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ON THE μ^i IN A MINIMAL INJECTIVE RESOLUTION

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A will always denote a (commutative) noetherian local ring with maximal ideal $\mathfrak{m}$ and residue class field $k=A/\mathfrak{m}$.

Introduction.

Let M be a finitely generated A-module, and let

$$0 \to M \to E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \ldots \to E^i \xrightarrow{d^i} \ldots$$

be a minimal injective resolution of M, that is, E^i is the injective envelope of $\operatorname{Ker} d^i = \operatorname{Im} d^{i-1}$ for all i. Then the cardinal number $\mu^i(\mathfrak{p},M)$, where $\mathfrak{p}$ is a prime ideal, denotes the number of indecomposable components of E^i isomorphic to $E(A/\mathfrak{p})$. We know that $\mu^i(\mathfrak{p},M)$ is finite and is only depending on i, $\mathfrak{p}$, and M (cf. Hyman Bass [1, §2]).

The depth (or co-dimension) of M turns out to be the smallest integer i for which $\mu^i(\mathfrak{m},M)>0$, while the injective dimension is the greatest i for which $\mu^i(\mathfrak{m},M)>0$ (if such an i exists). This means that if $E(k)$ is a direct summand in E^i (that is, $\mu^i(\mathfrak{m},M)>0$), then i must lay between the depth and the injective dimension of M.

The first part of this paper deals with the question: is $E(k)$ a direct summand in E^i for all i between the depth and the injective dimension? Since $\mu^i(\mathfrak{p},M)=\mu^i(\mathfrak{p}A_\mathfrak{p},M_\mathfrak{p})$, the answer of this question also tells us for which i, $E(A/\mathfrak{p})$ is a direct summand in E^i. Unfortunately I have only been able to answer the question, in the affirmative, in the following cases:

1) A is Cohen–Macaulay,
2) $\operatorname{depth} M \geqq \operatorname{depth} A$,
2′) $M=A$,
3) M is Cohen–Macaulay,
4) $\operatorname{inj\,dim} M$ finite.

In case 4) holds $\varphi(M)=\sum_i(-1)^{s-i}\mu^i(\mathfrak{m},M)\geqq 0$, where $s=\operatorname{inj\,dim} M$, and the following statements are equivalent:

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(i) $\varphi(M) > 0$, (ii) $\operatorname{Ann} M = 0$, (iii) $\operatorname{grad} M = 0$.

This is the analogue to the corresponding result about the Euler-characteristic of a finitely generated module of finite projective dimension.

In the second part (section 3) we will assume that there exists a Gorenstein module ([11] and [12]). Then μ^i of a Cohen–Macaulay module or a finitely generated module of finite injective dimension can be expressed by some Betti-numbers.

1. Notation and previous results.

NOTATION 1.1. Let L and M be finitely generated non-zero A-modules. Then the following notation will be used:

$\operatorname{grade}_M L$ = maximal length of a M-regular sequence in $\operatorname{Ann} L$. (Elements $a_1, \ldots, a_s \in \mathfrak{m}$ are called a M-regular sequence, if each a_i is not a zero-divisor in $M/(a_1, \ldots, a_{i-1})M$.)

$\operatorname{depth} M = \operatorname{grade}_M k$.

$\dim M$ = the Krull-dimension of M, that is (in this case), the Krull-dimension of the local ring $A/\operatorname{Ann} M$.

M Cohen–Macaulay, if $\operatorname{depth} M = \dim M$ (and M non-zero and finitely generated).

zM = the zero-divisors on M.

$\mu^i(M) = \mu^i(\mathfrak{m}, M)$ as defined in the introduction.

$\beta_i(M)$ = the ith Betti-number of M, that is, the dimension of $\operatorname{Tor}_i(k, M)$ considered as a vector space over k.

The results in Bourbaki [2, Chap. IV] are considered as well known and will hence be used without reference.

Some of the previous results, mainly from Bass [1, §§ 2–3], can be summarized in

PROPOSITION 1.2. *Let L and M be finitely generated non-zero A-modules. Then:*

(a) *$\mu^i(M)$ is the dimension of $\operatorname{Ext}^i(k, M)$ considered as a vector space over k. Further, $\mu^i(\mathfrak{p}, M) = \mu^i_{A_\mathfrak{p}}(M_\mathfrak{p})$ for all prime ideals $\mathfrak{p}$.*

(b) $\operatorname{depth} M = \inf\{i \mid \mu^i(M) > 0\} \leqq \dim M$
$\leqq \sup\{i \mid \mu^i(M) > 0\} = \operatorname{inj\,dim} M \quad (\leqq \infty)$.

(c) *$\mu_A{}^i(M) = \mu^{i-1}_{A/(a)}(M/aM)$ for all i, if $a \in \mathfrak{m} - (zA \cup zM)$.*

(d) *If $\operatorname{Ext}^i(L, M) \neq 0$ for some i and length $L < \infty$, then $\mu^i(M) > 0$.*

(e) *If $\operatorname{inj\,dim} M = \infty$, then $\mu^i(M) > 0$ for $i \geqq \dim A$.*

(f) *If* $\operatorname{inj\,dim} M < \infty$, *then* $\operatorname{inj\,dim} M = \operatorname{depth} A$, *and*

(g) $\operatorname{depth} A = \operatorname{depth} L + \sup\{i \mid \operatorname{Ext}^i(L,M) \neq 0\}$.

(h) $\operatorname{grade}_M L = \inf\{i \mid \operatorname{Ext}^i(L,M) \neq 0\} =$ (*length of a maximal* M*-regular sequence in* $\operatorname{Ann} L$) $= \operatorname{grade}_M A/\mathfrak{a}$, *where* $\mathfrak{a}$ *is the radical of* $\operatorname{Ann} L$.

(i) $\operatorname{depth} A \leqq \operatorname{grade}_A M + \dim M \leqq \dim A$.

For (g) and (i) we refer to [7], [8] and [9]:

(g): see [7, § 2] or [8, Theorem 3.1] (=[9, Theorem 1.1]).

(i): see [8, Lemma 3.8] (=[9, Lemme 1.4]).

Also the following results will be useful:

PROPOSITION 1.3. *Let* M *be a finitely generated non-zero* A*-module. Then*:

(a) *If*
$$\ldots \to F_i \xrightarrow{d_i} \ldots \to F_1 \xrightarrow{d_1} F_0 \to M \to 0$$
is a minimal free resolution of M (*that is,* $\operatorname{Im} d^i \subseteq \mathfrak{m}F_{i-1}$ *for all* i), *then the rank of* F_i *is* $\beta_i(M)$ *for all* i.

(b) $\operatorname{proj\,dim} M = \sup\{i \mid \beta_i(M) > 0\} = \inf\{i \geqq 0 \mid \beta_{i+1}(M) = 0\}$.

(c) *If* $\operatorname{proj\,dim} M < \infty$, *then the Euler-characteristic* $\chi(M) \geqq 0$ (*where* $\chi(M) = \sum_i (-1)^i \beta_i(M)$), *and the following statements are equivalent*:

$$\text{(i) } \chi(M) > 0, \quad \text{(ii) } \operatorname{Ann} M = 0, \quad \text{(iii) } \operatorname{grade}_A M = 0.$$

PROOF. (a) and (b): see Serre [10, IV appendix I]. (c): see Kaplansky [6, section 4-3].

2. Non-vanishing of μ^i.

PROPOSITION 2.1. *Let* A *be Cohen–Macaulay, and let* M *be a non-zero finitely generated* A*-module. Then*

$$\mu^i(M) > 0 \quad \textit{for} \quad \operatorname{depth} M \leqq i \leqq \operatorname{inj\,dim} M (\leqq \infty).$$

PROOF. By Proposition 1.2 (e) and (f) it is enough to show $\mu^i(M) > 0$ for depth $M \leqq i < s = \operatorname{depth} A = \dim A$. The proof will be by induction on s:

$s = 0$: nothing to prove.
$s-1 \to s$, $s \geqq 1$ (the inductive step): Divide in two cases:

1°. $\operatorname{depth} M > 0$. With $a \in \mathfrak{m} - (zA \cup zM)$ one obtains by 1.2 (c) that $\mu_A{}^i(M) = \mu^{i-1}_{A/(a)}(M/aM)$. The inductive hypothesis gives the desired assertion.

2°. $\operatorname{depth} M = 0$. Choose F finitely generated free and a submodule K of F such that $F/K = M$. The corresponding long-exact sequence for $\operatorname{Ext}(k, -)$ gives us $\operatorname{Ext}^i(k,M) = \operatorname{Ext}^{i+1}(k,K)$ for $i \leqq s-2$, and hence depth $K = 1 > 0$ and $\mu^i(M) = \mu^{i+1}(K) > 0$ for $0 \leqq i \leqq s-2$ (by case 1°).

The only thing left to show is now $\mu^{s-1}(M) > 0$.

Let $a_1, \ldots, a_s$ be an A-regular sequence such that $\operatorname{depth} M_i = 0$ for all i, $0 \leqq i \leqq s$, where $M_i = M/\mathfrak{a}_i M$ and $\mathfrak{a}_i = (a_1, \ldots, a_i)$, $\mathfrak{a}_0 = 0$. To see that this is possible assume that $\mathfrak{a}_{i-1} = (a_1, \ldots, a_{i-1})$ has been chosen. Pick $a \in \mathfrak{m} - z(A/\mathfrak{a}_{i-1})$, $x \in M_{i-1}$ such that $\operatorname{Ann} x = \mathfrak{m}$ (since $\operatorname{depth} M_{i-1} = 0$), and $t > 0$ such that $x \notin a^t M_{i-1}$. Put $a_i = a^t \in \mathfrak{m} - z(A/\mathfrak{a}_{i-1})$. Then

$$\mathfrak{m} = \operatorname{Ann} x \subseteq \operatorname{Ann} \bar{x} \subseteq \mathfrak{m},$$

where $\bar{x}$ is the residue class of x in $M_{i-1}/a_i M_{i-1} = M_i$. This gives $\operatorname{Ann} \bar{x} = \mathfrak{m}$, that is, $\operatorname{depth} M_i = 0$.

Since $\operatorname{proj\,dim} A/\mathfrak{a}_{i-1} = i-1$, there is an exact sequence

$$\operatorname{Ext}^{i-1}(A/\mathfrak{a}_{i-1}, M) \xrightarrow{a_i \cdot} \operatorname{Ext}^{i-1}(A/\mathfrak{a}_{i-1}, M) \to \operatorname{Ext}^i(A/\mathfrak{a}_i, M) \to 0 .$$

Now induction on i shows: $\operatorname{Ext}^i(A/\mathfrak{a}_i, M) = M_i$. In particular

$$a_s \in zM_{s-1} = z\operatorname{Ext}^{s-1}(A/\mathfrak{a}_{s-1}, M),$$

so from the exact sequence

$$\operatorname{Ext}^{s-1}(A/\mathfrak{a}_s, M) \to \operatorname{Ext}^{s-1}(A/\mathfrak{a}_{s-1}, M) \xrightarrow{a_s \cdot} \operatorname{Ext}^{s-1}(A/\mathfrak{a}_{s-1}, M)$$

it follows that $\operatorname{Ext}^{s-1}(A/\mathfrak{a}_s, M) \neq 0$, and hence $\mu^{s-1}(M) > 0$ (by Proposition 1.2 (d)), since the length of $A/\mathfrak{a}_s$ is finite.

Lemma 2.2. *If* $\operatorname{depth} A = 0$, *there exists* $t \geqq 0$ *such that* k *is a direct summand in* $\mathfrak{m}^t$.

Proof. Since $\operatorname{depth} A = 0$, $\operatorname{Ann} \mathfrak{m} \neq 0$. Choose $a_0 \in \operatorname{Ann} \mathfrak{m} - 0$ and $t \geqq 0$ such that $a_0 \in \mathfrak{m}^t - \mathfrak{m}^{t+1}$. Expand $\bar{a}_0 \in \mathfrak{m}^t/\mathfrak{m}^{t+1}$ to a basis $(\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_p)$ for the vector space $\mathfrak{m}^t/\mathfrak{m}^{t+1}$ over k. Then $(a_0, a_1, \ldots, a_p) = \mathfrak{m}^t$ (by Nakayama's lemma) and $\sum_i b_i a_i = 0$ implies that all $b_i \in \mathfrak{m}$.

If $b_0 a_0 = \sum_{i>0} b_i a_i \in (a_0) \cap (a_1, \ldots, a_p)$, then $b_0 \in \mathfrak{m}$, that is, $b_0 a_0 = 0$. This shows $(a_0) \cap (a_1, \ldots, a_p) = 0$, and hence

$$\mathfrak{m}^t = (a_0) \oplus (a_1, \ldots, a_p) = k \oplus (a_1, \ldots, a_p) .$$

PROPOSITION 2.3. *Let M be a finitely generated non-zero A-module with* $\operatorname{depth} M \geqq \operatorname{depth} A$. *Then*

$$\mu^i(M) > 0 \quad \textit{for} \quad \operatorname{depth} M \leqq i \leqq \operatorname{inf}\dim M (\leqq \infty)\,,$$

in particular,

$$\mu^i(A) > 0 \quad \textit{for} \quad \operatorname{depth} A \leqq i \leqq \operatorname{inj}\dim A (\leqq \infty)\,.$$

PROOF. By Proposition 1.2 (c), assume $\operatorname{depth} A = 0$ and use induction on i:

$i-1 \to i$, $i \geqq 2$: From the exact sequence

$$0 \to \mathfrak{m}^t \to A \to A/\mathfrak{m}^t \to 0\,,$$

with $t \geqq 0$ such that $\mathfrak{m}^t = k \oplus \mathfrak{a}$ (by Lemma 2.2), it follows for $i \geqq 2$ that

$$\operatorname{Ext}^i(A/\mathfrak{m}^t, M) = \operatorname{Ext}^{i-1}(\mathfrak{m}^t, M) = \operatorname{Ext}^{i-1}(k, M) \oplus \operatorname{Ext}^{i-1}(\mathfrak{a}, M)\,.$$

By Proposition 1.2 (d) this gives $\mu^i(M) > 0$, if $\mu^{i-1}(M) > 0$ and $i \geqq 2$.

$i = 1$: By Proposition 1.2 (b), (f) and (e) we can assume:

$$\operatorname{depth} M = 0, \quad \operatorname{inj}\dim M = \infty, \quad \dim A > 0\,.$$

Choose a prime ideal $\mathfrak{p}$ with $\dim A/\mathfrak{p} = 1$ and $a \in \mathfrak{m} - \mathfrak{p}$. From the exact sequence

$$0 \to A/\mathfrak{p} \xrightarrow{a\cdot} A/\mathfrak{p} \to A/(\mathfrak{p} + (a)) \to 0$$

it follows that also

$$\operatorname{Hom}(A/\mathfrak{p}, M) \xrightarrow{a\cdot} \operatorname{Hom}(A/\mathfrak{p}, M) \to \operatorname{Ext}^1\big(A/(\mathfrak{p} + (a)), M\big)$$

is exact. Here $\operatorname{Hom}(A/\mathfrak{p}, M) \neq 0$, since $\operatorname{depth} M = 0$, and thus

$$\operatorname{Ext}^1\big(A/(\mathfrak{p} + (a)), M\big) \neq 0$$

(by Nakayama's lemma), that is, $\mu^1(M) > 0$ (by Proposition 1.2 (d)).

PROPOSITION 2.4. *If M is a Cohen–Macaulay A-module, then $\mu^i(M) > 0$ for* $\operatorname{depth} M = \dim M \leqq i \leqq \operatorname{inj}\dim M (\leqq \infty)$.

PROOF. Induction on $d = \operatorname{depth} M = \dim M$:

$d = 0$ is equivalent to $\operatorname{Supp} M = \{\mathfrak{m}\}$, that is, $\mu^i(\mathfrak{p}, M) = 0$ for all i and all prime ideals $\mathfrak{p} \neq \mathfrak{m}$. This gives $E(k)^{\mu^i(M)} = E^i \neq 0$ (that is, $\mu^i(M) > 0$) for $0 \leqq i \leqq \operatorname{inj\,dim} M$, when

$$0 \to M \to E^0 \to E^1 \to \dots \to E^i \to \dots$$

is a minimal injective resolution of M.

$d-1 \to d$, $d \geqq 1$: By Proposition 2.3 we can assume depth $A > 0$. If $a \in \mathfrak{m} - (zA \cup zM)$, then M/aM is a Cohen–Macaulay $A/(a)$-module of dimension $d-1$ (see Serre [10, IV B]). The inductive hypothesis and Proposition 1.2 (c) now gives us the desired result.

The next result is a corollary of Theorem 3.4 of Peskine and Szpiro [8]. The equivalence between (ii) and (iii) in (c) is in fact well known [8, Cor. 3.5 of Th. 3.4] and is also proved by Levin and Vasconcelos [7, Th. 4.1] using different methods.

THEOREM 2.5. *Let M be a non zero finitely generated A-module of finite injective dimension s. Then*:

(a) $\mu^i(M) > 0$ *for* $\operatorname{depth} M \leqq i \leqq \operatorname{inj\,dim} M = s$,
(b) $\varphi(M) = \sum_i (-1)^{s-i} \mu^i(M) \geqq 0$.
(c) *The following statements are equivalent*:
(i) $\varphi(M) > 0$, (ii) $\operatorname{Ann} M = 0$, (iii) $\operatorname{grade}_A M = 0$.

REMARK. As pointed out before, this is the analogue of Proposition 1.3, the theorem about the Euler-characteristic of a finitely generated module of finite projective dimension. In fact we will use this proposition to prove Theorem 2.5.

PROOF. Let ^ denote completion with respect to the $\mathfrak{m}$-adic topology. Then $\hat{A}$ is again noetherian local with residue class field k, and further we know:

$$(\operatorname{Ann}_A M)\hat{} = \operatorname{Ann}_{\hat{A}} \hat{M}, \quad zA = A \cap z\hat{A}, \quad \hat{M} = M \otimes_A \hat{A}$$

and $\hat{A}$ is A-flat (see Bourbaki [2, Chap. III, § 3, no. 4, th. 3 et cor. 1]). Let also L be a finitely generated A-module. Then $\operatorname{Hom}_A(L, M)\hat{} = \operatorname{Hom}_{\hat{A}}(\hat{L}, \hat{M})$ (by Bourbaki [2, Chap. I, § 2, no. 10, prop. 11]) and hence (by an easy induction on i) $\operatorname{Ext}_A{}^i(L, M)\hat{} = \operatorname{Ext}_{\hat{A}}{}^i(\hat{L}, \hat{M})$, in particular $\mu_A{}^i(M) = \mu_{\hat{A}}{}^i(\hat{M})$, since $\hat{k} = k$, and thus

$$\operatorname{depth}_A M = \operatorname{depth}_{\hat{A}} \hat{M} \quad \text{and} \quad \operatorname{inj\,dim}_A M = \operatorname{inj\,dim}_{\hat{A}} \hat{M}\,.$$

Therefore we may assume that A is complete.

Peskine and Szpiro ([8, Th. 3.4]) have proved that $\mathrm{Ext}^i(E(k),N)=0$ for $i \neq s$, and that $N=\mathrm{Ext}^s(E(k),M)$ is finitely generated with $\mathrm{Supp}\,M=\mathrm{Supp}\,N$ and $\mathrm{proj\,dim}\,N<\infty$.

We want to show $\mu^i(M)=\beta_{s-i}(N)$ for all i. Assume that this is done. Then by Proposition 1.3 (b)

$$\mathrm{proj\,dim}\,N = s-\mathrm{depth}\,M \quad \text{and} \quad \mu^i(M) = \beta_{s-i}(N) > 0$$

for $\mathrm{depth}\,M \leqq i \leqq s$. By Proposition 1.3 (c) we get

$$\varphi(M) = \textstyle\sum_i(-1)^{s-i}\mu^i(M) = \sum_j(-1)^j\beta_j(N) = \chi(N) \geqq 0\,.$$

Now (a) and (b) are established. We prove (c) as follows.

(i) $\Rightarrow$ (ii): By Proposition 1.3 (c), $\chi(N)=\varphi(M)>0$ implies that $0=\mathrm{Ann}\,N \supseteqq \mathrm{Ann}\,M$.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i): Assume $\mathrm{grade}_A M=0$. Since $\mathrm{Supp}\,N=\mathrm{Supp}\,M$, also $\mathrm{grade}_A N=0$ by Proposition 1.2 (h). By Proposition 1.3 (c) this gives

$$0 < \chi(N) = \varphi(M)\,.$$

Now follows the proof of $\mu^i(M)=\beta_{s-i}(N)$ for all i: Let X, Y and Z be A-modules. Then there is a homomorphism

$$\sigma\colon\ X\otimes\mathrm{Hom}(Y,Z) \to \mathrm{Hom}(\mathrm{Hom}(X,Y),Z)$$

defined by

$$\sigma(x\otimes f)(g) = fg(x), \quad x \in X\,,$$

$$X \xrightarrow{\ g\ } Y \xrightarrow{\ f\ } Z\,.$$

The homomorphism σ is natural in X and Z, and σ is an isomorphism, if X is finitely generated and free (cf. [3, VI, Prop. 5.2]). Assume X finitely generated and let

$$\ldots \to F_p \to \ldots \to F_1 \to F_0 \to X \to 0$$

be a free resolution of X with each F_i finitely generated. Let

$$0 \to Z \to E^0 \to E^1 \to \ldots \to E^q \to \ldots$$

be an injective resolution of Z and define a double complex $K^{\cdot\cdot}$ by

$$K^{pq} = F_{-p}\otimes\mathrm{Hom}(Y,E^q) = \mathrm{Hom}(\mathrm{Hom}(F_{-p},Y),E^q)$$

(and obvious differentiations). By standard filtrations one gets two spectral sequences with the same limit and initial terms:

$$(*)\qquad \begin{aligned} I_2^{pq} &= \operatorname{Tor}_{-p}(X, \operatorname{Ext}^q(Y,Z)) ,\\ II_2^{pq} &= \operatorname{Ext}^p(\operatorname{Ext}^{-q}(X,Y),Z) . \end{aligned}$$

Let $X=k$, $Y=E(k)$, and $Z=M$ in these spectral sequences:

$$\begin{aligned} I^{pq} &= \operatorname{Tor}_{-p}(k,N) && \text{for } q=s ,\\ &= 0 && \text{otherwise} ,\\ II^{pq} &= \operatorname{Ext}^p(k,M) && \text{for } q=0 ,\\ &= 0 && \text{otherwise} , \end{aligned}$$

and hence we have $\operatorname{Tor}_{s-i}(k,N)=\operatorname{Ext}^i(k,M)$ for all i.

This ends the proof of Theorem 2.5.

REMARK. There are more direct ways to prove $\mu^i(M)=\beta_{s-i}(N)$, but we shall use the spectral sequences (*) later.

PROPOSITION 2.6. *If M is a non-zero finitely generated A-module, then*:

$$\mu^i(M) > 0 \quad \textit{for} \quad \dim M \leqq i \leqq \operatorname{inj\,dim} M \;\; (\leqq \infty) .$$

PROOF. It is enough to show: $\mu^{i+1}(M)=0$, if $\mu^i(M)=0$ and $i \geqq \dim M$. Assume therefore $\mu^i(M)=0$ and $i \geqq \dim M$, and hence also $\operatorname{Ext}^i(L,M)=0$ for all A-modules L of finite length (by Proposition 1.2 (d)). This gives that $\operatorname{Ext}^{i+1}(\cdot,M)$ is a left-exact contra-variant functor in the category of A-modules of finite length and hence

$$\begin{aligned} \operatorname{Ext}^{i+1}(k,M) &= \operatorname{Hom}(k, \varinjlim_n \operatorname{Ext}^{i+1}(A/\mathfrak{m}^n,M)) \\ &= \operatorname{Hom}(k,H_{\mathfrak{m}}^{i+1}(M)) = 0 , \end{aligned}$$

where $H_{\mathfrak{m}}^{i}$ is the ith derived of the local cohomology functor (see Hartshorne [4, Prop. 4.5, Th. 2.8, and Prop. 6.4]).

REMARK. Proposition 2.4 follows of course also from Proposition 2.6.

3. μ^i and Gorenstein modules.

Assume in this section that *G is a Gorenstein module of dimension s*, that is (Sharp [11]), G is a finitely generated non-zero A-module with

$$s = \operatorname{depth} G = \operatorname{inj\,dim} G \;\; (<\infty) .$$

Then G is a Cohen–Macaulay module of dimension s (by Proposition 1.2 (b)).

Also the ring A is Cohen–Macaulay of dimension s (Sharp [11, 3.9]).

PROPOSITION 3.1. *Let M be a finitely generated non-zero A-module of finite injective dimension. Then:*

(a) $\mu^i(M)\mu^s(G) = \beta_{s-i}(\operatorname{Hom}(G,M))$ *for all i, and* $\operatorname{proj\,dim}\operatorname{Hom}(G,M) = s\text{-}\operatorname{depth} M < \infty$,

(a′) $\operatorname{Hom}(G,G)$ *is free of rank* $\mu^s(G)^2$,

(b) $\operatorname{grade}_M L = \operatorname{grade}_{\operatorname{Hom}(G,M)} L$ *for each non-zero finitely generated module L, and* $\operatorname{Ass} M = \operatorname{Ass}\operatorname{Hom}(G,M)$.

REMARK. The first part of (b) follows, in the case $M = G$, from Sharp [11, 4.11].

PROOF. Consider the spectral sequences (*) from the proof of Theorem 2.5 and let $X = k$, $Y = G$ and $Z = M$. Since $\operatorname{Ext}^q(G,M) = 0$ for $q \neq 0$ (by Proposition 1.2 (g)) and $\operatorname{Ext}^{-q}(k,G) = 0$ for $q \neq -s$, the spectral sequences degenerate to isomorphisms:

$$\operatorname{Tor}_{s-i}(k,\operatorname{Hom}(G,M)) = \operatorname{Ext}^i(\operatorname{Ext}^s(k,G),M) \quad \text{for all } i .$$

This shows (a) and (a′).

Since $\operatorname{Ann} G \subseteq \operatorname{Ann}\operatorname{Hom}(G,G) = \operatorname{Ann} A = 0$, it follows that

$$\operatorname{Ass}\operatorname{Hom}(G,M) = \operatorname{Supp} G \cap \operatorname{Ass} M = \operatorname{Ass} M .$$

For $a \in \operatorname{Ann} L - zM$ we have

$$\operatorname{Hom}(G,M/aM) = \operatorname{Hom}(G,M)/a\operatorname{Hom}(G,M)$$

and therefore the rest of (b) is an easy induction.

COROLLARY 3.2. *Let A be Gorenstein of dimension s, and let M be a non-zero finitely generated A-module. Then*

(i) $$\operatorname{inj\,dim} M < \infty ,$$

if and only if

(ii) $$\operatorname{proj\,dim} M < \infty ,$$

and when these conditions are satisfied, then

$$\mu^i(M) = \beta_{s-i}(M) \quad \textit{for all } i .$$

REMARK. The equivalence between (i) and (ii) is well-known [7, Th. 2.2].

A stronger result is (cf. Jensen [5, Cor. 5]): A is Gorenstein, if and only if every (not necessarily finitely generated) A-module M of finite injective dimension has finite projective dimension. In fact, for a module M over a Gorenstein ring, (i) and (ii) are equivalent with

(iii) $$\text{weak}\dim M < \infty.$$

LEMMA 3.3. *Let M be a finitely generated non-zero A-module. Then M is Cohen-Macaulay of dimension d, if and only if* $\operatorname{Ext}^i(M,G)=0$ *for* $i \neq s-d$.

PROOF. $\operatorname{Ext}^i(M,G)=0$ for $i \neq s-d$, if and only if $\operatorname{depth} M = d$ and $\operatorname{grade}_G M = s-d$ (by Proposition 1.2 (g) and (h)). By Proposition 3.1 (b) it follows that $\operatorname{grade}_A M = \operatorname{grade}_G M$, and hence $\operatorname{depth} M = d$ and $\operatorname{grade}_G M = s-d$, if and only if M is Cohen–Macaulay of dimension d (by Proposition 1.2 (i), since A is Cohen–Macaulay).

PROPOSITION 3.4. *If M is a Cohen–Macaulay A-module of dimension d, then:*

(a) $\operatorname{Ext}^{s-d}(M,G)$ *is Cohen–Macaulay of dimension d too, and* $\operatorname{Ext}^{s-d}(\operatorname{Ext}^{s-d}(M,G),G) = M \otimes \operatorname{Hom}(G,G) = M^{\mu^s(G)^2}$.

(b) $\operatorname{Ass} M = \operatorname{Ass} \operatorname{Ext}^{s-d}(M,G)$.

REMARK. From Theorem 3.11 (v) of Sharp [11], (a) follows in the case $d=0$.

PROOF. Consider the spectral sequences (*) from the proof of Theorem 2.5 with $X=M$ and $Y=Z=G$. By Proposition 3.1 and Lemma 3.3 they degenerate to

$$\operatorname{Ext}^i(\operatorname{Ext}^{s-d}(M,G),G) = \begin{cases} M \otimes \operatorname{Hom}(G,G) & \text{for } i=s-d \\ 0 & \text{otherwise}, \end{cases}$$

and thus (a) follows from Lemma 3.3.

Let us now show $zM = z\operatorname{Ext}^{s-d}(M,G)$. Assume $a \in \mathfrak{m} - zM$. Then M/aM is Cohen–Macaulay of dimension $d-1$ (by Serre [10, IV B]), and hence $\operatorname{Ext}^{s-d}(M/aM,G)=0$. We have therefore the exact sequence

$$0 \to \operatorname{Ext}^{s-d}(M,G) \xrightarrow{a\cdot} \operatorname{Ext}^{s-d}(M,G),$$

and thus $a \notin z\operatorname{Ext}^{s-d}(M,G)$. Now $zM \subseteq z\operatorname{Ext}^{s-d}(M,G)$ is shown. The other inclusion follows by the duality of (a).

To show (b), it is enough to show $\mathrm{Ass}\, M \subseteq \mathrm{Ass}\,\mathrm{Ext}^{s-d}(M,G)$ (again by the duality of (a)). Assume therefore $\mathfrak{p} \in \mathrm{Ass}\, M$, that is,

$$\mathfrak{p} \subseteq zM = z\mathrm{Ext}^{s-d}(M,G)\ ,$$

and hence there exist $\mathfrak{p}' \in \mathrm{Ass}\,\mathrm{Ext}^{s-d}(M,G)$ such that $\mathfrak{p} \subseteq \mathfrak{p}'$ and $\mathfrak{q} \in \mathrm{Ass}\, M$ such that $\mathfrak{p}' \subseteq \mathfrak{q}$. We have $\mathfrak{p} \subseteq \mathfrak{p}' \subseteq \mathfrak{q}$, that is,

$$\mathfrak{p} = \mathfrak{p}' \in \mathrm{Ass}\,\mathrm{Ext}^{s-d}(M,G)\ ,$$

since $\mathrm{Ass}\,\mathrm{Ext}^{s-d}(M,G)$ is without embedded primes (see Prop. 13 of [10, IV B]).

PROPOSITION 3.5. *Let M be a Cohen–Macaulay A-module of dimension d. Then*

(a) $\mu^i(M)\mu^s(G) = \beta_{i-d}(\mathrm{Ext}^{s-d}(M,G))$ *and*
$\beta_i(M)\mu^s(G) = \mu^{i+d}(\mathrm{Ext}^{s-d}(M,G))$ *for all i,*
(b) $\mathrm{inj\,dim}\, M < \infty \Leftrightarrow \mathrm{proj\,dim}\,\mathrm{Ext}^{s-d}(M,G) < \infty$ *and*
$\mathrm{proj\,dim}\, M < \infty \Leftrightarrow \mathrm{inj\,dim}\,\mathrm{Ext}^{s-d}(M,G) < \infty$.

PROOF. Consider the spectral sequences (*) with $X = k$, $Y = M$, and $Z = G$. They degenerate to isomorphisms:

$$\mathrm{Tor}_{i-d}(k, \mathrm{Ext}^{s-d}(M,G)) = \mathrm{Ext}^s(\mathrm{Ext}^i(k,M),G) \quad \text{for all } i\ .$$

This gives the first part of (a). The second part follows from the first and Proposition 3.4 (a).

(b) is now obvious.

COROLLARY 3.6. *Let A be a Gorenstein ring of dimension s, and $l_A t$ M be a Cohen–Macaulay A-module of dimension d. Then*:

(a) $\mathrm{Ext}^{s-d}(M,A)$ *is Cohen–Macaulay of dimension d and*
$\mathrm{Ext}^{s-d}(\mathrm{Ext}^{s-d}(M,A),A) = M$.
(b) $\mu^i(M) = \beta_{i-d}(\mathrm{Ext}^{s-d}(M,A))$ *and*
$\beta_i(M) = \mu^{i+d}(\mathrm{Ext}^{s-d}(M,A))$ *for all i.*
(c) $\mathrm{inj\,dim}\, M < \infty \Leftrightarrow \mathrm{inj\,dim}\,\mathrm{Ext}^{s-d}(M,A) < \infty \Leftrightarrow$
$\mathrm{proj\,dim}\, M < \infty \Leftrightarrow \mathrm{proj\,dim}\,\mathrm{Ext}^{s-d}(M,A) < \infty$.

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