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INTEGRAL REPRESENTATION FORMULAS AND L^p-ESTIMATES FOR THE $\bar\partial$-EQUATION

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Introduction.

In this paper, we study the equation $\bar\partial v = u$ in strictly pseudoconvex domains, where u is a $(0,q)$-form and $\bar\partial u = 0$. The main result is a L^p-estimate for this equation. More precisely, we construct operators T_q from $(0,q)$-forms on D to $(0,q-1)$-forms on D, $q \geqq 1$, which are continuous in all L^p-norms, $1 \leqq p \leqq \infty$, and such that

$$(1) \qquad u = C_q[\bar\partial T_q u - T_{q+1}\bar\partial u]$$

where u is a $(0,q)$-form, and u and $\bar\partial u$ have L^1-coefficients. From this it follows immediately that $\bar\partial(C_q \cdot T_q u) = u$ when $\bar\partial u = 0$. The operators T_q are singular integral operators with kernels K_q continuous outside the diagonal. We prove the formula

$$(2) \qquad u(z) = C_0\left[\int_{\partial D} u(\zeta)K_0(\zeta,z) - (T_1\bar\partial u)(z)\right]$$

when $u \in C^1(\bar D)$. The regularity and boundary behaviour of $T_q u$ are also studied.

The plan of the paper is as follows: In Section 1, we fix the notation, and recall some simple but useful facts. The formulas (1) and (2) are special cases of (a slight generalization of) a formula of Koppelman, valid without the pseudoconvexity condition. It was announced in [8], and since no proof has appeared, we give one in Section 2. It is based on an idea of Leray, [9, ch. 7]. The kernels are pull-backs of a universal form ν on a bundle E over $\mathbf{C}^n \times \mathbf{C}^n \setminus \Delta$ by a suitable section s. To construct such a s, some special functions from Henkin [3] are used. In Section 3, a modification of Henkin's construction is given where only C^2 boundary is needed. In Section 4, we show how to extend and modify Henkin's functions to get the section s, while in Section 5 we see that if this is done with care, certain uniformity properties in ζ and z follows,

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from which the L^p-estimates and other properties of the T_q's are deduced in Section 6.

When $u \in A(\bar{D})$, formula (2) reduces to the representation formula of Henkin [3], which was the starting point of my investigation. I want to thank Professor Hörmander for the suggestion to extend Henkin's results, and for help and encouragement during the preparation of this work.

After I had obtained (2), and the results for (0,1)-forms, I learnt about the closely related work of Grauert–Lieb and Kerzman. I want to thank Dr. Lieb for sending me a preprint of the note [10], which made me aware of Koppelman's work, and showed how to extend my methods to forms of arbitrary degree. As this does not complicate the proofs much, I have decided to give the general version. I also want to thank Dr. Kerzman for sending me a copy of his New York University thesis [7], and a research announcement [6] with the same title. Kerzman's papers give both L^p and Hölder estimates for (0,1)-forms. After this manuscript was completed, Henkin's paper [4] appeared, which gives sup-norm estimates for (0,1)-forms.

1. Notation and preliminaries.

We will use the standard notation for differential forms in $\mathbf{C}^n$ (as in Hörmander [5, Section 2.1]). In the present paper, however, $I,J,\ldots$ will always be strictly increasing multiindices. If $u_1,\ldots,u_n$ are differential forms, and $I=(i_1,\ldots,i_k)$ is an increasing multi-index, u^I denotes $u_{i_1}\wedge\ldots\wedge u_{i_k}$, and $\bigwedge_{j\neq i_1,\ldots,i_k} u_j$ denotes $u_{j_1}\wedge\ldots\wedge u_{j_{n-k}}$, where $j_1<\ldots<j_{n-k}$ are the remaining integers in $\{1,\ldots,n\}$.

When M is a complex manifold, $A_{(p)}(M)$ is the space of holomorphic p-forms, or $(p,0)$-forms with holomorphic coefficients.

$\langle\cdot,\cdot\rangle$ denotes the $\mathbf{C}$-bilinear pairing $(z,w)\to\sum_{i=1}^n z_i w_i$, from $\mathbf{C}^n\times\mathbf{C}^n$ to $\mathbf{C}$, as well as the $\mathbf{R}$-bilinear pairing $(\varphi,v)\to\varphi(v)$ from $L_{\mathbf{R}}(V,\mathbf{C})\times V$ to $\mathbf{C}$, when V is a (real or complex) vectorspace, and $L_{\mathbf{R}}(V,\mathbf{C})$ the space of $\mathbf{R}$-linear maps from V to $\mathbf{C}$. Recall that if $g\in C^1(U,\mathbf{C})$, U open in $\mathbf{C}^n$, then

$$\partial g(z) = \sum_{i=1}^n \frac{\partial g(z)}{\partial z_i}\, dz_i$$

is the $\mathbf{C}$-linear part of $dg(z)$, that is,

$$\langle \partial g(z),t\rangle = \tfrac{1}{2}(\langle dg(z),t\rangle - i\langle dg(z), it\rangle)$$

A (real or Hermitian) scalar product determines norms on V^* and its exterior powers $\Lambda^p V^*$. These norms, as well as the modulus of complex numbers, are denoted by $|\cdot|$. We have $|u \wedge v| \leqq |u|\,|v|$. Notice that $L^p_{(r,q)}(D)$, the space of (r,q) forms with coefficients in $L^p(D)$, is the space of equivalence classes of (r,q)-forms u with measurable coefficients, for which $|u| \in L^p(D)$. We define $\|u\|_p = \|(|u|)\|_{L^p(D)}$. If M is an m-dimensional C^1-submanifold of a Euclidean space, $d\sigma$ denotes the volume measure associated with the induced Riemannian metric. If u is an integrable m-form on M, we have

$$\left|\int_M u\right| \leqq \int_M |u(z)|\, d\sigma(z)\,.$$

$d\lambda$ denotes the Lebesgue measure on C^n, and dx the Lebesgue measure on R^m.

We are going to consider differential forms $u = \sum_{I,J} u_{I,J}(x,y)\, dx^I \wedge dy^J$, on $X \times Y$, where the $u_{I,J}$'s are scalar functions, and $(x_1, \ldots x_n)$, $(y_1, \ldots y_m)$ are coordinates on X resp. Y. If c is an m-dimensional C^1-cycle in X, then

$$\int_c u = \sum_J \left(\int_c \sum_{|I|=m} u_{I,J}(x,y)\, dx^I\right) dy^J$$

is a form on Y. On $X \times Y$ we split $d = d_x + d_y$, where

$$d_x = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\, dx_i \quad \text{and} \quad d_y = \sum_{j=1}^{m} \frac{\partial}{\partial y_j}\, dy_j\,,$$

and in the same way $\bar\partial = \bar\partial_x + \bar\partial_y$ when X and Y carry complex structures. When $\alpha = (\alpha_1 \ldots, \alpha_n)$ and $= (\beta_1 \ldots, \beta_m)$ are multi-orders,

$$D_x^{\alpha} D_y^{\beta} u(x,y) = (\partial/\partial x_1)^{\alpha_1} \ldots (\partial/\partial x_n)^{\alpha_n} (\partial/\partial y_1)^{\beta_1} \ldots (\partial/\partial y_m)^{\beta_m} u(x,y)\,.$$

When X or Y carry complex structure, the symbols above still denote derivation with respect to the underlying real coordinates.

The space $C^k(X, C^\infty(Y))$ of C^k-functions on X with values in the Fréchet space $C^\infty(Y)$, is identified with

$$\{f \in C(X \times Y)\colon D_x^{\alpha} D_y^{\beta} f(x,y) \text{ exists and is continuous when } |\alpha| \leqq k;\ \beta \text{ arbitrary}\}\,,$$

by the map $g \to (f\colon (x,y) \to g(x)(y))$. When Y carries a complex structure, $C^k(X, A(Y))$ is the subspace where $D_x^{\alpha} f(x,y)$ is holomorphic in y when $|\alpha| \leqq k$.

If V is a complex vectorspace, we orient it by defining the real basis

$e_1, ie_1, \ldots, e_n, ie_n$ to be positively oriented for every complex basis $e_1, \ldots, e_n$.

Finally, when V is a normed space, $S(x,r)$ is the sphere, and $B(x,r)$ the open ball, with radius r and center x.

2. Koppelman's formulas.

Let $\xi_1, \ldots, \xi_n$ be the coordinate functions on $\mathbf{C}^n$. We define

$$\omega(\xi) = \bigwedge_{i=1}^{n} d\xi_i \in A_{(n)}(\mathbf{C}^n),$$

and

$$\omega'(\xi) = \sum_{i=1}^{n} (-1)^{i-1} \xi_i \bigwedge_{j \neq i} d\xi_j \in A_{(n-1)}(\mathbf{C}^n).$$

Note that $\omega(\xi)$ generates $A_{(n)}(\mathbf{C}^n)$ over $A(\mathbf{C}^n)$, while $\omega'(\xi)$ plays a similar role for the complex projective space.

Let $P^{n-1}(\mathbf{C})$ be the complex $n-1$ dimensional projective space and

$$\pi\colon\ \mathbf{C}^n \setminus \{0\} \to P^{n-1}(\mathbf{C})$$

the projection. For simplicity, we also write π for

$$\pi \times 1\colon\ (\xi, x) \to (\pi(\xi), x)$$

from $(\mathbf{C}^n \setminus \{0\}) \times X$ to $P^{n-1}(\mathbf{C}) \times X$. Let $\tilde{U}_i = \{\xi \in \mathbf{C}^n,\ \xi_i \neq 0\}$, and $U_i = \pi(\tilde{U}_i)$. The functions

$$\eta_j^{(i)}\colon\ \xi \to \xi_j/\xi_i, \quad j \neq i,$$

are complex coordinates on U_i, and a simple computation shows that

$$\pi^*\Big(\bigwedge_{j \neq i} d\eta_j^{(i)}\Big) = \bigwedge_{j \neq i} d(\xi_j/\xi_i) = (-1)^{i-1} \xi_i^{-n} \omega'(\xi)$$

on $\tilde{U}_i$.

Let $\tilde{U}$ be open in $(\mathbf{C}^n \setminus \{0\}) \times \mathbf{C}^m$, and $U = \pi(\tilde{U})$. As π^* is injective, it follows that the map

$$A_{(n+m-1)}(U) \ni u \to f \in A(\tilde{U}),$$

when $\pi^* u = f(\xi, w)\, \omega'(\xi) \wedge \omega(w)$, is a bijection on

$$\{f \in A(\tilde{U})\colon f \text{ is homogeneous in } \xi \text{ of degree } -n\}.$$

Let in particular

$$\tilde{E} = \{(\xi, \zeta, z) \in \mathbf{C}^{3n}\,;\ \langle \xi, \zeta - z \rangle \neq 0\},$$

and $E=\pi(\tilde{E})$. Let

$$\tilde{\nu} = \langle \xi,\zeta-z\rangle^{-n}\omega'(\xi) \wedge \omega(\zeta) \wedge \omega(z) .$$

By the remark above, $\tilde{\nu}=\pi^*\nu$, where $\nu \in A_{(3n-1)}(E)$. As E is open in $P^{n-1}(\mathbf{C}) \times \mathbf{C}^{2n}$, ν is holomorphic of maximal degree, and therefore

$$d\nu = 0 . \tag{2.1}$$

Let $p\colon E \to \mathbf{C}^n \times \mathbf{C}^n \setminus \Delta$ be the projection $(\pi(\xi),\zeta,z) \to (\zeta,z)$.

REMARK. It is easy to see that $(v,\zeta,z) \to (\pi(v+\overline{\zeta-z}),\zeta,z)$ is a homeomorphism from $\{(v,\zeta,z) \in \mathbf{C}^n \times (\mathbf{C}^n \times \mathbf{C}^n \setminus \Delta)\colon \langle v,\zeta-z\rangle=0\}$ to E, which gives $E \overset{p}{\to} \mathbf{C}^n \times \mathbf{C}^n \setminus \Delta$ a complex vector bundle structure.

Let U be open in $\mathbf{C}^n \times \mathbf{C}^n \setminus \Delta$, and $s\colon U \to E$ a p-section of class C^1. We define $K=K(s)$ as the unique $(n,n-1)$ form such that $K \wedge \omega(z)=s^*\nu$. We write $K=\sum_{q=0}^{n-1}K_q$, when K_q is of type $(n,n-q-1)$ in ζ and $(0,q)$ in z. Since $d(K \wedge \omega(z))=s^*d\nu$ and $K \wedge \omega(z)$ is of type $(2n,n-1)$, we must have

LEMMA 2.1. *$\bar\partial K=0$, and therefore $\bar\partial_\zeta K_q=-\bar\partial_z K_{q-1}$, $q=0,1,\ldots,n$, when we define $K_{-1}=0$ and $K_n=0$.*

In the applications, the sections will be of the form

$$s_f\colon\ (\zeta,z) \to (\pi \circ f(\zeta,z),\zeta,z) ,$$

where $f=(f_1,\ldots,f_n)\colon U \to \mathbf{C}^n$ satisfies $\langle f(\zeta,z),\zeta-z\rangle \neq 0$ in U. In fact, this is always true. Locally, this is clear, and we normalize the local functions f and patch them together by a partition of unity. For simplicity, instead of $K(s_f)$, we usually write $K(f)$, or even K when there is no danger of confusion. The important thing is, however, that $K(f)$ only depends on s_f. Now

$$s_f^*\nu = (f,1)^*\tilde{\nu} = F(\zeta,z)^{-n} f^*(\omega'(\xi)) \wedge \omega(\zeta) \wedge \omega(z) ,$$

where $F(\zeta,z)=\langle f(\zeta,z),\zeta-z\rangle$, so

$$K(\zeta,z) = F(\zeta,z)^{-n} \sum_{i=1}^{n} (-1)^{i-1} f_i \bigwedge_{j\neq i} (\bar\partial_\zeta f_j+\bar\partial_z f_j) \wedge \omega(\zeta) . \tag{2.2}$$

To explicit the kernels K_q, we introduce the following notation: When $0 \leqq q \leqq n-1$, P_q is the set of permutations $p\colon \{1,\ldots,n\} \to \{i_p,J_p,L_p\}$, where J_p and L_p are increasing multi-indices with $|J_p|=n-q-1$ and $|L_p|=q$. Let ε_p denote the signature of p. Separating the terms $\bar\partial_\zeta f_j$ from the terms $\bar\partial_z f_j$, we clearly get

$$(2.3)\qquad K_q(\zeta,z) = F(\zeta,z)^{-n} \sum_{p\in P_q} \varepsilon_p f_{i_p} \bigwedge_{j\in J_p} \bar\partial_\zeta f_j \wedge \bigwedge_{l\in L_p} \bar\partial_z f_l \wedge \omega(\zeta) .$$

An important example is $b\colon (\zeta,z) \to \bar\zeta - \bar z$ from $\mathbf{C}^n \times \mathbf{C}^n \setminus \Delta$ to $\mathbf{C}^n$.

As $\bar\partial_\zeta b_j = d\bar\zeta_j$ and $\bar\partial_z b_j = -d\bar z_j$, we get

$$K(b)(\zeta,z) = |\zeta - z|^{-2n} \sum_{i=1}^{n} (-1)^{i-1} (\bar\zeta_i - \bar z_i) \bigwedge_{j\neq i} (d\bar\zeta_j - d\bar z_j) \wedge \omega(\zeta) ,$$

and

$$K(b)_q(\zeta,z) = (-1)^{(n+1)q} |\zeta - z|^{-2n} \sum_{p\in P_q} \varepsilon_p (\bar\zeta_{i_p} - \bar z_{i_p}) \bigwedge_{j\in J_p} d\bar\zeta_j \wedge \omega(\zeta) \wedge \bigwedge_{l\in L_p} d\bar z_l ,$$

when $0 \leqq q \leqq n-1$. These kernels are called the Bochner–Martinelli kernels. Note, however, that they differ from those of Koppelman [8] by a constant factor. Clearly $K(b)$ is translation invariant, and

$$|K(b)(\zeta,z)| \leqq C_b |\zeta - z|^{-2n+1} ,$$

where C_b is a constant only depending on n. Introducing polar coordinates, we see that

$$\int_D |K(b)(\zeta,z)|\, d\lambda(\zeta) \leqq C_b \omega_{2n-1} \cdot \text{diameter } (D) ,$$

where ω_{2n-1} is the area of the unit sphere in $\mathbf{C}^n$, and from the translation invariance of $K(b)$ we get

$$D_z^\alpha\left(\int_D u(\zeta) \wedge K(b)_{q-1}(\zeta,z)\right) = \int_D (D^\alpha u(\zeta)) \wedge K(b)_{q-1}(\zeta,z) ,$$

when $u \in (C_0^{|\alpha|})_{(0,q)}(D)$.

We are now ready to give the main result of this section.

THEOREM 2.2. *Let D be a bounded set in $\mathbf{C}^n$ with C^1 boundary, and $s\colon \bar D \times D \setminus \Delta \to E$ a C^1-section, which is equal to s_b on a neighbourhood of Δ in $D \times D$. Write K_q for $K_q(s)$. Then we have*

$$(2.4)\qquad u(z) = C_q \left[\int_{\partial D} u(\zeta) \wedge K_q(\zeta,z) - \int_D \bar\partial u(\zeta) \wedge K_q(\zeta,z) + \bar\partial_z \left(\int_D u(\zeta) \wedge K_{q-1}(\zeta,z) \right) \right]$$

for all $z \in D$ and $u \in C^1_{(0,q)}(\bar D)$, $q = 0,1,\ldots,n$, where

$$C_q = (-1)^{\frac12 n(n-1)+q} (n-1)!\, (2\pi i)^{-n} .$$

PROOF. As $(C_0^\infty)_{(p,q)}(D)$ is dense in $L^1_{(p,q)}(D)$, $0 \leqq p,q \leqq n$, it is enough to show that

$$(2.5)\quad \int_D u(z) \wedge v(z) = C_q \Bigg[\int_{\partial D \times D} u(\zeta) \wedge K_q(\zeta,z) \wedge v(z) - \\ - \int_{D\times D} \bar\partial u(\zeta) \wedge K_q(\zeta,z) \wedge v(z) + \int_D \left(\bar\partial_z \int_D u(\zeta) \wedge K_{q-1}(\zeta,z)\right) \wedge v(z)\Bigg]$$

for all $v \in (C_0^\infty)_{(n,n-1)}(D)$. We also notice that K_q and K_{q-1} may be replaced by K in this formula, as the additional terms must vanish for degree reasons.

For all $\varepsilon > 0$, let

$$M_\varepsilon = \{((\zeta,z) \in \bar D \times \bar D : |\zeta - z| = \varepsilon\} ,$$
$$U_\varepsilon = \{(\zeta,z) \in \bar D \times \bar D : |\zeta - z| \geqq \varepsilon\} .$$

Then Stokes theorem gives

$$\int_{\partial D\times D} u(\zeta) \wedge K(\zeta,z) \wedge v(z) - \int_{M_\varepsilon} u(\zeta) \wedge K(\zeta,z) \wedge v(z)$$
$$= \int_{U_\varepsilon} \bar\partial u(\zeta) \wedge K(\zeta,z) \wedge v(z) + (-1)^{2n+q-1} \int_{U_\varepsilon} u(\zeta) \wedge K(\zeta,z) \wedge \bar\partial v(z) ,$$

as K is closed outside Δ, and the integrand vanishes on the rest of ∂U_ε. Let $T: \mathbf{C}^n \times \mathbf{C}^n \to \mathbf{C}^n \times \mathbf{C}^n$ be the map $(w,z) \to (z+w,z)$. When ε is chosen sufficiently small,

$$\int_{M_\varepsilon} u(\zeta) \wedge K(\zeta,z) \wedge v(z) = \int_{S(0,\varepsilon)\times D} T^*(u \wedge K) \wedge v(z) ,$$

and $K = K(b)$ on $M_\varepsilon \cap \operatorname{supp} v$. Writing

$$u(\zeta) = \sum_{|I|=q} u_I(\zeta)\, d\xi^I ,$$

we get

$$\int_{S(0,\varepsilon)\times D} T^*(u \wedge K) \wedge v$$
$$= \int_{S(0,\varepsilon)\times D} \sum_{|I|=q} u_I(z+w) d(\bar z + \bar w)^I \wedge |w|^{-2n} \omega'(\bar w) \wedge \omega(w+z) \wedge v(z)$$
$$= (-1)^{(2n-1)q} \int_D \sum_{|I|=q} \left\{ \int_{S(0,\varepsilon)} u_I(z+w)\, K(b)_0(w,0) \right\} d\bar z^I \wedge v(z)$$

by degree considerations. Observe that

$$\omega(z+w) = \omega(w) + \text{terms killed by } v(z)\,,$$

and

$$d(\bar{z}+\bar{w})^I \wedge \omega'(\bar{w}) \wedge \omega(w) = d\bar{z}^I \wedge \omega'(\bar{w}) \wedge \omega(w) + \\ + \text{terms of degree} > 2n-1 \text{ in } w\,.$$

We now need

LEMMA 2.3. *Let D be open in $\mathbf{C}^n$, and $g \in C^1(D)$. Then*

$$\int_{S(0,\varepsilon)} g(z+w)K(b)_0(w,0) \to C_0^{-1} g(z)$$

uniformly on compacts in D.

This follows easily from the proof of (56.1) in [9]. (The difference in sign follows from different orientation conventions.)

The Bochner–Martinelli kernel is absolutely integrable, so we may let $\varepsilon \to 0$, and get

$$\int_{\partial D\times D} u(\zeta) \wedge K(\zeta,z) \wedge v(z) - (-1)^q C_0 \int_D u(z) \wedge v(z)$$
$$= \int_{D\times D} \bar{\partial} u(\zeta) \wedge K(\zeta,z) \wedge v(z) + (-1)^q \int_D \left(\int_D u(\zeta) \wedge K(\zeta,z) \wedge \bar{\partial} v(z) \right).$$

After integrating by parts in the last term and rearranging, we get formula (2.5).

REMARK. Formula (2.4) may be valid without the condition $s=s_b$ near Δ in $D\times D$. It is well known how each C^1 homotopy h between s and s_b determines a form ω_h with $K(s)-K(b)=d\omega_h$. (Such a homotopy will always exist because E has a smooth vector bundle structure.) As v has compact support in D, integration by parts gives

$$\int_{M_s} u(\zeta) \wedge K(s)(\zeta,z) \wedge v(z) - \int_{M_s} u(\zeta) \wedge K(b)(\zeta,z) \wedge v(z)$$
$$= (-1)^{q+1} \int_{M_s} du(\zeta) \wedge \omega_h(\zeta,z) \wedge v(z) - \int_{M_s} u(\zeta) \wedge \omega_h(\zeta,z) \wedge dv(z)\,.$$

For quite a large class of sections, which may even be discontinuous at Δ, we may choose h so that

$$\int_{S(z,\varepsilon)} |\omega_h(\zeta,z)|\, d\sigma(\zeta) \to 0$$

uniformly on compacts as $\varepsilon \to 0$. If $K(s)$ is absolutely integrable, we get (2.4). However, we are not going to use this extension.

3. Henkin's construction.

Assume that we can find a C^1 section s of E over $\bar{D} \times \bar{D} \setminus \Delta$, which is holomorphic in $z \in D$ when $\zeta \in \partial D$. At least locally, $s = s_f$ with $f(\zeta,z)$ holomorphic in z when $\zeta \in \partial D$, so it follows from (2.2) that $K_0(\zeta,z)$ is analytic in $z \in D$, while $K_q(\zeta,z) = 0$ for $q \geqq 1$, when $\zeta \in \partial D$. If also s equals s_b near Δ in $D \times D$, we get at once from theorem 2.2 that

$$(3.1) \qquad u(z) = C_0 \left[\int_{\partial D} u(\zeta) K_0(\zeta,z) - \int_D \bar{\partial} u(\zeta) \wedge K_0(\zeta,z)\right]$$

where $u \in C^1(\bar{D})$, $z \in D$; and

$$(3.2) \qquad u(z) = C_q \left[\bar{\partial}_z \left(\int_D u(\zeta) \wedge K_{q-1}(\zeta,z)\right) - \int_D \bar{\partial} u(\zeta) \wedge K_q(\zeta,z)\right]$$

where $u \in C^1_{(0,q)}(\bar{D})$, $z \in D$ and $q \geqq 1$. Such a section can be constructed when D is bounded with C^2 strictly pseudoconvex boundary. This means that there exists a $\varrho \in C^2(U)$, where U is a neighbourhood of ∂D, with $\varrho < 0$ in $U \cap D$ and $\varrho > 0$ in $U \setminus \bar{D}$, and with $d\varrho(z) \neq 0$ and the Levi form

$$L_\varrho(z)t = \sum_{i,j=1}^{n} \frac{\partial^2 \varrho(z)}{\partial z_i \partial \bar{z}_j} t_i \bar{t}_j$$

positive definite when $z \in U$.

We write $V_\varepsilon = \{z \in U:\ |\varrho(z)| < \varepsilon\}$, and $D_\varepsilon = V_\varepsilon \cup D$. When $\varepsilon, \delta > o$,

$$O = O_{\varepsilon,\delta} = \{(\zeta,z) \in V_\varepsilon \times D_\varepsilon:\ \zeta \in V_\varepsilon \text{ and } |\zeta - z| < \delta\}.$$

Then the first step in the construction of an s with the properties above is

THEOREM 3.1. *With the assumptions and notations above we can find:* I. *Constants* $C, \varepsilon, \delta > 0$, $H \in C^1(V_\varepsilon, A(D_\varepsilon))$, *and* $A, B \in C^1(O_{\varepsilon,\delta})$, *such that* B *does not vanish on* O, *and*

(1) $H = AB$ *on* O; H *is bounded away from zero outside* O.
(2) $\operatorname{Re} A(\zeta,z) \geqq [\varrho(\zeta)-\varrho(z)]+C|\zeta-z|^2$
(3) $d_\zeta A(\zeta,z)|_{\zeta=z} = -d_z A(\zeta,z)|_{\zeta=z} = 2\cdot\partial\varrho(z)$.

II. *There exists* $h \in C^1(V_\varepsilon, A(D_\varepsilon))^n$, *such that*

$$H(\zeta,z) = \sum_{i=1}^{n} h_i(\zeta,z)(\zeta_i - z_i) \ (or\ \langle h(\zeta,z),\zeta-z\rangle)\,.$$

Apart from the fact that we only require C^2 boundary, the theorem above is almost identical to results in Henkin [3]. Ramirez de Arellano [11] gives related results. We shall indicate Henkin's argument, and give in detail the modifications needed.

Proof. Suppose first that D is strictly convex, that is, that the Hessian

$$H_\varrho(z)t = \sum_{i,j=1}^{2n} \frac{\partial^2\varrho(z)}{\partial x_i \partial x_j} t_i t_j$$

is positive definite near ∂D, where $(x_1, \ldots x_{2n})$ are the underlying real coordinates of $\mathbf{C}^n$. If

$$H_\varrho(\zeta)t \geqq 3C|t|^2 \quad \text{when } \zeta \in \partial D\,,$$

then, by Taylors theorem

$$\varrho(z)-\varrho(\zeta) \geqq \langle d\varrho(\zeta),\ z-\zeta\rangle + C|\zeta-z|^2$$

when ζ is near ∂D and $|\zeta - z| < \delta$, where δ is a suitable positive constant. We define

$$H(\zeta,z) = 2\langle\partial\varrho(\zeta),\zeta-z\rangle,\ h_i(\zeta,z) = 2\,\partial\varrho(\zeta)/\partial\zeta_i\,,$$

and get

$$\operatorname{Re} H(\zeta,z) = \langle d\varrho(\zeta),\zeta-z\rangle \geqq [\varrho(\zeta)-\varrho(z)]+C|\zeta-z|^2\,.$$

With $A = H$, the theorem clearly holds.

Combined with the constructions below, this gives elementary existence proofs and L^p-estimates for the $\bar\partial$-complex of a bounded domain with C^2 strictly convex boundary.

When ϱ is C^2 and strictly pseudoconvex at $v \in \mathbf{C}^n$, there exists a biholomorphic map T_v from a neighbourhood U_v of v to a neighbourhood $U(v)$ of 0, with inverse S_v; such that $\varrho_v = \varrho \circ S_v$ is strictly convex. Define T_v: $(z_1, \ldots, z_n) \to (w_1(z), z_2, \ldots, z_n)$, when

$$w_1(z) = 2\langle\partial\varrho(v),z-v\rangle + \sum_{i,j=1}^{n} \frac{\partial^2\varrho(v)}{\partial z_i \partial z_j}(z_i - v_i)(z_j - v_j)\,,$$

and $z_1,\ldots,z_n$ so chosen that $\partial w_1(v)/\partial z_1 \neq 0$. By Taylor's formula

$$\varrho_v(w) = \operatorname{Re} w_1 + L_\varrho(v)(S_v'(w)) + o(|w|^2),$$

so $H_{\varrho_v}(0)$ is positive definite.

We can therefore find $\delta_v' > 0$ and $C_v' > 0$, such that

$$\langle d\varrho_v(w'),\, w'-w\rangle \geqq \varrho_v(w') - \varrho_v(w) + C_v'|w'-w|^2,$$

when $|w'| < \delta_v'$ and $|w-w'| < \delta_v'$. As T_v is a diffeomorphism, there exists $\delta_v, C_v > 0$, with

$$\langle d\varrho_v(T_v(\zeta)), T_v(\zeta) - T_v(z)\rangle \geqq \varrho(\zeta) - \varrho(z) + C_v|\zeta - z|^2,$$

when $|\zeta - v| < \delta_v$ and $|\zeta - z| < \delta_v$. We define

$$\begin{aligned} A_v(\zeta,z) &= 2\langle \partial\varrho_v(T_v(\zeta)), T_v(\zeta) - T_v(z)\rangle \\ &= 2\langle \partial\varrho(\zeta), S_v'(T_v(\zeta))\,[T_v(\zeta) - T_v(z)]\rangle \end{aligned}$$

when $\zeta \in B(v,\delta_v)$ and $z \in B(\zeta,\delta_v)$. Choose $v_1,\ldots,v_N \in U$, such that $\{B(v_i,\delta_{v_i})\}_{i=1,\ldots,N}$ cover ∂D, and $\varphi_i \in C_0^\infty(B(v_i,\delta_{v_i}))$, $i=1,\ldots,N$, which form a partition of unity on a neighbourhood V of D. If $\delta = \min \delta_{v_1},\ldots \delta_{v_N}$, we define

$$A(\zeta,z) = \sum_{i=1}^{N} \varphi_i(\zeta) A_{v_i}(\zeta,z) \quad \text{on } O_{\varepsilon,\delta}. \tag{3.3}$$

Let $C = \min\{C_{v_1},\ldots,C_{v_N}\}$. Clearly,

$$\operatorname{Re} A(\zeta,z) = \varrho(\zeta) - \varrho(z) + C|\zeta - z|^2 \text{ on } O. \tag{3.4}$$

If $\varepsilon > 0$ is chosen small enough, we therefore get $\operatorname{Re} A(\zeta,z) > 0$ when $(\zeta,z) \in V_\varepsilon \times D_\delta$ and $\frac{1}{2}\delta < |\zeta - z| < \delta$. Finally we notice that

$$\partial_\zeta A(\zeta,z) = 2\partial\varrho(\zeta) + O(|\zeta - z|),$$

and

$$\partial_z A(\zeta,z) = -2\sum_{i=1}^{N} \varphi_i(\zeta)\cdot\partial\varrho(\zeta) \circ S_v'(T_v(\zeta)) \circ T_v'(z),$$

so $A(\zeta,z)$ clearly satisfies (2) and (3) in the Theorem.

We may now proceed as Henkin, and use the Oka–Cartan theory for domains of holomorphy to construct H and h.

Let $\varkappa \in C_0^\infty(B(0,\delta))$, with $\varkappa(z) = 1$ when $|z| \leqq \frac{1}{2}\delta$. We solve $\bar\partial_z C(\zeta,z) = B_1(\zeta,z)$, with $C \in C^1(V_\varepsilon, L^2(D_\varepsilon))$, where $B_1 = \bar\partial_z((\ln A)\varkappa(\zeta - z))$ on $\operatorname{supp} d_z\varkappa(\zeta - z)$ and 0 outside. (Notice that $\operatorname{Re} A(\zeta,z) > 0$ on $\operatorname{supp} d_z\varkappa(\zeta - z)$, so we may choose the principal branch of the logarithm.) This is done by composing $B_1\colon V \to C_{(0,1)}(D_\varepsilon)$ with a left inverse E to $\bar\partial$, going from $\ker\bar\partial$ to $L^2(D_\varepsilon)$, which exists by [5, Theorem 2.2.3]. (Project on $(\ker\bar\partial)^\perp$.)

Then

$$H(\zeta,z) = A(\zeta,z)\exp(-C(\zeta,z)) \text{ on } O_{\varepsilon,\frac{1}{2}\delta},$$
$$= \exp(\ln A(\zeta,z)\varkappa(\zeta-z)-C(\zeta,z)) \text{ on the complement },$$

is in $C^1(V_\varepsilon, A(D_\varepsilon))$. Now Henkin writes

$$G(\zeta,z,w) = H(\zeta,z)-H(\zeta,w) .$$

Clearly $G \in C^1(V_\varepsilon, A(D_\varepsilon \times D_\varepsilon))$. Further, $G(\zeta,-)$ vanishes on $\Delta \subset D_\varepsilon \times D_\varepsilon$, so by Taylor's formula it is a global section of the Ideal I generated by $w_1-z_1,\ldots,w_n-z_n$. By the last part of section 7 of [1], it follows that there exists $g(\zeta,z,w) \in C^1(V_\varepsilon, A(D_\varepsilon \times D_\varepsilon)^n)$, such that

$$G(\zeta,z,w) = \sum_{i=1}^{n} g_i(\zeta,z,w)(w_i-z_i) .$$

As $A(\zeta,\zeta)=0$ and therefore $H(\zeta,\zeta)=0$, we have $G(\zeta,z,\zeta)=H(\zeta,z)$, and we may define $h(\zeta,z)=g(\zeta,z,\zeta)$.

REMARK. If one is willing to replace D_ε by $D_\varepsilon' \subset\subset D_\varepsilon$, as we may, the map

$$E: (\ker\bar\partial: C^\infty_{(0,1)}(D_\varepsilon) \to C^\infty_{(0,2)}(D_\varepsilon)) \to L^2(D_\varepsilon'),$$

defined by

$$\bar\partial Eu = u|_{D_\varepsilon'}, \quad Eu \perp \ker\bar\partial ,$$

is in fact continuous by the open mapping theorem. In the same way, we may construct

$$S: I(D_\varepsilon \times D_\varepsilon) \to A(D_{\varepsilon'} \times D_{\varepsilon'})^n ,$$

by choosing $f' \in A(D_\varepsilon \times D_\varepsilon)^n$ satisfying

$$\langle f', w-z\rangle = F \in I(D_\varepsilon \times D_\varepsilon) ,$$

and take $S(F)$ to be the ortogonal projection of f' on $(\ker\langle -, w-z\rangle)^\perp$ in $(A(D_{\varepsilon'} \times D_{\varepsilon'}) \cap L^2(D_{\varepsilon'} \times D_{\varepsilon'}))^n$.

S is clearly linear, and it is continuous by the open mapping theorem.

4. How to construct s.

To obtain representing kernels, we need to extend and modify h to get a $f \in C^1(\bar D \times \bar D \setminus \Delta)^n$ satisfying

$$(4.1) \qquad F(\zeta,z) = \langle f(\zeta,z), \zeta-z\rangle \neq 0 \quad \text{outside } \Delta ,$$

with $s_f(\zeta,z)=s_h(\zeta,z)$ when $\zeta \in \partial D$, but $s_b=s_f$ near Δ in $D \times D$. This will be done by choosing a $\varphi \in C^\infty(\bar{D} \times \bar{D} \setminus \Delta)$, so that $0 \leqq \varphi \leqq 1$ and $\varphi = 0$ near the zeroes of h but $\varphi(\zeta,z) = 1$ when $z \neq \zeta \in \partial D$. Then we define

$$f(\zeta,z) = \varphi(\zeta,z)p(\zeta,z)h(\zeta,z) + (1-\varphi(\zeta,z))b(\zeta,z)\,, \tag{4.2}$$

where p is a C^∞-function on $\operatorname{supp}\varphi$, such that $\operatorname{Re} p\,\varphi H > 0$ when $\varphi > 0$.

To get f everywhere well defined, we must have $\varphi(\zeta,z) = 0$ when $\varrho(\zeta) \leqq -\varepsilon$, in which case we define $f(\zeta,z) = b(\zeta,z)$.

Recall that with the ε and δ of theorem 3.1, we have $\operatorname{Re} A(\zeta,z) > 0$ when $(\zeta,z) \in V_\varepsilon \times D_\varepsilon$ and $\frac{1}{2}\delta < |\zeta - z| < \delta$. With $\varkappa \in C_0^\infty(B(0,\delta))$, $\varkappa(z) = 1$ when $z \leqq \frac{1}{2}\delta$, we may choose

$$p(\zeta,z) = \varkappa(\zeta - z)\,B(\zeta,z)^{-1} + (1 - \varkappa(\zeta - z))\overline{H(\zeta,z)}\,.$$

If φ is supported by $\{(\zeta,z) \in 0 \colon \operatorname{Re} A(\zeta,z) > 0\} \cup (V_\varepsilon \times D_\varepsilon \setminus O)$, (4.2) makes sense and (4.1) is valid.

We next want to make the form $K(f)$ explicit. If $g_j = p h_j$, and $\beta_j = \varphi\, dg_j + (1-\varphi)\, db_j$, $j = 1,\ldots,n$, we have

$$df_j = \beta_j + (g_j - b_j)\, d\varphi\,.$$

Substituting this in (2.2), we get

$$K(f) = F^{-n} \sum_{i=1}^{n} (-1)^{i-1} f_i \left\{ \bigwedge_{j \neq i} \beta_j + d\varphi \wedge \left(\sum_{j<i} (-1)^{j-1} (g_j - b_j) \cdot \right.\right.$$
$$\left.\left. \cdot \bigwedge_{l \neq i,j} \beta_l + \sum_{k>i} (-1)^{k-2} (g_k - b_k) \bigwedge_{l \neq i,k} \beta_l \right) \right\} \wedge \omega(\zeta)\,.$$

Since

$$\sum_{i=1}^{n} \sum_{k>i} \cdots = \sum_{k=1}^{n} \sum_{i<k} \cdots,$$

and

$$f_i(g_j - b_j) - f_j(g_i - b_i) = b_i g_j - g_i b_j,$$

we get

$$K(f) = F^{-n} \sum_{i=1}^{n} (-1)^{i-1} \left\{ f_i \bigwedge_{j \neq i} \beta_j + d\varphi \wedge \sum_{j<i} (-1)^{j-1} (b_i g_j - g_i b_j) \cdot \right.$$
$$\left. \cdot \bigwedge_{l \neq i,j} \beta_l \right\} \wedge \omega(\zeta)\,. \tag{4.3}$$

Decreasing ε somewhat, we may assume that the h_i's and therefore the g_i's are bounded, together with their derivatives, so

$$|f_i| \leqq C_1(\varphi + |\zeta - z|), \quad |\beta_i| \leqq C_1\,, \quad |b_i g_j - g_i b_j| \leqq C_1 |\zeta - z|$$

for some suitable $C_1 > 0$; C_1 independent of (ζ,z). Substituting in (4.3), we get

$$(4.4)\quad |K(f)(\zeta,z)| \leqq C_2|F(\zeta,z)|^{-n}\Big(\varphi(\zeta,z)+|\zeta-z|\big(1+|d_\zeta\varphi(\zeta,z)|+|d_z\varphi(\zeta,z)|\big)\Big)$$

for all $(\zeta,z)\in \bar{D}\times\bar{D}\setminus\Delta$ and some $C_2>0$.

5. The estimates.

The L^p-estimates follows quite easily (see next section) provided we can prove the existence of a constant C such that

$$(5.1)\qquad \int_D |K(\zeta,z)|\, d\lambda(\zeta) \leqq C \quad \text{for all } z\in D\,,$$

and

$$(5.1)'\qquad \int_D |K(\zeta,z)|\, d\lambda(z) \leqq C \quad \text{for all } \zeta\in D\,.$$

We want to determine a function $g(r)$: $g(r)\to 0$ as $r\to 0$, such that

$$\int_{D\cap B(z,r)} |K(\zeta,z)|\, d\lambda(\zeta) \leqq g(r) \quad \text{for all } z\in \bar{D}\,,$$

and

$$\int_{D\cap B(\zeta,r)} |K(\zeta,z)|\, d\lambda(z) \leqq g(r) \quad \text{for all } \zeta\in \bar{D}\,,$$

when $r\leqq$ some r_1. This easily gives the inequalities above, as K is continuous on $\bar{D}\times\bar{D}\setminus\Delta$.

We first introduce certain local coordinate systems, in terms of which it is easy to estimate $F(\zeta,z)$.

From now on, all constants C_i, $i=1,2,\ldots$ will be independent of ζ and z. Choose $\varepsilon_0>0$ such that $(\zeta,z)\in O_{\varepsilon,\delta}$ when $\zeta\in\bar{V}_{\varepsilon_0}$ and $|\zeta-z|\leqq\varepsilon_0$ or $z\in\bar{V}_{\varepsilon_0}$ and $|\zeta-z|\leqq\varepsilon_0$.

Lemma 5.1. *There exist positive constants R_0 and C_3, and for each $z\in\bar{V}_{\varepsilon_0}$ a C^1-map u_z: $B(z,\varepsilon_0)\to \mathsf{R}^{2n}$ of the form*

$$u_z\colon\ \zeta\to(\varrho(\zeta)-\varrho(z),\ \operatorname{Im}A(\zeta,z),u_z''(\zeta))$$

with $u_z(z)=0$, such that u_z has a local C^1 inverse ζ_z: $B(0,R_0)\to B(z,\varepsilon_0)$, and such that

(i) $|\zeta_z(u)-z|>C_3^{-1}|u|$ *and* $|\zeta_z'(u)|<C_3$ *for all* $u\in B(0,R_0)$

(ii) $|u_z'(\zeta)|<C_3$ *for all* $\zeta\in\operatorname{im}u_z$.

PROOF. By Theorem 3.1 (3), $\langle d_\zeta \operatorname{Im} A(\zeta,z)|_{\zeta=z}, t\rangle = -\langle d\varrho(z), it\rangle$. Thus $d_\zeta \operatorname{Im} A(\zeta,z)|_{\zeta=z}$ is everywhere $\neq 0$ near $\overline{V}_{\varepsilon_0}$, and is clearly linearly independent of $d\varrho(z)$. For every $z_0 \in \overline{V}_{\varepsilon_0}$ we can therefore find $u_z''(\zeta)$ such that u_{z_0} is a diffeomorphism near z_0. Applying the inverse function theorem to

$$(\zeta,z) \to (u_z(\zeta),z) = \big((\varrho(\zeta)-\varrho(z), \operatorname{Im} A(\zeta,z), u''_{z_0}(\zeta)),z\big)$$

near (z_0,z_0), we see that u_z is locally invertible, ζ_z satisfies (i), and u_z satisfies (ii) for suitable R_0, C_3 when z is close to z_0. The lemma follows by a compactness argument.

As $d_z A(\zeta,z)|_{z=\zeta} = -2\partial\varrho(\zeta)$, there exist in the same way

$$v_\zeta\colon B(\zeta,\varepsilon_0) \to \mathsf{R}^{2n},\ z \to (\varrho(\zeta)-\varrho(z), \operatorname{Im} A(\zeta,z), v''(z))$$

with local inverse $z_\zeta(v)$ for all $\zeta \in \overline{V}_{\varepsilon_0}$, and satisfying the analogues of (i) and (ii) for suitable constants R_0', C_3'. We assume for simplicity that $R_0 = R_0'$ and $C_3 = C_3'$.

REMARK. Henkin used $\operatorname{Im} A(\zeta,z)$ as a local coordinate in the study of the boundary behaviour of his kernel. Related constructions are also used in the works of Kerzmann and Lieb.

We write $D_z^{(1)}$ for $\zeta_z^{-1}(D)$ and $D_\zeta^{(2)}$ for $z_\zeta^{-1}(D)$. If k is a function on some subset of $D_\varepsilon \times D_\varepsilon$, $k_z^{(1)}(u)$ denotes $k(\zeta_z(u),z)$ and $k_\zeta^{(2)}(v)$ denotes $k(\zeta,z_\zeta(v))$. Let V_ε' denote $V_\varepsilon \cap D$ and $O'_{\varepsilon,\delta} = O_{\varepsilon,\delta} \cap (V_\varepsilon' \times \overline{D})$.

In the proof of the estimates, further restrictions on φ are needed. It is convenient to require the existence of positive constants r_0, C_4, and C_5, such that $\varphi(\zeta,z) = 0$ when ζ or $z \notin V_\varepsilon'$ and $|\zeta - z| < r_0$, and such that

(1) $\operatorname{Re} A(\zeta,z) \geqq C_4|\zeta-z|^2$,

(2) $C_5^{-1}|d_\zeta\varphi(\zeta,z)| \leqq |d_z\varphi(\zeta,z)| \leqq C_5|d_\zeta\varphi(\zeta,z)|$

when $(\zeta,z) \in \operatorname{supp}\varphi$ and $|\zeta - z| \leqq r_0$,

(3) $|d\varphi_z^{(1)}(u)| \leqq C_5\,\partial\varphi_z^{(1)}(u)/\partial u_1$ when $z \in \overline{V}'_{\varepsilon_0}$ and $u \in B(0,r_0/C_3) \cap D_z^{(1)}$,

(3′) $|d\varphi_\zeta^{(2)}(v)| \leqq C_5\,\partial\varphi_\zeta^{(2)}(v)/\partial v_1$ when $\zeta \in \overline{V}'_{\varepsilon_0}$ and $v \in B(0,r_0/C_3) \cap D_\zeta^{(2)}$.

We are going to construct a φ satisfying (1)–(3′), which in addition belongs to $C^1(D, C^\infty(D))$. We define

$$\chi(\zeta,z) = \big(-\operatorname{Re} A(\zeta,z) + \tfrac{1}{2}C|\zeta-z|^2\big)/\varrho(\zeta)$$

in $O'_{\varepsilon,\delta}$ and pick a function $\psi \in C^\infty(\mathsf{R})$ such that $\psi(t) = 0$ when $t \leqq \frac{1}{2}$, $\psi'(t) > 0$ when $\frac{1}{2} < t < 1$, and $\psi(t) = 1$ when $t \geqq 1$. We want to have

$\varphi(\zeta,z)=\psi\circ\chi(\zeta,z)$ when $\zeta\in D$ and is close to ∂D, and $z\in\bar{D}$ and is close to ζ. Then we get

$$(*)\qquad 0< -\varrho(\zeta)/2 \leqq \operatorname{Re}A(\zeta,z)-\tfrac{1}{2}C|\zeta-z|^2 \quad \text{when } \varphi(\zeta,z)\neq 0\,,$$

so (1) will be satisfied. It follows from theorem 3.1 that

$$\operatorname{Re}A(\zeta,z)-\tfrac{1}{2}C|\zeta-z|^2 \geqq \varrho(\zeta)-\varrho(z)+\tfrac{1}{2}C|\zeta-z|^2 \quad \text{when } (\zeta,z)\in O_{\varepsilon,\delta}\,.$$

We have $\varrho(z)\leqq 0$ in $\bar{D}$. If we shrink ε so that $\varepsilon<C\delta^2/4$, we therefore see that $\chi(\zeta,z)>1$ when $(\zeta,z)\in O'_{\varepsilon,\delta}$ and $|\zeta-z|$ is close to δ. We may also choose $\varepsilon\leqq\varepsilon_0$. If $\psi_1\in C^\infty(\mathsf{R})$ has support in $]-\varepsilon,\infty[$ and equals one near zero, and $\varphi(\zeta,z)$ is defined as $(\psi\circ\chi(\zeta,z))\,\psi_1\circ\varrho(\zeta)$ in $O'_{\varepsilon,\delta}$ and $\psi_1\circ\varrho(\zeta)$ in $(D\times\bar{D})\setminus O'_{\varepsilon,\delta}$, we have $\varphi\in C^1(D,C^\infty(\bar{D}))$. Finally we put $\varphi(\zeta,z)\equiv 1$ on $(\partial D\times\bar{D})\setminus\Delta$. When $\zeta\in\partial D$ and $z\in\bar{D}\setminus\{\zeta\}$, $|\zeta-z|<\delta$; note that $-\varrho(\zeta')$ is small and positive while $\psi_1\circ\varrho(\zeta')=1$ and

$$\operatorname{Re}A(\zeta',z')-\tfrac{1}{2}C|\zeta'-z'|^2 \geqq \varrho(\zeta')+\tfrac{1}{2}C|\zeta'-z'|^2$$

and has a positive lower bound when $(\zeta',z')\in D\times\bar{D}$ is close to (ζ,z), so $\varphi\equiv 1$ on a neighbourhood of (ζ,z) in $\bar{D}\times\bar{D}$. Thus φ is C^1-function on

$$(\bar{D}\times\bar{D})\setminus\{(\zeta,\zeta):\zeta\in\partial D\}\,.$$

If r_0 is chosen small enough, it follows from (*) that $\varphi(\zeta,z)=0$ when $\psi_1\circ\varrho(\zeta)<1$ and $|\zeta-z|<r_0$, and also that $\varphi(\zeta,z)=0$ when ζ or $z\notin V_{\varepsilon_0}$ and $|\zeta-z|<r_0$. When verifying (2)–(3′), we may therefore assume $\varphi(\zeta,z)=\psi\circ\chi(\zeta,z)$ on $\operatorname{supp}\varphi$. By theorem 3.1,

$$d_z\operatorname{Re}A(\zeta,z)|_{\zeta=z} = -d\varrho(z) \quad\text{and}\quad d_\zeta\operatorname{Re}A(\zeta,z)|_{\zeta=z} = d\varrho(z)\,,$$

so we may write

$$\operatorname{Re}A(\zeta,z) = \varrho(\zeta)-\varrho(z)+o_1(\zeta,z)\,,$$

where o_1 and do_1 vanish on Δ. We can therefore get $|do_1(\zeta,z)|<$ any positive constant on O'_{ε_0,r_0}, by choosing r_0 small enough. Putting

$$o(\zeta,z) = -o_1(\zeta,z)+\tfrac{1}{2}C|\zeta-z|^2\,,$$

we get

$$\chi(\zeta,z) = [\varrho(\zeta)-\varrho(z)+o(\zeta,z)]/\varrho(\zeta)$$

and

$$(**)\qquad d_\zeta\varphi(\zeta,z) = (\psi'\circ\chi)[(-\chi(\zeta,z)-1)d\varrho(\zeta)+d_\zeta o(\zeta,z)]/\varrho(\zeta)\,,$$

$$(***)\qquad d_z\varphi(\zeta,z) = (\psi'\circ\chi)[d\varrho(z)+d_z o(\zeta,z)]/\varrho(\zeta)\,.$$

As $\tfrac{1}{2}\leqq\chi\leqq 1$ on $\operatorname{supp}\psi'\circ\chi$, and $|d\varrho|$ has positive upper and lower bounds on $\bar{V}'_{\varepsilon_0}$ which dominate $\sup|do|$ when r_0 is small, (2) follows. To prove (3), we pull back the equation (**) by $\zeta_z{}^*$, to get

$$d\varphi_z^{(1)}(u) = \psi'(\chi_z^{(1)}(u))[(-\chi_z^{(1)}(u)-1)du_1+(\zeta_z^*do)(u)]/(u_1+\varrho(z))$$

when $z \in \overline{V}_{\varepsilon_0}$ and $u \in B(0,r_0/C_3)\cap D_z^{(1)}$. A linear map and its adjoint have the same norm, so by lemma 5.1 we have $|\zeta_z^*(u)|<C_3$, and the discussion above shows that the coefficient of du_1 dominates the other terms when r_0 is chosen small enough. In the same manner, (3′) follows from (***).

Remark. Clearly there exists a large class of functions satisfying (1) to (3′). A simpler one is

$$\psi\big((\varrho(z)-\tfrac{1}{2}C|\zeta-z|^2)/3\varrho(\zeta)\big)\ \psi_1\circ\varrho(\zeta)\ ,$$

but this is only a C^2-function in z. In the examples above $\partial\varphi_z^{(1)}(u)/\partial u_1$, resp. $\partial\varphi_\zeta^{(2)}(v)/\partial v_1$, *dominate* the other derivatives as u, resp. v, tend to zero, and this property is independent of the other basis vectors.

When $F(\zeta,z)$ is defined by (4.1),

$$F(\zeta,z) = \varphi(\zeta,z)\,A(\zeta,z)+\big(1-\varphi(\zeta,z)\big)\,|\zeta-z|^2$$

when $\zeta \in V_\varepsilon'$ and $z \in \overline{D}$ with $|\zeta-z|<\tfrac{1}{2}\delta$. By property (1) of φ, this gives

$$|F(\zeta,z)| \geqq C^*[\varphi(\zeta,z)|\operatorname{Im}A(\zeta,z)|+\min(1,C_4)|\zeta-z|^2]$$

when $(\zeta,z)\in O'_{\varepsilon,r_0}$ and combined with lemma 5.1,

$$|F_z^{(1)}(u)| \geqq C_6[\varphi_z^{(1)}(u)\,|u_2|+|u|^2]$$

when $z \in \overline{V}_{\varepsilon_0}$ and $u \in D^{(1)}\cap B(0,r_0/C_3)$. Here C_6 and C^* are suitable constants. Under these assumptions on z and u, lemma 5.1 also gives

$$|(d_\zeta\varphi)(\zeta_z(u),z)| = |(u_z^*d\varphi_z^{(1)})(u)| \leqq C_3\,|d\varphi_z^{(1)}(u)|\ .$$

Substituting this in (4.4) and using properties (2) and (3) of φ, we get

$$|K_z^{(1)}(u)| \leqq C_7[\varphi_z^{(1)}(u)\,|u_2|+|u|^2]^{-n}\left(\varphi_z^{(1)}(u)+|u|\big(1+\partial\varphi_z^{(1)}(u)/\partial u_1\big)\right)$$

when $z \in \overline{V}'_{\varepsilon_0}$ and $u \in D_z^{(1)}\cap B(0,r_0/C_3)$. By lemma 5.1, $|\det\zeta_z'(u)|\leqq C_3^{2n}$, and the change of variables formula for integrals gives

$$(5.2)\quad \int\limits_{B(z,r)\cap D} |K(\zeta,z)|\,d\lambda(\zeta) \leqq$$

$$\leqq C_8 \int\limits_{B(0,C_3 r)} \big(\varphi_z^{(1)}(u)|u_2|+|u|^2\big)^{-n}\big(\varphi_z^{(1)}(u)+|u|\big(1+\partial\varphi_z^{(1)}(u)/\partial u_1\big)\big)\,du\ ,$$

provided $rC_3^2\leqq r_0$, $C_3 r\leqq R_0$, and $z\in\overline{V}'_{\varepsilon_0}$. ($C_8$ is a suitable constant.) On the other hand, by our assumption on φ, we have $K(\zeta,z)=K(b)(\zeta,z)$ when $z\notin V_{\varepsilon_0}$ and $|\zeta-z|\leqq r_0$, and thus

$$\int_{B(z,r)\cap D} |K(\zeta,z)| \leqq \int_{B(0,r)} |K(b)(0,w)|\, d\lambda(w) = C_b' r\,,$$

provided $z \notin \overline{V}'_{\varepsilon_0}$ and $r \leqq r_0$.

In the same way we get

$$(5.2') \quad \int_{B(\zeta,r)\cap D} |K(\zeta,z)| d\lambda(z) \leqq$$

$$\leqq C_8 \int_{B(0,C_3 r)} (\varphi_\zeta^{(2)}(v)\ |v_2| + |v|^2)^{-n}\big(\varphi_\zeta^{(2)}(v) + |v|(1 + \partial\varphi_\zeta^{(2)}(v)/\partial v_1)\big)\, dv$$

when $\zeta \in \overline{V}'_{\varepsilon_0}$ and $rC_3^2 \leqq r_0$, $rC_3 \leqq R_0$, while

$$\int_{B(\zeta,r)\cap D} |K(\zeta,z)|\, d\lambda(z) \leqq C_b' r$$

when $\zeta \notin \overline{V}'_{\varepsilon_0}$ and $r \leqq r_0$. This means that we have to consider

$$\int_D N(u)\big(\varphi(u) + |u|(1 + \partial\varphi(u)/\partial u_1)\big)\, du\,,$$

where D is a subdomain of $B(0,R)$, $\varphi \in C^\infty(D)$ and φ takes values in $[0,1]$, $\partial\varphi(u)/\partial u_1$ is positive when $d\varphi(u) \neq 0$, and $N(u) = [\varphi(u)|u_2| + |u|^2]^{-n}$.

I. Let ω_p denote the p-dimensional area of the unit sphere S^p in R^{p+1}. Then

$$\int_D N(u)|u|\, du \leqq \int_0^R \left(\int_{|u|=s} |u|^{-(2n-1)}\, d\sigma(u) \right) ds = \omega_{2n-1} R\,.$$

Let $v = (v_1, v') \in R^m$, $m \geqq 3$, and $\tau, s > 0$; $N \geqq 1$. Let

$$I(s,\tau) = \int_{|v|=s} \frac{d\sigma(v)}{[\tau|v_1| + |v|^2]^N}.$$

Using polar coordinates, we get

$$I(s,\tau) < 2\int_0^s \frac{dv_1}{[\tau v_1 + s^2]^N} \left(\int_{|v'|=s} d\sigma(v') \right) = 2\omega_{m-2} s^{m-2N} \int_0^s \frac{dv_1}{\tau v_1 + s^2},$$

so

$$(5.3) \qquad I(s,\tau) < 2\omega_{m-2} (s^{m-2N}/\tau) \log(1 + \tau/s)\,.$$

II. Consider

$$I_2 = \int_D N(u)\varphi(u)du = \int_0^R \left(\int_{S(0,s)\cap D} N(u)\varphi(u)d\sigma(u) \right) ds .$$

We subdivide $S(0,s)\cap D$ as $D_s' \cup D_s''$, where

$$D_s' = \{u \in S(0,s) \cap D:\ \varphi(u) < s^{\frac{1}{2}}\} ,$$
$$D_s'' = \{u \in S(0,s) \cap D:\ \varphi(u) \geqq s^{\frac{1}{2}}\} .$$

Then

$$\int_{D\cap S(0,s)} N(u)\varphi(u)d\sigma(u) < \int_{|u|=s} s^{-2n}s^{\frac{1}{2}}\, d\sigma(u) + \int_{u=s} [s^{\frac{1}{2}}|u_2| + |u|^2]^{-n}\, d\sigma(u) .$$

By (5.3), the last term is less than $2\omega_{2n-2}s^{-\frac{1}{2}}\log(1+s^{-\frac{1}{2}})$ while the first is clearly $\omega_{2n-1}s^{-\frac{1}{2}}$.

Thus $I_2 \leqq C_9 R^{\frac{1}{2}} \log(1/R)$.

III. Consider

$$I_3 = \int_D N(u)|u| \frac{\partial\varphi(u)}{\partial u_1}\, du ,$$

and write $u=(u_1,u')$. The mapping $u \to (\varphi(u),u')$ from $\operatorname{supp} d\varphi$ to $[0,1]\times\{u' \in R^{2n-1};\ |u'|<R\}$ has nonvanishing Jacobian $\partial\varphi(u)/\partial u_1$, and is clearly injective. Thus we get

$$I_3 \leqq \int_0^1 \left(\int_{|u'|<R} [\varphi|u_2| + |u'|^2]^{-n+\frac{1}{2}} du' \right) d\varphi$$
$$\leqq \int_0^1 \left(\int_0^R \left(\int_{|u'|=s} [\varphi|u_2| + |u'|^2]^{-n+\frac{1}{2}} d\sigma(u') \right) ds \right) d\varphi ,$$

and by (5.3),

$$I_3 < 2\omega_{2n-3} \int_0^1 \int_0^R \varphi^{-1} \log(1+\varphi/s)\, ds d\varphi .$$

A simple integration gives $I_3 < C_{10} R \log(1/R)^2$.

Remark. With more information on φ, it is easy to improve II. The examples above satisfies $\varphi(\zeta,z) \leqq C_{11}|d_\zeta\varphi|\,|\zeta - z|$, on $0 \leqq \varphi \leqq \frac{1}{2}$, as is easily checked, and this gives

$$\varphi_z \leqq C_{12} \frac{\partial \varphi_z}{\partial u_1} |u| \quad \text{on } 0 \leqq \varphi_z \leqq \tfrac{1}{2} \quad \text{for all } z \in \bar{D} .$$

Then

$$\int_{\varphi \geqq \frac{1}{2}} N(u)\varphi(u)\, du \leqq C_{13} R ,$$

while the other part is estimated by III.

To sum up, we have shown that, with $g(r) = C_{14} r^{\frac{1}{2}} \log(r^{-1})$, or even $g(r) = C_{15} r (\log(r^{-1}))^2$, we have

$$\int_{D \cap B(z,r)} |K(\zeta,z)|\, d\lambda(\zeta) \leqq g(r) , \tag{5.4}$$

and

$$\int_{D \cap B(\zeta,r)} |K(\zeta,z)|\, d\lambda(z) \leqq g(r) , \tag{5.4'}$$

for $\zeta, z \in \bar{D}$, and $r C_3^2 \leqq r_0$, $C_3 r \leqq R_0$.

Finally, we study the stability of the estimates when D is perturbed.

Proposition 5.2. *Let ϱ: $U \to \mathsf{R}$ be as above, and F a compact neighbourhood of ∂D in U. Let*

$$N_c = \{\varrho \in C^2(U) \colon \max_{|\alpha| \leqq 2,\, z \in F} |D^\alpha \varrho'(z) - D^\alpha \varrho(z)| < c\} .$$

When $\varrho' \in C^2(U)$, we define

$$D' = D(\varrho') = (D \setminus F) \cup \{z \in F : \varrho'(z) < 0\} .$$

There exist constants C and c, such that D' is open and we can find a continuous kernel $K' = K(s')$ on $\bar{D}' \times \bar{D}' \setminus \Delta$ with

$$\int_{D'} |K'(\zeta,z)|\, d\lambda(\zeta) < C$$

for all $z \in \bar{D}'$ and

$$\int_{D'} |K'(\zeta,z)|\, d\lambda(z) < C$$

for all $\zeta \in \bar{D}'$, when $\varrho' \in N_c$.

We shall only indicate the proof: When c is small enough,

$$A'(\zeta,z) = 2 \sum_{i=1}^{N} \varphi_i(\zeta) \langle \partial \varrho'(\zeta),\, S'_{v_i}(T_{v_i}(\zeta))[T_{v_i}(\zeta) - T_{v_i}(z)] \rangle$$

satisfies conditions (1) and (2) of theorem 3.1 in $O_{\varepsilon,\delta}$, provided we decrease C and δ somewhat. For ε and c small enough, we may apply the open mapping theorem to function spaces over D_ε to find $C'>0$, $B' \in O_{\varepsilon,\delta}$, $H' \in C^1(V_\varepsilon, A(D_\varepsilon))$, and $h' \in C^1(V_\varepsilon, A(D_\varepsilon))^n$ with $C'^{-1} \leqq |B'| < C'$, $C'^{-1} < |H'| < C'$ outside $O_{\varepsilon,\frac{1}{2}\delta}$, and h' and the 1. order derivatives of h', H' and B' bounded by C'; which satisfies theorem 3.1 with respect to $\varrho' \in N_c$. Then C_2 of formula (4.4) may be chosen independently of ϱ', and also ε_0, R_0, and C_3 of lemma 3.1. This means that ψ and ψ_1 used in the construction of φ may be chosen independently of ϱ', and an examination of the preceeding estimates shows that the constant C_0 exists.

6. Deduction of the main results.

We assume that D is a bounded domain, with C^2 strictly pseudoconvex boundary. Let K_q, $q=0,1,\ldots,n-1$, be the kernels constructed above. We have proved that

$$\int_D |K_q(\zeta,z)|\, d\lambda(\zeta) \leqq C$$

for all $z \in \overline{D}$, and

$$\int_D |K_q(\zeta,z)|\, d\lambda(z) \leqq C$$

for all $\zeta \in \overline{D}$, where C is a suitable constant.

Theorem 6.1. *If $u \in L^1_{(0,q)}(D)$, $q \geqq 1$, the integral $\int_D u(\zeta)\wedge K_{q-1}(\zeta,z)$ is absolutely convergent for almost all $z \in \overline{D}$. The operator T_q, defined by*

$$T_q u(z) = \int_D u(\zeta)\wedge K_{q-1}(\zeta,z)\,,$$

maps $L^p_{(0,q)}(D)$ into $L^p_{(0,q-1)}(D)$ with norm $\leqq C$, when $1 \leqq q \leqq n$ and $1 \leqq p \leqq \infty$.

Proof. If $u \in L^1_{(0,q)}(D)$,

$$\int_D |u(\zeta)| \left(\int_D |K_{q-1}(\zeta,z)|\, d\lambda(z)\right) d\lambda(\zeta) \leqq C\, \|u\|_1 .$$

As

$$|u(\zeta)\wedge K_{q-1}(\zeta,z)| \leqq |u(\zeta)|\, |K_{q-1}(\zeta,z)|\,,$$

we get from Fubini's theorem that $T_q(z)$ converges absolutely for almost all $z \in \overline{D}$, and

$$\int_D |T_q u(z)|\, d\lambda(z) \leqq \int_{D\times D} |K_{q-1}(\zeta,z)|\; |u(\zeta)|\; d\lambda(\zeta)\, d\lambda(z) \leqq C\|u\|_1 \,.$$

On the other hand, if $u \in L^\infty_{(0,q)}(D)$, then

$$\left| \int_D u(\zeta) \wedge K_{q-1}(\zeta,z) \right| \leqq \int_D |u(\zeta)|\; |K_{q-1}(\zeta,z)|\; d\lambda(\zeta) \leqq C\|u\|_\infty \,,$$

so $\|T_q\|_p \leqq C$ for $p=1$ or ∞. The result for all p between 1 and ∞ follows by the M. Riesz–Thorin theorem. (See [2, ch. VI, theorem (10.11)]).

Remark. Combining the result above with proposition 5.2, we see that we can use a fixed C for all domains D' that are sufficiently close to D in the C^2-sense (with appropriate kernels on D').

Proposition 6.2. *T_q maps $L^\infty_{(0,q)}(D)$ into $C_{(0,q-1)}(\bar{D})$, when $q \geqq 1$.*

Proof. If $\varkappa \in C_0^\infty(C^n)$, with $\varkappa$ identically one near 0, we define

$$K^{(k)}_{q-1}(\zeta,z) = \big(1-\varkappa(k(\zeta+z))\big)\, K_{q-1}(\zeta,z) \,.$$

Let the corresponding integral operator be $T_q^{(k)}$. As $K^{(k)}_{q-1}$ is continuous on $\bar{D}\times\bar{D}$, $T_q^{(k)}$ has the wanted property. When $u \in L^\infty_{(0,q)}(D)$, it follows from (5.4) and the inequality

$$|T_q u(z) - T_q^{(k)}(z)| \leqq \left(\int_{k|\zeta-z| \in \operatorname{supp}\varkappa} |K_{q-1}(\zeta,z)|\; d\lambda(\zeta) \right) \|u\|_\infty$$

that $T_q^{(k)} u \to T_q u$ uniformly on $\bar{D}$.

When $\varphi \in C^1(D, C^\infty(D))$, it follows from the construction of the kernels, that for all multi-indices α, $D_z^\alpha K_q(\zeta,z)$ exists and is continuous in $D\times D \setminus \Delta$. In this case we have

Proposition 6.3. *T_q maps $C^k_{(0,q)}(D)$ into $C^k_{(0,q-1)}(D)$, for $1 \leqq q \leqq n$ and $0 \leqq k \leqq \infty$.*

Proof. Assume that $u \in C^k_{(0,q)}(D)$, and $z \in D$. Choose $\varkappa \in C_0^\infty(D)$, with $\varkappa = 1$ near z, and so small support that $K_{q-1}(\zeta,z') = B_{q-1}(\zeta,z')$ when $\zeta \in \operatorname{supp}\varkappa$, and z' is close to z. By the remark preceding theorem 2.2,

$$D^\alpha T_q u(z) = \int_D D_\zeta^\alpha(\varkappa(\zeta)\cdot u(\zeta)) \wedge K(b)_{q-1}(\zeta,z) + \int_D (1-\varkappa(\zeta))u(\zeta) \wedge D_z^\alpha K_{q-1}(\zeta,z).$$

Finally, we would like to extend the formulas to more general forms. We have

PROPOSITION 6.4. *If* $u \in L^1_{(0,q)}(D)$ *and* $\bar\partial u \in L^1_{(0,q+1)}(D)$; $q \geqq 1$, *then*

$$u = C_q[\bar\partial T_q u - T_{q+1}\bar\partial u] .$$

PROOF. It is well known how to construct a sequence $u_k \in C^\infty_{(0,q)}(\bar D)$, such that $\|u_k - u\|_1$ and $\|\bar\partial u_k - \bar\partial u\|_1 \to 0$ as $k \to \infty$. (If u is supported by a small neighbourhood of $w \in \partial D$, we take $u = \varphi_k * u$, when $\varphi_k(z) = k^{2n}\varphi(kz)$ and $\varphi \in C_0{}^\infty(B(0,1))$ is supported by a narrow come around the exterior normal to ∂D at w. By the definition of weak derivatives, $\bar\partial u_k = \varphi_k * (\bar\partial u)$ in D. The general case follows by a partition of unity argument.) By theorem 6.1,

$$T_q u_k \to T_q u \quad \text{in } L^1_{(0,q-1)}(D) ,$$

and the formula follows by the continuity of

$$\bar\partial + T_{q+1}\colon\ (v,w) \to \bar\partial v + T_{q+1} w$$

from $L^1_{(0,q-1)}(D) \oplus L^1_{(0,q+1)}(D)$ into $\mathscr{D}'_{(0,q)}(D)$.

COROLLARY. *Assume* $q \geqq 1$, $u \in L^1_{(0,q)}(D)$, *and* $\bar\partial u = 0$. *Then*

$$\bar\partial(C_q T_q u) = u \text{ in } L^1_{(0,q)}(D).$$

In exactly the same way, we prove

PROPOSITION 6.5. *If* $u \in C(\bar D)$, *and* $\bar\partial(u|_D) \in L^1_{(0,1)}(D)$, *we have*

$$u(z) = C_0\left[\int_{\partial D} u(\zeta)\, K_0(\zeta,z) - \int_D \bar\partial u(\zeta) \wedge K_0(\zeta,z)\right]$$

almost everywhere in D.

ADDED IN PROOF. A detailed version of [10], giving sup-norm and Hölder estimates for the $\bar\partial$-complex in strictly pseudoconvex domains, has appeared in Math. Ann. 190 (1970–71), 6–44, while G. M. Henkin (Uspehi Mat. Nauk 26 (1971) nr. 3, 211–12) and P. L. Polyakov (ibid. nr. 4, 243–44) have announced sup-norm estimates for the $\bar\partial$-complex in certain analytic polyhedra.

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