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ON THE HOMOMORPHIC IMAGE OF THE CENTER OF A C^*-ALGEBRA

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Let A and A' be C^*-algebras with centers Z and Z'. If $\varphi: A \to A'$ is a surjective *-homomorphism, we generally have $\varphi(Z) \subset Z'$. In this paper we study under which conditions equality holds in the above inclusion. It is known that the corresponding question for complex algebras and homomorphisms has a negative answer, even if A is finite dimensional. If, in addition to being finite dimensional, A is semi-simple equality holds. It turns out, that strong semi-simplicity does not suffice to insure equality in the case of C^*-algebras, and equality depends on the joint separation properties of $\operatorname{Prim} A$ and $\operatorname{Prim} A'$.

Let A, A' and φ be as above. We shall always assume that A and A' have a unit. $\operatorname{Prim} A$ denotes the quasi-compact primitive ideal space of A. The map

$$\check{\varphi}: \operatorname{Prim} A' \to \operatorname{Prim} A$$

given by $\check{\varphi}(J) = \varphi^{-1}(J)$ imbeds $\operatorname{Prim} A'$ as a closed subspace of $\operatorname{Prim} A$ (see [2, p. 61]).

The crucial result for our work is the Dauns and Hofman theorem [3], which, in our set-up, we formulate as follows: The continuous map

$$\eta_A = \eta: \operatorname{Prim} A \to \operatorname{Prim} Z$$

defined by $\eta(J) = J \cap Z$ induces a *-isomorphism of $C(\operatorname{Prim} Z)$ onto $C(\operatorname{Prim} A)$, where $C(T)$ denotes the set of continuous complex valued functions on the topological space T. As pointed out by Dixmier [3] this means that $(\operatorname{Prim} Z, \eta)$ is the Stone–Čech compactification of $\operatorname{Prim} A$.

Proposition 1. *Let A, A', Z, Z', and φ be as above. Then the following three statements are equivalent:*

(i) $\varphi(Z) = Z'$.

(ii) $(\varphi|_Z)^\vee : \operatorname{Prim}(Z') \to \operatorname{Prim}(Z)$ *is injective.*

(iii) *If J_1 and J_2 are primitive ideals of A', which can be separated by*

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continuous functions on Prim A', *then* J_1 *and* J_2 *can be separated by continuous functions on* Prim A.

PROOF. The map $\varphi|_Z: Z \to Z'$ induces a map $(\varphi|_Z)^\vee: \operatorname{Prim} Z' \to \operatorname{Prim} Z$ such that the diagram

$$\begin{array}{ccc} \operatorname{Prim} Z & \xleftarrow{(\varphi|_Z)^\vee} & \operatorname{Prim} Z' \\ \uparrow \eta & & \uparrow \eta \\ \operatorname{Prim} A & \xleftarrow{\check{\varphi}} & \operatorname{Prim} A' \end{array} \tag{1}$$

is commutative. In fact,

$$\eta \circ \check{\varphi}(J) = \varphi^{-1}(J) \cap Z = \varphi^{-1}(J) \cap \varphi^{-1}(Z') \cap Z = \varphi^{-1}(J \cap Z') \cap Z = (\varphi|_Z)^\vee(\eta(J)).$$

From this fact and from the observation that J_1 and J_2 are separated by continuous functions if and only if they are separated by η the equivalence (ii) $\Leftrightarrow$ (iii) follows. The equivalence (i) $\Leftrightarrow$ (ii) is well known.

COROLLARY 1. *If* A, A', Z, Z', *and* φ *are as above and if* A *has Hausdorff primitive ideal space, then* $\varphi(Z) = Z'$.

REMARK. One motivation for this paper is Lemma 1.1 of [5] in which the author proved that $\varphi(Z) = Z'$ under strong conditions on A and A'. However, it has been pointed out to us by E. Kehlet, that we have always $\varphi(Z) = Z'$, whenever A is a von Neumann algebra. This is proved by means of Dixmier's approximation theorem [1]. In the following we shall prove a result which generalizes Kehlet's result and Corollary 1.

We recall that a C^*-algebra is called weakly central if for any two different maximal ideals J_1 and J_2 we have $J_1 \cap Z \neq J_2 \cap Z$. Thus A is weakly central if and only if the map $\eta: \operatorname{Prim} A \to \operatorname{Prim} Z$ restricted to the set of maximal ideals $\operatorname{Max}(A)$ is 1-1.

THEOREM 1. *Let* A, A', Z, Z', *and* φ *be as above. If* A *is weakly central, then* $\varphi(Z) = Z'$.

PROOF. Consider the diagram

$$\begin{array}{ccc} \operatorname{Max}(A') & \longrightarrow & \operatorname{Max}(A) \\ \downarrow i' & & \downarrow i \\ \operatorname{Prim} A' & \xrightarrow{\check{\varphi}} & \operatorname{Prim} A \\ \downarrow \eta & & \downarrow \eta \\ \operatorname{Prim} Z' & \xrightarrow{(\varphi|_Z)^\vee} & \operatorname{Prim} Z, \end{array} \tag{2}$$

where $\mathrm{Max}(A') \to \mathrm{Max}(A)$ is the restriction of $\check{\varphi}$. Clearly, (2) is commutative. The map $\eta \circ i$ is $1-1$ by assumption. Therefore $\eta \circ i \circ \check{\varphi}$ is $1-1$. By commutativity $(\varphi|_Z)^{\vee} \circ \eta \circ i'$ is $1-1$, and since $\eta \circ i'$ is surjective it follows that $(\varphi|_Z)^{\vee}$ is 1–1. Hence by Proposition 1 we have $\varphi(Z) = Z'$.

COROLLARY 2. *If A, A', Z, Z', and φ are as above, and if A is a von Neumann algebra, then $\varphi(Z) = Z'$.*

PROOF. A is weakly central by the result of Misonou [4].

As a converse of Theorem 1 we present the following

THEOREM 2. *Let A be a C^*-algebra with unit. Suppose that for any uniformly closed two-sided ideal J of A the center of A is mapped onto the center of A/J under the canonical map $A \to A/J$. Then A is weakly central.*

PROOF. Let J_1 and J_2 be maximal ideals of A with $J_1 \neq J_2$. Let $A' = A/J_1 \cap J_2$. Then $\mathrm{Prim}\,A'$ is the two point set $\{J_1/J_1 \cap J_2, J_2/J_1 \cap J_2\}$, and coincides with $\mathrm{Max}(A')$. Since $\mathrm{Max}(A')$ is a T_1-space ([2]), $\mathrm{Prim}\,A'$ is discrete. Thus the two points in $\mathrm{Prim}\,A'$ can be separated by continuous functions, hence by the assumption and the equivalence (i) $\Leftrightarrow$ (iii) of Proposition 1 J_1 and J_2 can be separated in $\mathrm{Prim}\,A$, but that means that J_1 and J_2 are separated by η, that is, $J_1 \cap Z \neq J_2 \cap Z$. Thus A is weakly central.

EXAMPLE. Let A be the C^*-algebra of all bounded sequences $x = (x_n)$ of 2×2 complex matrices with $\lim_{n\to\infty} x_n =$ a diagonal matrix with entries $a(x)$ and $b(x)$. Let $J = \{x \in A \mid a(x) = 0, b(x) = 0\}$. Then one observes that A/J is 2-dimensional and commutative, whereas the canonical image of the center is the 1-dimensional scalar algebra.

REFERENCES

1. J. Dixmier, *Les anneaux d'opérateurs de classe finie*, Ann. École Norm. Sup. 66 (1949), 209–261.
2. J. Dixmier, *Les C*-algèbres et leurs représentations* (Cahiers Scientifiques 29), Gauthier-Villars, Paris, 1964.
3. J. Dixmier, *Ideal center of a C*-algebra*, Duke Math. J. 35 (1968), 375–382.
4. Y. Misonou, *On a weakly central operator algebra*, Tôhoku Math. J. 4 (1952), 194–202.
5. J. Vesterstrøm, *Quotients of finite W*-algebras*, J. Functional Analysis 9 (1972), 322–335.

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