tau_{n-1}<\dots<\tau_1<t} B^{(n)}(\tau^{(n)})\, d\lambda^{(n)} .$$

The series converges uniformly since

$$\|T^{(n)}(s,t)\| \leqq n!^{-1}\left[\int_s^t \|B(\cdot)\|\, d\lambda\right]^n .$$

Proof. The last estimate is obvious since, because of symmetry considerations

$$n! \int\limits_{s<\tau_n<\dots<\tau_1<t} \|B^{(n)}(\tau^{(n)})\|\, d\lambda^{(n)} \leqq \int\limits_{s<\tau_1,\tau_2,\dots,\tau_n<t} \|B(\tau_1)\| \dots \|B(\tau_n)\|\, d\lambda^{(n)} .$$

It is also easy to verify directly that the integral equation is satisfied by $T(s,t)$ as defined above. The uniqueness of a locally bounded solution to the equation is proved by means of the usual argument for linear ordinary differential equations: Assume that T and T' are two solutions, and put

$$M = \sup\{\|T'(s,r)-T(s,r)\|;\ s\leqq r\leqq t\},$$

where $[s,t)$ is some fixed interval. Using the equation, it is proved by induction that

$$\|T'(s,r)-T(s,r)\| \leqq Mn!^{-1}\left(\int\limits_{(s,r)} \|B(\cdot)\|\, d\lambda\right)^n$$

for all n, whence $M=0$.

The uniqueness being established, the relation

$$T(s,t) = T(r,t)\, T(s,r) \quad \text{for } s\leqq r\leqq t$$

follows from the observation that if we define (for fixed r)

$$T'(s,t) = \begin{cases} T(s,t) & \text{if } r\notin[s,t], \\ T(r,t)\, T(s,r) & \text{if } s\leqq r\leqq t, \end{cases}$$

then $T'(s,t)$ satisfies the integral equation if $T(s,t)$ does.

Now we prove the continuous dependence of the evolution operators on their generating function.

Proposition 4. *Suppose $B_n(\cdot)$ and $B(\cdot)$ are $B(X)$ valued functions as in Proposition 3, and let $T_n(\cdot,\cdot)$ and $T(\cdot,\cdot)$ be the corresponding evolution operators. Suppose that for any finite interval J*

$$C_J = \sup_n \int_J \|B_n(\cdot)\|\, d\lambda \leqq \infty$$

and for any x in X

$$\int_J \|[B_n - B]x\| \, d\lambda \to 0 \quad \text{as } n \to \infty .$$

Then for each x, $T_n(\cdot,\cdot)x$ converges to $T(\cdot,\cdot)x$ uniformly on compact subsets of $R_+{}^2$.

PROOF. From Proposition 3 we have

$$T_n(s,t)x = \sum_{m=0}^{\infty} T_n{}^{(m)}(s,t)\bigl(I + B_n(s)\lambda(\{s\})\bigr)x$$

and

$$T(s,t)x = \sum_{m=0}^{\infty} T^{(m)}(s,t)\bigl(I + B(s)\lambda(\{s\})\bigr)x .$$

We shall prove the dominated termwise convergence of the former series to the latter.

Using a symmetry argument once more, we get for s and t in J

$$\|T_n{}^{(m)}(s,t)\| = \left\| \int_{s<\tau_m<\ldots<\tau_1<t} B_n{}^{(m)}(\tau^{(m)}) \, d\lambda^{(m)} \right\|$$

$$\leqq m!^{-1} \left[\int_{(s,t)} \|B_n(\tau)\| \, d\lambda \right]^m \leqq m!^{-1} \, C_J{}^m ,$$

where C_J is the constant appearing in the hypothesis. Since, moreover, $B_n(s)\lambda(\{s\})x$ converges to $B(s)\lambda(\{s\})x$, it is clear that the series for $T_n(s,t)$ is dominated termwise by a convergent series. Thus it is enough to prove that for each m

$$T_n{}^{(m)}(s,t)[I + B_n(s)\lambda(\{s\})]x \to T^{(m)}(s,t)[I + B(s)\lambda(\{s\})]x$$

as n tends to infinity. In order to prove this we show by an induction argument that if x_n converges to x,

$$\int_{J^m} \|B_n{}^{(m)}(\cdot)x_n - B^{(m)}(\cdot)x\| \, d\lambda^{(m)} \to 0 \quad \text{as } n \to \infty . \tag{8}$$

For $m = 1$ we have

$$\int_J \|B_n(\cdot)x_n - B(\cdot)x\| \, d\lambda \leqq \int_J \|B_n(\cdot)(x_n - x)\| \, d\lambda + \int_J \|[B_n(\cdot) - B(\cdot)]x\| \, d\lambda$$

$$\leqq C_J \|x_n - x\| + \int_J \|[B_n(\cdot) - B(\cdot)]x\| \, d\lambda ,$$

which converges to zero as n becomes infinite. Assuming (8) as induction hypothesis, we now prove the convergence for $m+1$. By Fubini

$$\int_{J^{m+1}} \|B_n^{(m+1)}(\cdot)x_n - B^{(m+1)}(\cdot)x\| \, d\lambda^{(m+1)}$$

$$= \int_{J^m}\left[\int_J \|B_n(\cdot)B_n^{(m)}(\cdot)x_n - B(\cdot)B^{(m)}(\cdot)x\| \, d\lambda\right] d\lambda^{(m)}$$

$$\leqq \int_{J^m}\left[\int_J \|B_n(\cdot)[B_n^{(m)}(\cdot)x_n - B^{(m)}(\cdot)x]\| \, d\lambda\right] d\lambda^{(m)} +$$

$$+ \int_{J^m}\left[\int_J \|[B(\cdot)-B_n(\cdot)]B^{(m)}(\cdot)x\| \, d\lambda\right] d\lambda^{(m)} .$$

The first of these integrals is dominated by

$$C_J \int_{J^m} \|B_n^{(m)}(\cdot)x_n - B^{(m)}(\cdot)x\| \, d\lambda^{(m)} ,$$

which converges to zero by hypothesis. The dominated convergence theorem tells us that the second integral also converges to zero: for

$$\int_J \|[B(\cdot)-B_n(\cdot)]B^{(m)}(\cdot)x\| \, d\lambda \leqq 2C_J\|B^{(m.}(\cdot)x\|$$

and

$$\int_J \|[B(\cdot)-B_n(\cdot)]B^{(m)}(\cdot)x\| \, d\lambda \to 0 \quad \text{as } n \to \infty .$$

This completes the proof.

The culmination of all this lies in the following theorem.

THEOREM 5. *Let* $A(\cdot) : \mathsf{R}^1 \to B(X)$ *be strongly integrable over every finite interval with respect to a Borel measure* λ. *Let* $\{S_n(\cdot)\}$ *be a sequence of time-ordered simple functions such that for any finite interval* J

$$\sup_n \int_J \|S_n(\cdot)\| \, d\lambda < \infty$$

and for each x *in* X

$$\int_J \|[S_n(\cdot)-A(\cdot)]x\| \, d\lambda \to 0 \quad \textit{as } n \to \infty .$$

Let $B(\cdot) : \mathsf{R}^1 \to B(X)$ *be defined by*

$$B(t) = \begin{cases} A(t) & \text{if } A(t)\lambda(\{t\})=0\,, \\ \{\exp[A(t)\lambda(\{t\})]-I\}\lambda(\{t\})^{-1} & \text{if } A(t)\lambda(\{t\})\neq 0\,, \end{cases}$$

and let $B_n(\cdot)$ be defined similarly with reference to $S_n(\cdot)$. Let $T_n(\cdot,\cdot)$ and $T(\cdot,\cdot)$ denote the families of evolution operators corresponding to $B_n(\cdot)$ and $B(\cdot)$ respectively. Then

$$T_n(s,t) = \bigcap\nolimits_{[s,t)} \exp[S_n(\cdot)d\lambda]$$

and $T_n(\cdot,\cdot)x$ converges to $T(\cdot,\cdot)x$ uniformly on compact subsets of R_+^2.

We precede the proof of Theorem 5 by the definition which is now justified.

Definition. With $A(\cdot)$ and $T(\cdot,\cdot)$ as in Theorem 5 we define

$$\bigcap\nolimits_{[s,t)}\exp[A(\cdot)d\lambda] = T(s,t);$$

for any bounded Borel set E (contained in $[s,t)$ say) we define

$$\bigcap\nolimits_E\exp[A(\cdot)d\lambda] = \bigcap\nolimits_{[s,t)}\exp[\chi_E(\cdot)A(\cdot)d\lambda]$$

(When $A(\cdot)$ is integrable over all of R^1 the multiplicative integral can also be defined over unbounded Borel sets E.)

Proof of Theorem 5. The theorem is an immediate consequence of Propositions 2, 3 and 4 once we prove that $B_n(\cdot)$ and $B(\cdot)$ are integrable over finite intervals, and that for any finite interval J and each x in X

$$\sup_n \int_J \|B_n(\cdot)\|\, d\lambda < \infty$$

and

$$\int_J \|[B(\cdot)-B_n(\cdot)]x\|\, d\lambda \to 0 \quad \text{as } n\to\infty\,.$$

The integrability of $B(\cdot)$ (and $B_n(\cdot)$) is easy: measurability is immediate, and that $\int_J \|B(\cdot)\|d\lambda<\infty$ follows from

$$\int_J \|B(\cdot)\|\, d\lambda_c = \int_J \|A(\cdot)\|\, d\lambda_c < \infty$$

and

$$\int_J \|B(\cdot)\|\, d\lambda_d = \Sigma_j \|\exp[A_j\lambda_j]-I\| \leqq \Sigma_j\|A_j\|\lambda_j < \infty\,,$$

where $A_j=A(t_j)$ and $\lambda_j=\lambda(\{t_j\})$, $\{t_j\}_{j=1}^{\infty}$ being the atoms of λ in J.

The domination property is easily verified, but the convergence needs more care. Clearly

$$\int_J \|[B(\cdot)-B_n(\cdot)]x\|\,d\lambda_c = \int_J \|[A(\cdot)-S_n(\cdot)]x\|\,d\lambda_c\,,$$

which converges by hypothesis. With respect to λ_d we have

$$\int_J \|[B(\cdot)-B_n(\cdot)]x\|\,d\lambda_d = \Sigma_j \|[\exp[A_j\lambda_j]-\exp[S_{nj}\lambda_j]]x\|\,. \tag{9}$$

We now have to exploit fully the convergence of

$$\int_J \|[A(\cdot)-S_n(\cdot)]x\|\,d\lambda_d = \Sigma_j \|[A_j-S_{nj}]x\|\,\lambda_j$$

as n tends to infinity. From the domination property we get

$$\Sigma_j \|S_{nj}x\|\lambda_j \leqq K\|x\|, \quad \text{and} \quad \Sigma_j \|A_jx\|\lambda_j \leqq K\|x\|\,, \tag{10}$$

and hence also

$$\|S_{nj}\|\lambda_j \leqq K, \quad \text{and} \quad \|A_j\|\lambda_j \leqq K\,. \tag{11}$$

Now, using (11), we can estimate (9) as follows:

$$\begin{aligned}
&\Sigma_j \|[\exp[A_j\lambda_j]-\exp[S_{nj}\lambda_j]]x\| \\
&\quad = \sum_j \left\| \int_0^1 \exp[S_{nj}\lambda_j\tau][S_{nj}-A_j]\lambda_j \exp[A_j\lambda_j(1-\tau)]x\,d\tau \right\| \\
&\quad \leqq e^K \int_0^1 \sum_j \|[S_{nj}-A_j]\lambda_j \exp[A_j\lambda_j(1-\tau)]x\|\,d\tau \\
&\quad \leqq e^K \sum_j \|[S_{nj}-A_j]x\|\lambda_j + e^K \int_0^1 \sum \|S_{nj}-A_j]\lambda_j\{\exp[A_j\lambda_j(1-\tau)]-I\}x\|\,d\tau\,.
\end{aligned}$$

The first term converges to zero by hypothesis. So does the second one. For we have, using (10), that it is less than

$$e^K \int_0^1 \sum_{j=1}^N \|[S_{nj}-A_j]\lambda_j\,\{\exp[A_j\lambda_j(1-\tau)]-I\}x\|\,d\tau + 2Ke^K\|x\| \sum_{j=N+1}^{\infty} \|A_j\|\lambda_j\,.$$

The latter sum can be made arbitrarily small for N large. For N fixed we then apply the dominated convergence theorem to the integral term to see that it too converges. This completes the proof.

As a corollary of the proof we obtain the following generalization of the dominated convergence theorem for multiplicative integrals.

Theorem 6. *Let $A_n(\cdot)$ and $A(\cdot)$ be $B(X)$ valued functions strongly integrable over an interval J, with $\sup_n \int_J \|A_n(\cdot)\| d\lambda < \infty$, and such that*

$$\int_J \|[A_n(\cdot)-A(\cdot)]x\| \, d\lambda \to 0 \quad \text{as } n \to \infty ,$$

for each x in X. Then

$$\bigcap_J \exp[A_n(\cdot)d\lambda]x \to \bigcap_J \exp[A(\cdot)d\lambda]x .$$

The process of multiplicative integration which we have outlined allows us to generate a large class of families of evolution operators. We must however still take into account the existence of operators which cannot be expressed as an exponential. We first note the following easy proposition.

Lemma 7. *Given $A(\cdot)$ locally strongly integrable with respect to λ, and a sequence of operators $\{T_n\}$ associated with points $\{t_n\}$ such that*

$$A(t_n)\lambda(\{t_n\}) = 0 \quad \text{and} \quad \Sigma\{\|T_n-I\|;\ t_n \in J\} < \infty .$$

Redefining λ to have mass one at each t_n we let

$$B(t) = \begin{cases} A(t) & \text{when } t \notin \{t_n\} \text{ and } \lambda(\{t\})=0 \\ [\exp[A(t)\lambda(\{t\})]-I]\lambda(\{t\})^{-1} & \text{when } A(t)\lambda(\{t\})>0 \\ T_n-I & \text{when } t=t_n \end{cases}$$

Then $B(\cdot) : \mathsf{R}^1 \to B(X)$ is locally strongly integrable with respect to λ.

This justifies the following:

Definition. Let $A(\cdot)$, T_n and λ be as above. Let $T(\cdot,\cdot)$ be the family of evolution operators corresponding to B. Then

$$\bigcap_{[s,t)}\{\exp[A(\cdot)d\lambda], T_n\} = T(s,t)$$

If $E \subset [s,t)$ we define

$$\bigcap_E\{\exp[A(\cdot)d\lambda], T_n\} = \bigcap_{[s,t)}\{\exp[\chi_E(\cdot)A(\cdot)d\lambda], T_n'\} ,$$

where $\{T_n'\}$ is the sequence of T_n's lying in E.

It is of course possible to go through an approximation procedure analogous to that carried out previously. We do not dwell on the details.

We now consider the role of bounded variation and absolute continuity for multiplicative integrals.

Definitions. A function $F : J \to B(X)$ is said to be *of bounded variation* (*absolutely continuous*) if and only if it satisfies the classical conditions for bounded variation (absolute continuity) with absolute values replaced by norms.

A family of evolution operators $T : \mathsf{R}_+{}^2 \to B(X)$ is said to be of *bounded variation* on an interval J if

$$\sup\{\sum\|T(t_i,t_{i+1})-I\|;\ t_1<t_2<\ldots<t_n\in J\} < \infty .$$

It is said to be *absolutely continuous* if for any $\varepsilon>0$ there exists $\delta>0$ such that when $(a_i,b_i,\ i=1,\ldots,n$, are disjoint intervals with $\sum_{i=1}^n|b_i-a_i|<\delta$ we have $\sum_{i=1}^n\|T(a_i,b_i)-I\|<\varepsilon$.

We now elucidate the relationship between these notions.

PROPOSITION 8. *Let $T(\cdot,\cdot) : \mathsf{R}_+{}^2 \to B(X)$ be a family of evolution operators.*

a) *$T(\cdot,\cdot)$ is of bounded variation over an interval J if and only if the following conditions are satisfied:*

(i) *$T(\cdot,\cdot)$ is of bounded variation separately in each variable with the other fixed.*

(ii) *There exist points $t_0,\ldots,t_n$ (in increasing order with t_0 and t_n the end points of J) and a constant M such that when s and t lie in any (t_i,t_{i+1}), then $T(s,t)^{-1}$ exists and $\|T(s,t)^{-1}\|\leqq M$.*

b) *$T(\cdot,\cdot)$ is absolutely continuous over J if and only if*

(i′) *$T(\cdot,\cdot)$ is absolutely continuous in each variable.*

(ii′) *$T(s,t)^{-1}$ exists for s and t in J and $\|T(s,t)^{-1}\|\leqq M$.*

(This is satisfied, for example, when $T(s,t)$ is jointly continuous.)

PROOF. We need consider only the case of bounded variation (absolute continuity is analogous but easier).

We suppose first that the evolution operators $T(\cdot,\cdot)$ have bounded variation over J. Then $\|T(s,t)\|$ is uniformly bounded for s and t in J, since

$$\|T(s,t)\| \leqq \|T(s,t)-I\|+1 .$$

Consequently the bounded variation of $T(s,t)$ in t follows from

$$\|T(s,t_{i+1})-T(s,t_i)\| \leqq \|T(s,t)\|\ \|T(t_i,t_{i+1})-I\| .$$

The bounded variation in s is similar. We verify next that (ii) is satisfied. Since $T(\cdot,\cdot)$ is of bounded variation in each interval we know that left and right hand limits exist. For definiteness we can assume that $T(\cdot,\cdot)$ is left continuous in each variable. Now let t_j be the points at which $\|T(t_j,t_j+)-I\|>0$. Since

$$\sum_{j=1}^{\infty}\|T(t_j,t_j+)-I\| < \infty$$

we can find n such that

$$\sum_{j=n}^{\infty} \|T(t_j,t_j+)-I\| < \tfrac{1}{4}.$$

To the points $t_1,\ldots,t_{n-1}$ we then add the end points of J and denote the points in this set by $t_0',\ldots,t_n'$ where the numbering is such that $t_0' < t_1' < \ldots < t_n'$. We now restrict our attention to a particular interval (t_i',t_{i+1}') from which big jumps are excluded. We can cover that interval by finitely many open intervals $I_k (k=1,\ldots,m)$ such that on each interval $\|T(p,q)-I\| < \frac{1}{2}$ (first choose such intervals (t_i',a) and (b,t_{i+1}'), and then use a compactness argument on $[a,b]$). Thus for p and q in I_k, $T(p,q)^{-1}$ exists and $\|T(p,q)^{-1}\| < 2$. For arbitrary s and t in (t_i',t_{i+1}') we then find points p_k, $k=1,\ldots,l$ say, between s and t such that points following each other lie in the same interval — one needs at most $m-1$ such points. Then

$$T(s,t)^{-1} = T(s,p_1)^{-1}T(p_1,p_2)^{-1}\ldots T(p_l,t)^{-1}$$

and

$$\|T(s,t)^{-1}\| \leqq 2^l \leqq 2^{m-1}.$$

This concludes one half of the proof.

The converse is easy. It is clearly enough to show that $T(\cdot,\cdot)$ is of bounded variation over each (t_i',t_{i+1}'). If $t_i' < \tilde{t}_1 < \ldots < \tilde{t}_n < t_{i+1}'$ and we choose any s with $t_i' < s < \tilde{t}_1$ we have

$$\begin{aligned}\sum\|T(\tilde{t}_k,\tilde{t}_{k+1})-I\| &\leqq \|T(s,\tilde{t}_k)^{-1}\| \sum\|T(s,\tilde{t}_{k+1})-T(s,\tilde{t}_k)\| \\ &\leqq M \sum\|T(s,\tilde{t}_{k+1})-T(s,\tilde{t}_k)\|.\end{aligned}$$

It is interesting to note that a family of evolution operators can indeed be of bounded variation in each variable without being of bounded variation in the stronger sense. The example which follows demonstrates this. Let $X=\oplus X_j$ be the countable direct sum of Banach spaces and let P_k denote the projection onto $\oplus_{j\leqq k} X_j$. Let $\{\alpha_k\}$ be an increasing sequence of real numbers converging to a finite limit α. We then define

$$T(s,t) = \begin{cases} I & \text{if } [s,t)\cap\{\alpha_k\}=\emptyset \\ 2^{k-l}P_k & \text{if } \alpha_{k-1}<s\leqq\alpha_k<\ldots<\alpha_l<t\leqq\alpha_{l+1} \\ 0 & \text{if } s<\alpha_k<\alpha\leqq t. \end{cases}$$

Then

$$\sum_{i=1}^{n} \|T(\alpha_i,\alpha_{i+1})-I\| = n$$

but, for any $s \leqq t_1 < t_2 < \ldots < t_n$,

$$\sum_{i=1}^{n} \|T(s,t_{i+1})-T(s,t_i)\| \leqq 1$$

and for any $s_1 < s_2 < \ldots < s_n \leqq t$,

$$\sum_{i=1}^{n} \|T(s_{i+1},t) - T(s_i,t)\| \leqq 2 .$$

It would be interesting to develop some counterexample for the case of absolute continuity as well.

We come now to the main differentiation theorems.

THEOREM 9. *Let X be a separable dual space of a Banach space Y. Suppose that $T(\cdot,\cdot)$: $\mathsf{R}_+^2 \to B(X)$ is a family of evolution operators. The following statements are equivalent:*

(1) *There exists A: $\mathsf{R}^1 \to B(X)$ strongly integrable on each finite interval with respect to a Borel measure λ, and a sequence $\{T_n\}$ of operators associated with points $\{t_n\}$ at which $A(t_n)\lambda(\{t_n\}) = 0$ and satisfying*

$$\Sigma\{\|T_n - I\|;\, t_n \in J\} < \infty$$

for each finite interval J, such that

$$T(s,t) = \textstyle\bigcap_{[s,t)}\{\exp[A(\cdot)\,d\lambda], T_n\}$$

(2) *There exists $B(\cdot)$: $\mathsf{R}^1 \to B(X)$ strongly integrable on finite intervals with respect to a Borel measure λ and such that for all x in X*

$$T(s,t)x - x = \int_{[s,t)} B(\cdot)T(s,\cdot)x\, d\lambda;$$

(3) *$T(\cdot,\cdot)$: $\mathsf{R}_+^2 \to B(X)$ is of bounded variation, and left continuous in each variable.*

If T is continuous in the second variable, or if each $T(s,t)$ is normal and invertible, (1) can be replaced by (1′):

(1′) *There exists A : $\mathsf{R}^1 \to B(X)$ strongly integrable with respect to a Borel measure λ over each finite interval, such that*

$$T(s,t) = \textstyle\bigcap_{[s,t)}\exp[A(\cdot)d\lambda] .$$

THEOREM 10. *Let m denote Lebesgue measure and let T : $\mathsf{R}^2 \to B(X)$ be a family of evolution operators. Then the following are equivalent:*

(1) *There exist A : $\mathsf{R}^1 \to B(X)$ strongly integrable with respect to m on finite intervals such that*

$$T(s,t) = \bigcap_{s}^{t} \exp[A(\cdot)\, dm] .$$

(2) *There exists A : $\mathsf{R}^1 \to B(X)$ strongly integrable with respect to m on finite intervals such that*

$$T(s,t)x - x = \int_s^t A(\cdot)T(s,\cdot)x \, dm$$

(3) $T(\cdot,\cdot)$ *is absolutely continuous.*

Theorem 10 is an immediate consequence of Theorem 9, as will be clear from the proof of the latter.

Proof of Theorem 9. The equivalence of 1 and 2 is obvious from the previous considerations. That (2) implies (3) is also evident, if we take into account the fact that $\|T(s,t)\|$ is uniformly bounded on any finite interval.

That (3) implies (2) is a consequence of the differentiation theorem of Appendix 2. We see this as follows. Let λ_s be the measure associated with the variation function

$$V([s,t);\, T(s,\cdot)) = \sup\{\sum \|T(s,t_{i+1}) - T(s,t_i)\|;\ s \leqq t_0 < t_1 < \ldots < t_n < t\}\,.$$

Then, taking into account the left continuity,

$$T(s,t)x - x = \int_{[s,t)} G_s(\cdot)x \, d\lambda_s$$

for some suitable locally strongly integrable function G_s: $[s,\infty) \to B(X)$. From the proof of Proposition 8 we see that λ_s is absolutely continuous with respect to the measure λ associated with the variation function

$$V([s,t);\, T(\cdot,\cdot)) = \sup\{\sum \|T(t_i,t_{i+1}) - I\|;\ s \leqq t_0 < t_1 < \ldots < t_n < t\}\,.$$

Thus we also have

$$T(s,t)x - x = \int_{[s,t)} H_s(\cdot)x \, d\lambda\,,$$

which has the advantage that the integration is with respect to a measure independent of s. When $T(s,t)$ is invertible and locally norm bounded in $\mathsf{R}_+{}^2$ the proof is quickly concluded. For then $T(\cdot,\cdot)$ can be extended to all of R^2 by putting $T(s,t) = T(t,s)^{-1}$ when $s < t$ and the identity $T(s,t) = T(p,t)T(s,p)$ (which continues to hold) implies that

$$H_s(\cdot) = H_p(\cdot)T(s,p)\lambda \quad \text{a.e.}$$

In this case therefore $H_s(\cdot)T(\cdot,s) = H_p(\cdot)T(\cdot,p)$; so $H_s(\cdot)T(\cdot,s)$ is independent of s. Defining $B(\cdot) = H_s(\cdot)T(\cdot,s)$ we have $H_s(\cdot) = B(\cdot)T(s,\cdot)$ as required. The local integrability of $B(\cdot)$ is implied by the local integrability of $H_s(\cdot)$ and norm boundedness of $[T(s,\cdot)]^{-1}$.

For the general case we use the property of evolution operators of bounded variation described in Proposition 8a (ii). Recalling the notation used there, we have

$$T(t_j,t_j+)x = x + \int_{\{t_j\}} [T(t_j,t_j+)-I]\lambda(\{t_j\})^{-1}x\,d\lambda\,. \tag{12}$$

Moreover we shall prove that for $t_j < t \leqq t_1$,

$$T(t_j+,t)x = x + \int_{(t_j,t)} B(\cdot)T(t_j+,t)x\,d\lambda\,. \tag{13}$$

The integral equation is then obtained for arbitrary s and t by using the same argument which concluded the proof of Proposition 2. To prove (13), we simply note that $T(s,t)$ can be defined for any s and t in (t_j,t_{j+1}) and that as before $H_s(\cdot)T(\cdot,s)$ is independent of s on (t_j,t_{j+1}) so that we can put $B(\cdot)=H_s(\cdot)T(\cdot,s)$ and consequently obtain

$$T(s,t)x - x = \int_{[s,t)} B(\cdot)T(s,\cdot)x\,d\lambda$$

from which (13) follows, when s decreases to t_j.

It remains to show that in some circumstances (1) can be replaced by (1′). Clearly when $T(\cdot,\cdot)$ is continuous so is the measure λ, in which case (1) reduces to (1′). When each $T(s,t)$ is normal and invertible we simply apply the following lemma.

LEMMA 11. *Suppose that* $\{T_n\}_{n=1}^{\infty}$ *is a sequence of normal operators with bounded inverses such that* $\sum\|T_n-I\|<\infty$. *Then there exists a sequence* $\{A_n\}_{n=1}^{\infty}$ *of operators such that* $T_n=\exp[A_n]$, *and* $\sum\|A_n\|<\infty$.

PROOF. By spectral theory, we can certainly find bounded normal operators $A_n=\log(T_n)$ such that $T_n=\exp(A_n)$. Furthermore if $\|T_n-I\| < 1$, then use of the principal branch of the logarithm on the spectrum of T_n gives an A_n with

$$\|A_n\| \leqq -\log(1-\|T_n-I\|)\,,$$

and the result follows.

In conclusion we note that the definition of multiplicative integrals has generally involved limits of products of the form $\prod(1+A(t_k)\Delta_k)$. Since $(1+A/n)^n$ converges to $\exp[A]$ as n becomes infinite in most situations our definition of multiplicative integrals leads to the same concept. We suggest now that the simplest approach to multiplicative integration would be not to use a limiting process at all, but simply to utilize the "generalized exponential" which is described in Theorem 3. This allows a very direct transition from the well developed theory of vector valued integration to the theory of multiplicative integration.

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Appendix 1. Differentiation with respect to a continuous Borel measure.

For the sake of completeness we present a proof of the following theorem for which we lack a suitable reference.

THEOREM. *Let λ be a continuous Borel measure on* R^1. *Let* $f\colon \mathsf{R}^1 \to X$ *be strongly integrable with respect to* λ (*i.e.* f *is* λ-a.e. *approximable by simple functions and* $\int \|f\| d\lambda < \infty$. *Then we have for* λ-a.e. t *in* R^1 *that*

$$\lim_{\lambda(J)\to 0} \lambda(J)^{-1} \int_J \|f(\cdot)-f(t)\| \, d\lambda = 0 ,$$

where J is an open interval containing t and having $\lambda(J) > 0$.

PROOF. The proof depends on a change of variables which reduces this to the standard theorem involving Lebesgue measure m, rather than λ.

Let the function $\varphi : \mathsf{R}^1 \to \mathsf{R}^1$ be chosen in such a way that $\lambda((a,b)) = \varphi(b)-\varphi(a)$ for every interval (a,b). Then φ is a continuous non-decreasing function, and it is constant on an interval if and only if that interval has λ-measure 0. Let I_n, $n=1,2,\ldots$, denote the at most countably many closed intervals of constancy of φ, and put $A=\mathsf{R}^1-\bigcup I_n$. Then φ is a measure-preserving homeomorphism of A with measure λ onto $\varphi(A)$ with measure m, and $\mathsf{R}^1-A=\bigcup I_n$ has λ-measure 0 while $\varphi(\mathsf{R}^1)-\varphi(A)=\varphi(\bigcup I_n)$ is at most countable and thus has Lebesgue measure 0. Consequently, if ψ denotes any function with $\varphi(\psi(t))=t$ for all $t \in \varphi(\mathsf{R}^1)$, then the following change of variables formula holds for simple functions, and hence in general:

$$\int_E f(\cdot)\, d\lambda = \int_{\varphi(E)} f\circ\psi(\cdot)\, dm .$$

Now, when $t \in A$ and J is an open interval containing t we have $\lambda(J) > 0$, and

$$\lambda(J)^{-1} \int_J \|f(\cdot)-f(t)\| \, d\lambda = m(\varphi(J))^{-1} \int_{\varphi(J)} \|f(\psi(\cdot))-f\big(\psi(\varphi(t))\big)\| \, dm .$$

We notice that $\varphi(J)$ is an interval and that $\varphi(t)$ lies in the interior of $\varphi(J)$, and recall that the correspondence between t in A and $\varphi(t)$ in $\varphi(A)$ is 1-1. Using the standard theorem for the case of Lebesgue measure (see [2, p. 217]) we get convergence to zero as $\lambda(J)$ tends to zero.

Appendix 2. On the absolute continuity and bounded variation of operator-valued functions.

We precede the main result of this Appendix by a definition.

DEFINITION. Let Z be a Banach space. Then $F : \mathsf{R}^1 \to Z^*$ is said to be weak*-measurable if for each z in Z, $\langle z, F\rangle$ is Borel measurable. F is said to be weak*-integrable if it is weak*-measurable and $\|F\| : \mathsf{R}^1 \to \mathsf{R}^1$ is both measurable and integrable.

THEOREM. *Let* $F : \mathsf{R}^1 \to Z^*$ ($B(X)$, *with* X *separable and* $X = Y^*$) *be weak*-integrable* (*strongly integrable*). *Then*:

(a) $F(t)-F(s) \;=\; \text{weak*-}\int_{[s,t)} G(\cdot)d\lambda \qquad \left(\text{strong-}\int_{[s,t)} G(\cdot)d\lambda\right)$

for some Borel measure λ *and a weak*-integrable* (*strongly integrable*) $G : \mathsf{R}^1 \to Z^*$ ($B(X)$) *if and only if* F *is of bounded variation and left continuous*; λ *may be taken as the Stieltjes measure associated with the variation function of* F.

(b) $F(t)-F(s) \;=\; \text{weak*-}\int_s^t G(\cdot)dm \qquad \left(\text{strong-}\int_s^t G(\cdot)dm\right)$

(*where* m *is Lebesgue measure*), *if and only if* F *is absolutely continuous.*

PROOF. The proof for Z^*-valued functions can be found in [7, Theorem 6.5] or in [6]. The proof for $B(X)$ valued functions (with $X = Y^*$) is obtained by expressing $B(X)$ as a dual space Z^* and by proving that weak*-convergence in Z^* is equivalent to strong convergence in $B(X)$. For this purpose tensor products are appropriate. The basic facts we need about tensor products are recalled in the following lemma.

LEMMA. *Let* X *and* Y *be Banach spaces. Let* $X \otimes Y$ *be their algebraic tensor product. Any element of* $X \otimes Y$ *may be represented as a finite sum* $\sum x_i \otimes y_i$ *where* $x \otimes y$ *may be thought of as a bilinear functional on* $X^* \times Y^*$, *defined by* $x\otimes y(x^*, y^*) = x^*(x)y^*(y)$. *A norm may be defined on* $X \otimes Y$ *by*

$$\gamma\left(\sum_{i=1}^{n} x_i \otimes y_i\right) = \inf\left\{\sum_{i=1}^{m} \|\bar{x}_i\|\,\|\bar{y}_i\|;\ \sum \bar{x}_i \otimes \bar{y}_i = \sum x_i \otimes y_i\right\}.$$

Let $X \otimes_\gamma Y$ *be the completion of* $X \otimes Y$ *in the norm* γ. *Any element of* $X \otimes_\gamma Y$ *can be represented* (*but not uniquely*) *by a* γ-*convergent series* $\sum_{i=1}^{\infty} x_i \otimes y_i$; γ *extends to all of* $X \otimes_\gamma Y$ *and* $\gamma(x \otimes y) = \|x\|\,\|y\|$. $B(X, Y^*)$ (*the space of continuous linear maps from* X *into* Y^*) *is isometrically* *isomorphic to* $(X\otimes_\gamma Y)^*$ *by the map* $T \to L_T$ *where* $L_T(\sum x_i\otimes y_i) = \sum_i Tx_i(y_i)$.

The missing step in our proof of the theorem is now supplied by

LEMMA. *Let* $X = Y^*$ *be separable. Then* $B(X)$ *is isometrically isomorphic to* $(Y^*\otimes_\gamma Y)^* = Z^*$. *A function* $F(\cdot)\colon \mathsf{R}^1 \to B(X) = Z^*$ *is weak*-measureable if and only if it is strongly measurable. Consequently it is weak*-integrable if and only if it is strongly integrable.*

PROOF. The first assertion is a consequence of the previous lemma. If $F(\cdot)$ is weak*-measurable then we have in particular that $\langle y, F(\cdot)y^*\rangle$ is measurable for any y in Y and y^* in Y^*. Thus $F(\cdot)y^* : \mathsf{R}^1 \to Y^*$ is weak*-measurable. But $F(\cdot)y^*$ is also separable valued ($X = Y^*$) being separable). Thus, according to a lemma of Hoffman-Jørgensen (Lemma 2 of [5] or Lemma 4 of [6]), $F(\cdot)y^*$ is in fact approximable a.e. by a series of simple functions, and so $F(\cdot)$ is strongly measurable. The converse and the statement concerning integrability are trivial.

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