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HOMOGENEOUS UNIVERSAL MODULES

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Introduction.

The purpose of this paper is to improve the theorem of B. Jónsson [6], [7] on the existence of homogeneous universal structures in the case where the class of structures considered is the class of modules over a fixed ring.

We prove that for any ring Λ and any infinite cardinal $\varkappa > \operatorname{Card}\Lambda$, there exists a homogeneous universal Λ-module of cardinality $\varkappa$ if and only if $\varkappa = \varkappa^{\underline{\gamma}}$, where γ is the smallest cardinal such that every ideal of Λ has a basis of cardinality $<\gamma$. (Theorem 3. This and the other results do *not* depend on the Generalized Continuum Hypothesis.)

If a homogeneous universal module of cardinality $> \operatorname{Card}\Lambda$ exists, then it is injective (Lemma 3). Thus the injective modules will play an important role in our considerations. In particular, we make use of the following result on the cardinality of the injective envelope of a module:

> For any infinite $\varkappa \geqq \operatorname{Card}\Lambda$, every module of cardinality $\varkappa$ has injective envelope of cardinality $\varkappa$ if and only if $\varkappa = \varkappa^{\underline{\gamma}}$ (Theorem 2).

The proof of the above result uses an equational definition of injective; this definition and other preliminaries are discussed in Section 1. The main theorems are proved in Section 2.

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1. Preliminaries.

Throughout this paper Λ denotes a fixed associated ring with $1 \neq 0$. The word "ideal" means left ideal of Λ; "module" means left Λ-module;

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"homomorphism" means Λ-module homomorphism; and "embedding" means one-one homomorphism.

Following Jónsson [6], [7], we call a module M *universal* if every module of cardinality $\leqq \operatorname{Card} M$ is isomorphic to a submodule of M. M is called *homogeneous* if every isomorphism $f: N \to N'$ of submodules of M of cardinality $< \operatorname{Card} M$ extends to an automorphism of M. If $\varkappa > \operatorname{Card} \Lambda$ it can be proved, as in Theorem B of [7], that any two homogeneous universal modules of cardinality $\varkappa$ are isomorphic.

We denote by γ the smallest cardinal such that every ideal of Λ may be generated (as Λ-module) by strictly fewer than γ elements; e.g. Λ is noetherian if and only if $\gamma \leqq \aleph_0$. Note that $\gamma \leqq (\operatorname{Card} \Lambda)^+$, the successor of $\operatorname{Card} \Lambda$.

We recall some definitions from [5]. If M is a module and α a cardinal, an *α-system* in M is a set of strictly fewer than α equations in a single variable x, each of the form $\lambda x = a$, where $\lambda \in \Lambda$, $a \in M$. An α-system in M is *consistent* in M if it has a solution in an extension of M. M is called *α-injective* if every consistent α-system in M has a solution in M. A proof of the following is given in [5].

LEMMA 1. *An α-system $\mathscr{S} = \{\lambda_\nu x = a_\nu : \nu < \beta < \alpha\}$ in a module M is consistent in M $\langle = \rangle$ whenever $\sum \varrho_\nu \lambda_\nu = 0$ then $\sum \varrho_\nu a_\nu = 0$.*

It follows from Lemma 1 that if $M \subseteq N$, an α-system in M is consistent in M if and only if it is consistent in N (as an α-system in N). This may also be proved directly from the definition of consistency, using the fact that the class of modules satisfies the "amalgamation property", that is, if $M_0 \subseteq M_1$, $M_0 \subseteq M_2$, then there exists M_3 such that $M_1 \subseteq M_3$, $M_2 \subseteq M_3$ and

$$M_0 \begin{smallmatrix} \subseteq M_1 \subseteq \\ \subseteq M_2 \subseteq \end{smallmatrix} M_3$$

commutes (cf. [5, §2]).

A module M is called *injective* if it is a direct summand of every module containing it. The following is proved in [5].

LEMMA 2. *M is injective if and only if M is γ-injective.*

The reason for our interest in injectives is given by:

LEMMA 3. *If M is a homogeneous universal module of infinite cardinality $\varkappa > \operatorname{Card} \Lambda$, then M is injective.*

Proof. Let $\mathscr{S} = \{\lambda_\nu x = a_\nu : \nu < \beta\}$ be a consistent γ-system in M (so that $\beta < \gamma \leqq \varkappa$). Let N be the submodule of M generated by $\{a_\nu : \nu < \beta\}$. Then $\operatorname{Card} N < \varkappa$ and there is an extension N' of N containing a solution b of $\mathscr{S}$. We can assume $\operatorname{Card} N' < \varkappa$. Since M is universal there is an embedding $f : N' \to M$, and since M is homogeneous the isomorphism $f|N : N \to f(N)$ extends to an automorphism g of M. It is clear that $g^{-1}f(b)$ is a solution of $\mathscr{S}$ in M.

An injective module E containing N is an *injective envelope* of N if any embedding of N into an injective E' extends to an embedding of E into E'. A module M containing N is an *essential extension* of N if for every non-zero submodule P of M, $P \cap N \neq \{0\}$.

Lemma 4 (Eckmann–Schopf [4]). (i) *An injective module E containing N is an injective envelope of N if and only if it is an essential extension of N.*

(ii) *Every module M has an injective envelope, which is unique up to isomorphism over M.*

For every module M, $E(M)$ will denote a fixed injective envelope of M. By *abus de langage* we say that $E(M)$ is *the* injective envelope of M.

The remainder of this section will not be used in the proof of the main results of Section 2. We want to remark that Lemma 3 is a special case of a more general result. Let $\mathscr{M}$ be the class of models of a first order theory T; and let λ be the cardinality of the language L of T. A structure $M \in \mathscr{M}$ is said to be *α-algebraically closed* (relative to $\mathscr{M}$) if every set of equations (i.e. atomic formulas) of cardinality $< \alpha$ with constants from M which has a solution in an extension P of M, where $P \in \mathscr{M}$, also has a solution in M. M is said to be *α-existentially closed* (relative to $\mathscr{M}$) if every set of equations and inequations of cardinality $< \alpha$ with constants from M which has a solution in an extension P of $\mathscr{M}$, where $P \in \mathscr{M}$, also has a solution in M. (Cf. [5, §7] where $\aleph_0$-algebraically closed and $\aleph_0$-existentially closed are defined).

The notions of $\mathscr{M}$-universal and $\mathscr{M}$-homogeneous structures are defined as in [6] and [7]. The proof of the following is similar to that of Lemma 3.

Proposition 1. *Let $M \in \mathscr{M}$ be an $\mathscr{M}$-homogeneous $\mathscr{M}$-universal structure of cardinality $\alpha > \lambda$. Then M is α-existentially closed.*

Corollary 1. *Suppose that T has a model companion T^** (see [5, § 2]). *Then every $\mathscr{M}$-homogeneous $\mathscr{M}$-universal structure of cardinality $\alpha > \lambda$ is a model of T^*.*

PROOF. It is shown in [5] that the models of T^* are precisely the models of T which are $\aleph_0$-existentially closed.

Corollary 1 is also a consequence of the following observation of G. Sabbagh.

PROPOSITION 2. *Any two infinite $\mathscr{M}$-homogeneous $\mathscr{M}$-universal structures M_1 and M_2 such that* $\operatorname{Card} M_1 > \lambda$, $\operatorname{Card} M_2 > \lambda$ *satisfy the same sentences of $L_{\infty\omega}$. If $M_1 \subseteq M_2$, then $M_1 \prec_{\infty\omega} M_2$.*

The proof is an easy consequence of Karp's criterion [8].

E. Fisher has pointed out the following corollary (a generalization of Corollary 1).

COROLLARY 2. *If $T^* \supset T$ is inductive and model-consistent relative to T, then every $\mathscr{M}$-homogeneous $\mathscr{M}$-universal structure of cardinality $> \lambda$ is a model of T^*.*

PROOF. We construct a chain

$$M = M_1 \subseteq N_1 \subseteq M_2 \subseteq N_2 \subseteq \ldots ,$$

where each M_i is a copy of M and each N_i is a model of T^* containing M_i and such that $\operatorname{Card} N_i = \operatorname{Card} M_i$. The union is an elementary extension of M and a model of T^*.

2. Homogeneous universal modules.

We will make use of the following notation (cf. [1, § 34]). If $\varkappa$ and α are cardinals

$$\varkappa^{\underline{\alpha}} = \sum_{\beta<\alpha} \varkappa^{\beta} \quad \text{(cardinal exponentiation)} .$$

THEOREM 1. *Let $\varkappa$ be an infinite cardinal $\geqq \operatorname{Card} \Lambda$. Let M be a module of cardinality $\varkappa$ and $E(M)$ the injective envelope of M. Then*

$$\operatorname{Card} E(M) \leqq \begin{cases} \varkappa^{\underline{\gamma}} & \text{if } \gamma \text{ is regular} , \\ \varkappa^{\gamma} & \text{if } \gamma \text{ is singular} . \end{cases}$$

Furthermore if $\varkappa^{\underline{\gamma}} = \varkappa$, then $\operatorname{Card} E(M) = \varkappa$.

PROOF. We first prove:

(*) Every module N of cardinality $\varkappa \geqq \operatorname{Card} A$ may be embedded in a module N^* of cardinality $\varkappa^{\underline{\gamma}}$ such that every consistent γ-system in N has a solution in N^*.

Let $\{\mathscr{S}_\beta : \beta < \delta\}$ be an enumeration of all the consistent γ-systems in N. It is clear that $\delta \leqq \varkappa^{\underline{\gamma}}$. We construct N^* as the union $\bigcup_{\beta<\delta} N_\beta$ of an increasing sequence of modules. Define $N_0 = N$, $N_\lambda = \bigcup_{\beta<\lambda} N_\beta$ if λ is a limit ordinal, and $N_{\beta+1}$ = an extension of N_β of cardinality $= \operatorname{Card} N_\beta$ in which $\mathscr{S}_\beta$ has a solution. It is easily seen that N^* has the desired property.

Now, given M of cardinality $\varkappa \geqq \operatorname{Card} A$ define $M_0 = M$, $M_{\nu+1} = (M_\nu)^*$ for any ordinal ν, $M_\lambda = \bigcup_{\nu<\lambda} M_\nu$ for any limit ordinal λ. Let ϱ be the first regular infinite cardinal $\geqq \gamma$ and let $E = \bigcup_{\nu<\varrho} M_\nu$. Then it is easy to see that E is γ-injective and hence injective (Lemma 2). Now if γ is regular, $\varrho = \gamma + \aleph_0 \leqq \varkappa^{\underline{\gamma}}$; and if γ is singular then $\gamma \leqq \operatorname{Card} A \leqq \varkappa$, so that $\varrho = \gamma^+ \leqq \gamma^\gamma \leqq \varkappa^\gamma$. Therefore, since

$$\begin{aligned}(\varkappa^{\underline{\gamma}})^{\underline{\gamma}} &= \varkappa^{\underline{\gamma}} && \text{if } \gamma \text{ is regular}\,,\\ &= \varkappa^{\gamma} && \text{if } \gamma \text{ is singular}\end{aligned}$$

([1, § 34, Satz 7]), we have

$$\begin{aligned}\operatorname{Card} E &\leqq \varkappa^{\underline{\gamma}} && \text{if } \gamma \text{ is regular}\,,\\ &\leqq \varkappa^{\gamma} && \text{if } \gamma \text{ is singular}\,.\end{aligned}$$

Furthermore if $\varkappa^{\underline{\gamma}} = \varkappa$ and γ is regular, then $\operatorname{Card} E \leqq \varkappa^{\underline{\gamma}} = \varkappa$; if $\varkappa^{\underline{\gamma}} = \varkappa$ and γ is singular, then $\varkappa = (\varkappa^{\underline{\gamma}})^{\underline{\gamma}} = \varkappa^\gamma$ so $\operatorname{Card} E \leqq \varkappa^\gamma = \varkappa$. The proof is complete.

Remark. Theorem 1 is a special case of a more general result. Let $\mathscr{M}$ be the class of models of an inductive (i.e. $\forall\,\exists$) first order theory T and let λ be the cardinality of the language L of T. Let α be an infinite cardinal. Then every $M \in \mathscr{M}$ can be embedded in a α-existentially closed structure $P \in \mathscr{M}$; furthermore if $\operatorname{Card}(M) = \varkappa \geqq \sup(\lambda, \alpha)$, we may choose P such that

$$\operatorname{Card}(P) \leqq \begin{cases} \varkappa^{\underline{\alpha}} & \text{if } \alpha \text{ is regular or if } \varkappa = \varkappa^{\underline{\alpha}}\,,\\ \varkappa^{\alpha} & \text{if } \alpha \text{ is singular}\,.\end{cases}$$

Theorem 2. *Let $\varkappa$ be an infinite cardinal $\geqq \operatorname{Card} A$. A necessary and sufficient condition that the injective envelope of every module of cardinality $\varkappa$ be of cardinality $\varkappa$ is that $\varkappa^{\underline{\gamma}} = \varkappa$.*

Proof. Sufficiency follows from Theorem 1. As for necessity, we have to prove that for any infinite cardinal $\alpha < \gamma$, $\varkappa^\alpha = \varkappa$. We prove this as follows. We construct a module M of cardinality $\varkappa$, and a consistent α^+-system $\mathscr{S} = \{\lambda_\nu x = a_\nu : \nu < \alpha\}$ in M such that there exist $\varkappa^\alpha$ automorphisms $\{f_\sigma : \sigma < \varkappa^\alpha\}$ of M with the property that if $\sigma \neq \sigma'$, f_σ differs from $f_{\sigma'}$ on an element of $\{a_\nu : \nu < \alpha\}$. Then for every $\sigma < \varkappa^\alpha$, $\mathscr{S}_\sigma$ defined by $\mathscr{S}_\sigma = \{\lambda_\nu x = f_\sigma(a_\nu) : \nu < \alpha\}$ is consistent in M (because of Lemma 1) and therefore has a solution b_σ in $E(M)$. But if $\sigma \neq \sigma'$, $b_\sigma \neq b_{\sigma'}$; and by hypothesis $\operatorname{Card} E(M) = \varkappa$; therefore $\varkappa^\alpha = \varkappa$.

We begin the construction of M. Since $\alpha < \gamma$, there is a sequence $\{\lambda_\nu : \nu < \alpha\}$ of elements of Λ such that for every $\nu < \alpha$, λ_ν is not in the ideal generated by $\{\lambda_\mu : \mu < \nu\}$. We will define inductively an ascending chain $\{M_\nu : \nu < \alpha\}$ of injective modules of cardinality $\leqq \varkappa$ and a sequence of elements $\{a_\nu : \nu < \alpha\}$ such that $a_\nu \in M_{\nu+1} - M_\nu$ and such that

$$\{\lambda_\mu x = a_\mu : \mu < \nu\}$$

is consistent in M_ν. Define $M_0 = \{0\}$; if λ is a limit ordinal define $M_\lambda = E(\bigcup_{\nu<\lambda} M_\nu)$, which is of cardinality $\leqq \varkappa$ by hypothesis. If M_ν and $\{a_\mu : \mu < \nu\}$ have been defined, let

$$A_{\nu+1} = (M_\nu \oplus \Lambda)/K_{\nu+1} ,$$

where $K_{\nu+1}$ is the set of elements of $M_\nu \oplus \Lambda$ of the form

$$(\textstyle\sum_{\mu<\nu} \varrho_\mu a_\mu, \varrho_\nu)$$

where $\sum_{\mu\leqq\nu} \varrho_\mu \lambda_\mu = 0$. We assert:

(1) the canonical map $\iota : M_\nu \to A_{\nu+1}$ is one-one .

It suffices to prove that if $(m, 0) \in K_{\nu+1}$ then $m = 0$. But if $(m, 0) = (\sum_{\mu<\nu} \varrho_\mu a_\mu, \varrho_\nu)$ then $\varrho_\nu = 0$ and $\sum_{\mu<\nu} \varrho_\mu \lambda_\mu = 0$. Therefore since

$$\{\lambda_\mu x = a_\mu : \mu < \nu\}$$

is consistent in M_ν by the inductive hypothesis, we have $m = \sum_{\mu<\nu} \varrho_\mu a_\mu = 0$ (cf. Lemma 1).

Thus because of (1) we can identify M_ν with a submodule of $A_{\nu+1}$, and, since M_ν is injective, we can write

$$A_{\nu+1} = M_\nu \oplus B_{\nu+1} .$$

Let $a_\nu =$ the image of $(0, 1) \in M_\nu \oplus \Lambda$ in $A_{\nu+1}$. By the construction and Lemma 1, $\{\lambda_\mu x = a_\mu : \mu \leqq \nu\}$ is consistent in $A_{\nu+1}$. We claim:

(2) $a_\nu \notin M_\nu$.

We have to show that $(m, 0) - (0, 1) = (m, -1)$ is not an element of $K_{\nu+1}$ for any $m \in M_\nu$. But if $(m, -1) = (\sum_{\mu<\nu} \varrho_\mu \lambda_\mu, \varrho_\nu)$ then $\varrho_\nu = -1$, and

$\sum_{\mu \leqq \nu} \varrho_\mu \lambda_\mu = 0$ implies $\lambda_\nu = \sum_{\mu < \nu} \varrho_\mu \lambda_\mu$ which contradicts the choice of $\{\lambda_\nu : \nu < \alpha\}$.

Thus because of (2) we can write $a_\nu = m_\nu + b_{\nu+1}$ where $m_\nu \in M_\nu$ and $0 \neq b_{\nu+1} \in B_{\nu+1}$. Define

$$M_{\nu+1} = E(A_{\nu+1} \oplus B_{\nu+1}^{(\varkappa)})$$

(where $B_{\nu+1}^{(\varkappa)}$ denotes the direct sum of $\varkappa$ copies of $B_{\nu+1}$). $M_{\nu+1}$ has cardinality $\varkappa$ by hypothesis.

Furthermore, there exist $\varkappa$ automorphisms $\{g_{\nu+1,\xi} : \xi < \varkappa\}$ of $M_{\nu+1}$ which are the identity on M_ν and which differ on a_ν. (We can define $g_{\nu+1,\xi}$ so that it takes the copy of $B_{\nu+1}$ contained in $A_{\nu+1}$ onto the ξth copy of $B_{\nu+1}$ in $B_{\nu+1}^{(\varkappa)}$).

Finally, we let $M = \bigcup_{\nu<\alpha} M_\nu$. By construction $\{\lambda_\nu x = a_\nu : \nu < \alpha\}$ is consistent in M. Moreover for each $\sigma \in \varkappa^\alpha$ the sequence $(g_{\nu+1,\sigma(\nu+1)})_{\nu<\alpha}$ determines an automorphism f_σ of M such that for $\sigma \neq \sigma'$, f_σ differs from $f_{\sigma'}$ on an element of $\{a_\nu : \nu < \alpha\}$. Thus M satisfies all the properties stated in the first paragraph of the proof and therefore the proof is complete.

Remark. 1. If $\gamma > \aleph_0$, there are arbitrarily large cardinals such that $\varkappa^{\underline{\gamma}} > \varkappa$, since there are arbitrarily large cardinals of cofinality $\aleph_0$. It then follows from Theorem 2 that the injective envelope of every module of cardinality $\varkappa$ is of cardinality $\varkappa$ for every infinite $\varkappa \geqq \operatorname{Card} \Lambda$ if and only if Λ is noetherian. It would be easy to derive from this result some consequences on the definability of the notion of injective module, which will not be discussed here since the question has been completely solved in [5] by a different argument.

Remark 2. Let us denote by γ_1 the smallest cardinal such that every γ_1-injective module is injective. It is clear that $\gamma_1 \leqq \gamma$. One may easily extract from the proof of Theorem 2 a proof of the fact that if γ is an uncountable cardinal which is not the successor of a singular cardinal, then $\gamma_1 = \gamma$. (Note that the consistent α^+-system $\mathscr{S}$ in M constructed in the proof of Theorem 2 does not have a solution in M.) The following questions remain open:

i) What happens if γ is finite? One cannot hope to establish $\gamma = \gamma_1$. For example, if Λ is a Dedekind domain which is not principal, one has $\gamma = 3$ and $\gamma_1 = 2$. One the other hand, if Λ is a commutative integral domain such that $\gamma_1 = 2$, then by a classical result due to Cartan and Eilenberg ([3, Chap. VII, § 5]), Λ is a Dedekind domain and $\gamma \leqq 3$. One may then ask if there exists a function $f : \omega \to \omega$ such that for every ring Λ (or at least for every commutative integral domain Λ) with γ_1 finite one has $\gamma \leqq f(\gamma_1)$.

ii) What happens if $\gamma=\aleph_0$? A positive answer to the question raised in (i) would imply that in this case $\gamma_1=\aleph_0$.

iii) What happens if γ is the successor of a singular (infinite) cardinal? In particular, if $\gamma=\aleph_{\omega+1}$, is there a module which is $\aleph_\omega$-injective and not injective?

The following lemma is a refinement of Lemma 4.3 of [5].

Lemma 5. *Let A and B be modules. Suppose $\mathscr{F}(B)$ is a family of submodules of A such that for any $B'\in\mathscr{F}(B)$, $B'\cong B$, and for $B', B''\in\mathscr{F}(B)$, either $B''=B'$ or $B''\cap B'=\{0\}$. Suppose $\mathscr{F}(B)$ has cardinality $\varkappa$ and $A=C_1\oplus C_2$ where C_1 is an essential extension of a module D of cardinality $<\varkappa$. Then C_2 contains a submodule isomorphic to B.*

Proof. We may suppose $A=C_1\oplus C_2$ is an internal direct sum so that C_i is a submodule of A, $i=1,2$. If $\pi_2: A\to C_2$ is the canonical projection, it suffices to prove that there exists $B'\in\mathscr{F}(B)$ such that $\pi_2|B'$ is one-one. Assume this is false; we obtain a contradiction by defining a one-one function $g:\mathscr{F}(B)\to D$. In fact, for any $B'\in\mathscr{F}(B)$ there exists, by assumption, $b'\in B'-\{0\}$ such that $\pi_2(b')=0$ i.e. $b'\in C_1$. Since C_1 is an essential extension of D, $\Lambda b'\cap D\neq\{0\}$, so let $g(b')\in(\Lambda b'\cap D)-\{0\}$. Because the elements of $\mathscr{F}(B)$ have trivial intersection, g is one-one. This completes the proof.

Theorem 3. *Let $\varkappa$ be an infinite cardinal.*

(i) *If $\varkappa>\operatorname{Card}\Lambda$ and if there exists a homogeneous universal module of cardinality $\varkappa$, then $\varkappa^{\underline{\gamma}}=\varkappa$.*

(ii) *If $\varkappa\geqq\operatorname{Card}\Lambda$ and if $\varkappa^{\underline{\gamma}}=\varkappa$, then there exists a homogeneous universal module of cardinality $\varkappa$.*

Proof. (i) If M is a homogeneous universal module of cardinality $\varkappa>\operatorname{Card}\Lambda$, then by Lemma 3 M is injective. Since M is universal, and by the definition of injective envelope, every module of cardinality $\varkappa$ has injective envelope of cardinality $\varkappa$. We then apply Theorem 2.

(ii) Let $\mathscr{C}=\{\Lambda/I : I \text{ is an ideal of } \Lambda\}$ i.e. $\mathscr{C}$ is the set of all (up to isomorphism) cyclic modules. By the definition of γ,

$$\operatorname{Card}\mathscr{C}\leqq\varkappa^{\underline{\gamma}}=\varkappa\,.$$

Let $M=E(\oplus_{C\in\mathscr{C}}C^{(\varkappa)})$. Since $\varkappa^{\underline{\gamma}}=\varkappa$, Theorem 1 implies $\operatorname{Card}M=\varkappa$. We claim that M is homogeneous universal. To prove that M is universal we have to show that any module N of cardinality $\leqq\varkappa$ can be embedded in M. Because of Theorem 1 we can assume N is injective. By ([$1\frac{1}{2}$], Prop. 7.9]) N is the injective envelope of a direct sum S of cyclic modules. Since S is embeddable in $\bigoplus_{C\in\mathscr{C}} C^{(\varkappa)}$, N is embeddable in M. To prove that M is homogeneous, consider an isomorphism $f\colon N \to N'$ of submodules of M such that $\operatorname{Card} N < \varkappa$; f extends to an isomorphism $\bar{f}: E(N) \to E(N')$. Write $M = E(N) \oplus P$, $M = E(N') \oplus P'$; to prove that f extends to an automorphism of M, it suffices to prove that $P \cong P'$. Since P and P' are injective, to prove $P \cong P'$ it suffices, by the "Schroeder–Bernstein theorem for injectives" (see [2]), to prove P can be embedded in P' and P' can be embedded in P. We use Lemma 5: since M is universal, $P^{(\varkappa)}$ can be embedded in M; and since $E(N')$ is an essential extension of a module N' of cardinality $<\varkappa$, Lemma 5 implies P' contains a submodule isomorphic to P. Similarly P contains a copy of P', and therefore $P \cong P'$. The proof of the theorem is complete.

Remark 3. If $\varkappa > \operatorname{Card} \Lambda$ (and $\varkappa^{\gamma} = \varkappa$) we can give a proof that $M = E(\bigoplus_{C\in\mathscr{C}} C^{(\varkappa)})$ is homogeneous universal which does not use [$1\frac{1}{2}$] or [2]. In fact by Theorem 2.5 of [10] it suffices to prove the following: if Q is a Λ-module, X a subset of Q of cardinality $<\varkappa$, and q an element of Q, then any embedding f of the submodule $\langle X\rangle$ of Q generated by X, can be extended to an embedding of $\langle X\rangle + \Lambda q$ into M. We can assume Q is injective, and that $E(\langle X\rangle) \subseteq Q$, and we can extend f to $E(\langle X\rangle)$. Write $M = f(E(\langle X\rangle)) \oplus N$ and $Q = E(\langle X\rangle) \oplus P$. We can assume $q \in P$; to prove that f can be extended to $\langle X\rangle \oplus \Lambda q$ it suffices to show that Λq can be embedded in N. But since $\operatorname{Card} \Lambda < \varkappa$, $\operatorname{Card}\langle X\rangle < \varkappa$ and Lemma 5 implies that N contains a copy of Λq.

Remark 4. If Λ is a noetherian ring, then $\gamma \leqq \aleph_0$ and for every infinite cardinal $\varkappa$, $\varkappa^{\gamma} = \varkappa$. Hence there is a homogeneous universal module in every infinite cardinal $\geqq \operatorname{Card} \Lambda$. Using [9], it is easy to describe the structure of homogeneous universal Λ-modules. For example, if Λ is commutative noetherian, the homogeneous universal module of cardinality $\varkappa$ is isomorphic to $\bigoplus_{P\in\mathscr{P}} E(\Lambda/P)^{(\varkappa)}$ where $\mathscr{P}$ is the set of *prime* ideals of Λ. In particular, if $\Lambda = \mathsf{Z}$, the homogeneous universal module of cardinality $\varkappa$ is $\bigoplus_p \mathsf{Z}(p^\infty)^{(\varkappa)} \oplus \mathsf{Q}^{(\varkappa)}$.

Remark 5. When we consider the problem of the existence of homogeneous universal modules of infinite cardinality $\lambda = \operatorname{Card} \Lambda$, we find that many of the previous arguments fail; the principal difficulty is that condition VI_λ of [6] fails to hold: i.e. a subset X of cardinality $<\lambda$ of a module Q is not necessarily contained in a submodule of Q of cardinality $<\lambda$. However, the previous arguments yield the following result:

PROPOSITION 3. *If* $\lambda = \operatorname{Card} \Lambda \geqq \aleph_0$, *the following are equivalent:*

(i) *there exists an injective universal module of cardinality* λ;

(ii) $\lambda^{\underset{\smile}{\gamma}} = \lambda$;

(iii) *there exists an injective homogeneous universal module of cardinality* λ, *and any injective universal module is isomorphic to it.*

The following example, due to E. Fisher, shows that a homogeneous universal module of cardinality λ is not necessarily injective nor unique up to isomorphism: Let Λ be an integral domain which is not a field. If there is an injective homogeneous universal module M of cardinality $\lambda = \operatorname{Card} \Lambda$, let $M' = M \oplus \Lambda$. M' is certainly universal and it is homogeneous because any submodule of cardinality $<\lambda$ is torsion and hence contained in M. But M' is not injective, since Λ is not injective, and M' is, therefore, not isomorphic to M.

REMARK 6. As a special case of a result of E. Fisher [5½] we have: every homogeneous universal module of cardinality $> \operatorname{Card} \Lambda$ is saturated if and only if the theory of Λ-modules has a model-completion. In view of [5] therefore, every homogeneous universal module of cardinality $> \operatorname{Card} \Lambda$ is saturated if and only if Λ is coherent.

REFERENCES

1. H. Bachmann, *Transfinite Zahlen* (Ergebnisse. Math. Grenzgebiete 1), Zweite Auflage, Springer-Verlag, Berlin, 1967.

1½. H. Bass, *Finitistic dimension and a homological generalization of semi-primary rings*, Trans. Amer. Math. Soc. 95 (1960), 466–488.

2. R. T. Bumby, *Modules which are isomorphic to submodules of each other*, Arch. Math. 16 (1965), 184–185.

3. H. Cartan and S. Eilenberg, *Homological algebra* (Princeton Mathematical Series 19), Princeton University Press, Princeton, 1956.

4. B. Eckmann and A. Schopf, *Über injecktive Moduln*, Arch. Math. 4 (1953), 75–78.

5. P. Eklof and G. Sabbagh, *Model-completions and modules*, Ann. Math. Logic 2 (1971), 251–295.

5½. E. Fisher, *Homogeneous universal models revisited*, to appear.

6. B. Jónsson, *Universal relational systems*, Math. Scand. 4 (1956), 193–208.

7. B. Jónsson, *Homogeneous universal relational systems*, Math. Scand. 8 (1960), 137–142.

8. C. Karp, *Finite-quantifier equivalence*, in: *The theory of models*, Proceedings of the 1963 International Symposium at Berkeley, North-Holland, Amsterdam, 1965, 407–412.

9. E. Matlis, *Injective modules over noetherian rings*, Pacific J. Math. 8 (1958), 511–528.

10. M. Morley and R. Vaught, *Homogeneous universal models*, Math. Scand. 11 (1962), 37–57.

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