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SHELLABLE DECOMPOSITIONS OF CELLS AND SPHERES

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1. Introduction.

The second part of this note, especially proposition 2, may be considered an appendix to L. Schläfli's computation of the Euler characteristic for convex polytopes. His computation is easy and elegant, but he assumes that the facets of each polytope can be ordered in a certain favourable way. Later it became doubtful whether such an ordering is always possible, especially since the discovery of combinatorial n-balls whose facets don't allow a corresponding favourable arrangement. We were surprised to find that Schläfli's assumption can be justified in an almost trivial manner. On the other hand we don't know whether an analogous assumption is valid for all polyhedral spheres; our only result in this direction is contained in proposition 1.

2. Notations.

By a complex $\mathfrak{A}$ we understand a finite set of polytopes, all of which are contained in the same euclidean space E^d, with the usual closedness- and intersection-properties. $\Delta^i(\mathfrak{A})$ is the set of i-dimensional members in $\mathfrak{A}$, and $f^i(\mathfrak{A})$ their number. $\mathfrak{A}$ is an n-complex, if each element of $\mathfrak{A}$ is a face of some n-dimensional element of $\mathfrak{A}$. By $|\mathfrak{A}|$ we denote the corresponding polyhedron, $|\mathfrak{A}| = \bigcup_{X \in \mathfrak{A}} X$. On the other hand, given a set $A \subset E^d$, a decomposition of A is a complex $\mathfrak{A}$ such that $A = |\mathfrak{A}|$. The complex $\mathfrak{A}$ is called a simplicial decomposition, or a triangulation, of A, if all elements in $\mathfrak{A}$ are simplexes. $\mathfrak{B}$ is a subdivision of $\mathfrak{A}$ if $|\mathfrak{B}| = |\mathfrak{A}|$ and each element of $\mathfrak{B}$ is contained in some element of $\mathfrak{A}$. We are especially interested in the complexes $\mathfrak{C}(P)$, consisting of all faces of a polytope P, and in the boundary complex $\hat{\mathfrak{C}}(P) = \mathfrak{C}(P) \setminus \{P\}$ of P. A polyhedron is called an n-cell if it allows a triangulation which is isomorphic with a triangulation of an n-dimensional simplex T^n. As far

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as we know, it is still undecided whether each polyhedron homeomorphic to T^n is an n-cell, but this is certainly true for all convex n-polytopes. Similarly, a polyhedron is called an n-sphere if one of its triangulations is isomorphic with a triangulation of the boundary $\hat{T}^{n+1}$. We say that the n-cell B lies in the complex $\mathfrak{A}$ if there exists a subcomplex $\mathfrak{B} \subset \mathfrak{A}$ with $|\mathfrak{B}| = B$. Schläfli assumed for his computation that the boundary complex of every polytope is shellable. This notion may be described as follows.

Definition 1. Every decomposition of a 0-cell is shellable. A decomposition $\mathfrak{A}$ of an n-cell A is shellable provided that the n-dimensional elements of $\mathfrak{A}$ can be arranged in a sequence $S_1, S_2, \ldots, S_r$, $r = f^n(\mathfrak{A})$, such that, for each i with $2 \leqq i \leqq r$, $S_i \cap (\bigcup_{j<i} S_j)$ is a shellable $(n-1)$-cell contained in the boundary complex $\hat{\mathfrak{C}}(S_i)$. A decomposition $\mathfrak{B}$ of an n-sphere is shellable if there exists an n-cell $S \in \Delta^n(\mathfrak{B})$ such that $\mathfrak{B} \setminus \{S\}$ is a shellable decomposition of the n-cell $|\mathfrak{B}| \setminus \operatorname{relint} S$. The above definitions are more restrictive than the usual ones; we have chosen them here to fit our inductive arguments. Those terms which we have not defined explicitely shall be used here in the same sense as in the books [1] and [2].

3. Subdivisions with nice properties.

The geometry of n-dimensional cells and spheres is, for $n \geqq 3$, much more complicated than for the lower dimensions. Let us mention two phenomena which do not occur in the simpler cases:

For each $n \geqq 3$ there is a decomposition of an n-sphere, which is not isomorphic with the boundary complex of a polytope.

For each $n \geqq 3$ there is a non-shellable decomposition of an n-cell.

Further information about the above facts is contained in [5], [4], [3], and especially in [1]. W. Sanderson has shown in [5] that these phenomena disappear, for $n = 3$, if we choose appropriate subdivisions of the complexes in question, and we want to give first a short alternate proof of his theorem, which is valid in all dimensions. We need a lemma about stellar subdivisions. Let $\mathfrak{A}$ be a simplicial complex, $X \in \Delta^i(\mathfrak{A})$, $0 \leqq i \leqq \dim \mathfrak{A}$ an i-face in $\mathfrak{A}$ and $x \in \operatorname{relint} X$ an arbitrary point. The complex, which arises from $\mathfrak{A}$ by starring X at x, shall be denoted by $\operatorname{st}(x, X)[\mathfrak{A}]$. Of course, if $i = 0$, we have $\operatorname{st}(x, X)[\mathfrak{A}] = \mathfrak{A}$, but in the other cases we obtain proper subdivisions of $\mathfrak{A}$.

Lemma 1. *Let T be an n-simplex, $A \subset T$ a k-face of T, $0 \leqq k \leqq n$, and a a point in* relint$A$. *Set $\mathfrak{H} = \operatorname{st}(a, A)[\mathfrak{C}(T)]$ and consider an $(n-1)$-ball Y*

lying in $\hat{\mathfrak{C}}(T)$. *The n-simplexes of* $\mathfrak{H}$ *can be arranged in a sequence* $S_1,\ldots,S_{k+1}$ *such that each of the intersections* K_i *is an* $(n-1)$*-ball lying in* $\hat{\mathfrak{C}}(S_i)$, *where we set* $K_1 = S_1 \cap Y$ *and* $K_i = S_i \cap (Y \cup \bigcup_{j<i} S_j)$ *for* $2 \leqq i \leqq r+1$.

Proof. We denote by B the intersection of all facets of T contained in Y, and proceed by induction on the number

$$l(A,Y) = \min\{f^0(A) - f^0(A \cap B), f^0(A) - 1\}.$$

First case, $l = 0$. Our assertion being trivial for $f^0(A) = 1$, assume $A \subset B$, and arrange the simplexes of $\mathfrak{H}$ arbitrarily in a sequence $S_1,\ldots,S_{k+1}$. Clearly every intersection K_i is a proper subset of the boundary $\hat{S}_i$, therefore it is enough to show that K_i is the polyhedron of an $(n-1)$-complex $\mathfrak{K}_i \subset \hat{\mathfrak{C}}(S_i)$. Since $S_i \cap S_j$ is always a common $(n-1)$-face, we only have to consider the intersection of S_i with a facet $F \subset Y$ of T. We find

$$S_i = \operatorname{conv}\{\{a\} \cup \Delta^0(T) \setminus \{t_i\}\},$$

for some $t_i \in \Delta^0(A)$ and

$$F = \operatorname{conv}\{\Delta^0(T) \setminus \{t_j\}\},$$

for some $t_j \notin \Delta^0(A)$, and consequently

$$S_i \cap F = \operatorname{conv}\{\{a\} \cup \Delta^0(T) \setminus \{t_i, t_j\}\},$$

which implies the desired conclusion. Second case, $l > 0$. We choose $a_0 \in \Delta^0(A)$, not contained in B. Since $f^0(A) > 1$, we have $a_0 \neq a$. Let S_1 be the n-simplex in $\mathfrak{H}$ which does not contain a_0, and denote by $\mathfrak{H}'$ the complex $\mathfrak{H} \setminus \{S_1\}$. We set

$$A' = \operatorname{conv}\{\Delta^0(A) \setminus \{a_0\}\}$$

and choose a point $a' \in \operatorname{relint} A'$. Let π be the isomorphism between $\mathfrak{H}'$ and $\overline{\mathfrak{H}} = \operatorname{st}(a', A')[\mathfrak{C}(T)]$ which carries a into a' and leaves the other vertices of $\mathfrak{H}'$ fixed. An easy computation shows that

(1) $S_1 \cap Y$ is an $(n-1)$-ball in $\hat{\mathfrak{C}}(S_1)$,

(2) for each $S \in \Delta^n(\mathfrak{H}')$, $S \cap (Y \cup S_1) = \pi(S) \cap Y$.

(2), together with the inductive hypothesis, applied to $\overline{\mathfrak{H}}$ and the $(n-1)$-ball Y in $\hat{\mathfrak{C}}(T)$, allows us to arrange the simplexes of $\mathfrak{H}'$ in a sequence $S_2,\ldots,S_{k+1}$ such that $S_i \cap (Y \cup \bigcup_{j<i} S_j)$ is always an $(n-1)$-ball in $\hat{\mathfrak{C}}(S_i)$. This, in conjunction with (1), means that the sequence $S_1, S_2,\ldots,S_{k+1}$ has the properties required by our lemma.

REMARK. If Y is empty, we may arrange the simplexes in $\mathfrak{H}$ arbitrarily, and will find that K_1 is empty, whereas each K_i, $2 \leqq i \leqq k+1$, is an $(n-1)$-ball as described in the lemma. Similarly, if Y is the whole boundary of T, the sets K_i, $1 \leqq i \leqq k$, are as in the lemma, and K_{k+1} is the whole boundary of S_{k+1}.

PROPOSITION 1. *Every decomposition of an n-cell and every decomposition of an n-sphere contains a shellable subdivision.*

PROOF. Let $\mathfrak{A}$ be a complex whose polyhedron $|\mathfrak{A}|$ is an n-cell. Clearly there exists a subdivision of $\mathfrak{A}$ which is isomorphic with a subdivision $\mathfrak{A}'$ of $\mathfrak{C}(T^n)$. A well known result in piecewise linear topology (see corollary 1.6. in [3]) guarantees that there is a positive integer r such that some rth derived subdivsion $\mathfrak{A}_r$ of $\mathfrak{C}(T^n)$ is also a subdivision of $\mathfrak{A}'$, and since $\mathfrak{A}_r$ arises from $\mathfrak{C}(T^n)$ by repeated starring operations it remains to prove: if $\mathfrak{B}$ is a shellable simplicial decomposition of an n-ball, and if $\mathfrak{B}'$ arises from $\mathfrak{B}$ by starring a face $X \in \Delta^i(\mathfrak{B})$, $i \geqq 1$, then $\mathfrak{B}'$ is shellable, too.

Denote by $x \in \operatorname{relint} X$ the new vertex of $\mathfrak{B}'$. Assume that $<$ is a linear ordering of $\Delta^n(\mathfrak{B})$, fulfilling the conditions of definition 1. Consider a simplex $S \in \Delta^n(\mathfrak{B})$ with $X \subset S$. The complex $\mathfrak{C}(S) \subset \mathfrak{B}$ is replaced in $\mathfrak{B}'$ by $\mathfrak{S}' = \operatorname{st}(x, X)[\mathfrak{C}(S)]$. By our assumption about $\mathfrak{B}$, $Y = S \cap B(S)$ is an $(n-1)$-ball lying in $\mathfrak{C}(S)$, except when S is the first simplex in $\Delta^n(\mathfrak{B})$. Here $B(S)$ stands for the union of all n-simplexes of $\mathfrak{B}$ preceding S. We apply lemma 1 (or, if S happens to be the first simplex in $\Delta^n(\mathfrak{B})$, the remark following lemma 1) to the simplex S, the decomposition $\mathfrak{S}'$ of S and the $(n-1)$-ball Y, and denote by $<_s$ a linear ordering of $\Delta^n(\mathfrak{S}')$ with the properties described in lemma 1. Let $\ll$ be the following ordering of $\Delta^n(\mathfrak{B}')$: If $X, Y \in \Delta^n(\mathfrak{B}')$ are contained in the same n-simplex S of $\mathfrak{B}$, we set

$$X \ll Y \Leftrightarrow X <_s Y .$$

If $X, Y \in \Delta^n(\mathfrak{B}')$ are contained in different simplexes of $\Delta^n(\mathfrak{B})$, say $X \subset S$, $Y \subset T$, we set

$$X \ll Y \Leftrightarrow S < T.$$

The relation $\ll$ clearly satisfies the conditions of definition 1, and $\mathfrak{B}'$ is shellable.

The proof for decompositions of n-spheres is almost literally the same, the remark following lemma 1 guarantees that no trouble arises if the last simplex in such a complex has to be subdivided.

By the above construction, each decomposition of an n-sphere has a subdivision which arises from $\hat{\mathfrak{C}}(T^{n+1})$ by repeated starring operations. Clearly such a subdivision is isomorphic with the boundary complex of a polytope.

4. Boundary complexes of polytopes.

Before we can state our result, we need a few more definitions. If p and q are (not necessarily different) points in E^n, denote by

$$[p,q] = \{x \mid x = \alpha p + (1-\alpha)q,\ 0 \leqq \alpha \leqq 1\}$$

the closed segment and by $[p,q) = ([p,q] \setminus \{q\}) \cup \{p\}$ one of the halfopen segments between them. A point $p \in E^n \setminus P$ is admissible with respect to an n-polytope $P \subset E^n$ if there is no face F of $\hat{P}$ whose affine hull $\operatorname{aff} F$ contains p. Let $P \subset E^n$ be an n-polytope and $p \in E^n \setminus P$ an admissible point. The set $S(P,p)$ of visible points on the boundary $\hat{P}$ is defined by

$$S(P,p) = \{x \in \hat{P} \mid [x,p] \cap P = \{x\}\}\,.$$

Similarly

$$U(P,p) = \{x \in \hat{P} \mid \text{there exists no point } y \in \hat{P} \text{ such that } x \in [p,y)\}$$

stands for the (closure of the) set of invisible points. Both, $S(P,p)$ and $U(P,p)$, are $(n-1)$-balls in $\hat{\mathfrak{C}}(P)$. Let $\mathfrak{S}(P,p)$ and $\mathfrak{U}(P,p)$ be the underlying subcomplexes of $\hat{\mathfrak{C}}(P)$, such that we have $|\mathfrak{S}(P,p)| = S(P,p)$, $|\mathfrak{U}(P,p)| = U(P,p)$. Notice that $\mathfrak{G}(P,p) = \mathfrak{S}(P,p) \cap \mathfrak{U}(P,p)$ is isomorphic to the boundary complex of an $(n-1)$-dimensional polytope P'. The latter can be found by intersecting the polytope $Q = \operatorname{conv}(P \cup \{p\})$ with a hyperplane which separates p from the remaining vertices of Q.

Definition 2. Let $P \subset E^n$ be an n-polytope. A line $G \subset E^n$ is admissible with respect to P if

(3) no hyperplane $\operatorname{aff} F$, F a facet of P, is parallel to G;

(4) whenever F_1 and F_2 are different facets of P, the points $(\operatorname{aff} F_1) \cap G$ and $(\operatorname{aff} F_2) \cap G$ are different.

The next two lemmas are immediate consequences of the corresponding definitions. We omit their proofs.

Lemma 2. *Let $P \subset E^n$ be an n-polytope and $p \in E^n$ an admissible point. A facet F of P belongs to $\mathfrak{S}(P,p)$ if and only if p is beyond F.*

LEMMA 3. *Let P and p be as in lemma 2. If $E \subset E^n$ is a flat containing p, set $\bar{P} = P \cap E$, and assume $\dim \bar{P} = \dim E$. p is admissible with respect to $\bar{P}$, and $S(P,p) \cap E = S(\bar{P},p)$, $U(P,p) \cap E = U(\bar{P},p)$.*

PROPOSITION 2. *Let $P \subset E^n$ be an n-polytope and $p \in E^n$ an admissible point. $\mathfrak{S}(P,p)$ and $\mathfrak{U}(P,p)$ are both shellable decompositions of an $(n-1)$-cell.*

PROOF. Set

$$\alpha_k^n = \{\mathfrak{A} \mid \mathfrak{A} = \mathfrak{S}(P,p) \text{ or } \mathfrak{A} = \mathfrak{U}(P,p) \text{ for some } n\text{-polytope } P \subset E^n \text{ and some admissible point } p \in E^n, \text{ and } f^{n-1}(\mathfrak{A}) = k\}\,.$$

We prove our assertion for all members $\mathfrak{A} \in \alpha_k^n$ by induction on n and, for fixed n, on k. The cases $n=1$ and $k=1$ are trivial. Let P be an n-polytope, $n \geqq 2$, and consider $\mathfrak{A} \subset \hat{\mathfrak{C}}(P)$ with $f^{n-1}(\mathfrak{A}) \geqq 2$, where $\mathfrak{A} = \mathfrak{S}(P,p)$ or $\mathfrak{A} = \mathfrak{U}(P,p)$ for some admissible point $p \in E^n$. It is easy to see that there are many lines in E^n which contain p, meet the interior $\underline{P}$ of P and are admissible with respect to P. Let G be an arbitrary line of this kind. By G_1 and G_2 we denote the two closed rays whose union is the set $G \setminus \underline{P}$. We assume $p \in G_1$ and, if p_i is the endpoint of G_i, let $<$ be the following linear ordering of $G_1 \cup G_2$:

$$\begin{aligned} x \in G_1,\ y \in G_2 &\Rightarrow x < y\,, \\ x,y \in G_1,\ x \in [p_1,y) &\Rightarrow x < y\,, \\ x,y \in G_2,\ y \in [p_2,x) &\Rightarrow x < y\,. \end{aligned}$$

Denote by $\Delta^{n-1}(P)$ the set of facets of P. Since G is admissible, each hyperplane $\operatorname{aff} F$, $F \in \Delta^{n-1}(P)$, meets G in exactly one point $g(F)$, and if F,F' are different, $g(F)$ and $g(F')$ are different, too. We arrange the facets of P in a finite sequence $(F_i)_{1 \leqq i \leqq r}$, $r = \operatorname{card} \Delta^{n-1}(P)$, such that $g(F_i) < g(F_j)$ is equivalent to $i < j$.

First case, $\mathfrak{A} = \mathfrak{S}(P,p)$. By lemma 2, $\mathfrak{A}$ is the subcomplex of $\hat{\mathfrak{C}}(P)$ generated by those facets $F \in \Delta^{n-1}(P)$ for which $g(F) < p$. Therefore $\max\{i \mid g(F_i) < p\} = k \geqq 2$, and we choose a point $t \in G_1$ for which $g(F_{k-1}) < t < g(F_k)$. We set $g(F_k) = g_k$ and $\mathfrak{S}(P,t) = \mathfrak{A}'$. By the inductive hypothesis, $\mathfrak{A}$ is shellable if we can show that

$$\mathfrak{A} = \mathfrak{A}' \cup \mathfrak{C}(F_k), \quad \mathfrak{C}(F_k) \cap \mathfrak{A}' = \mathfrak{S}(F_k, g_k)\,. \tag{5}$$

The first of the above relations immediately follows from lemma 1. In order to prove the second, we observe that by the property (4) of G,

g_k is admissible, in $\operatorname{aff} F_k$, with respect to F_k, so $\mathfrak{S}(F_k, g_k)$ is defined. We set

$$|\mathfrak{A}| = A, \quad |\mathfrak{A}'| = A', \quad S(F_k, g_k) = S_k, \quad U(F_k, g_k) = U_k, \quad S_k \cap U_k = G_k .$$

Obviously $\operatorname{relint} F_k \cap A' = \emptyset$, so it is enough to establish the relations

$$(6) \qquad S_k \setminus G_k \subset A', \quad (U_k \setminus G_k) \cap A' = \emptyset .$$

Consider a point $x \in (\operatorname{relbd} F_k) \setminus G_k$. The line $\operatorname{aff}\{g_k, x\}$ intersects P in a nondegenerate segment $\bar{F}_k \subset F_k$. Set

$$E = \operatorname{aff}\{G \cup \{x\}\} \text{ and } \bar{P} = P \cap E .$$

Since G is admissible with respect to P, it is also admissible with respect to $\bar{P}$, and, furthermore, every facet of $\bar{P}$ contains an interior point of some facet of P. The last remark implies that x is an endpoint of the segment $\bar{F}_k$. By lemma 2 it suffices to prove that x belongs to $A' \cap E$ if and only if $[x, g_k] \cap \bar{F}_k = \{x\}$. This assertion, which corresponds to the case $n = 2$ of (6), is easily verified.

Second case, $\mathfrak{A} = \mathfrak{U}(P, p)$, and there is no facet $F \in \Delta^{n-1}(\mathfrak{A})$ with $g(F) \in G_1$. Choose $q \in G_2$ such that $q < g(F)$ for all $F \in \Delta^{n-1}(\mathfrak{A})$. Lemma 1 implies that

$$\mathfrak{A} = \mathfrak{U}(P, p) = \mathfrak{S}(P, q) ,$$

and by reversing the ordering of $G_1 \cup G_2$ we are lead back to the first case.

Third case, $\mathfrak{A} = \mathfrak{U}(P, p)$, and $g(F) \in G_1$ for some facet $F \in \Delta^{n-1}(\mathfrak{A})$. $\mathfrak{A}$ is generated by those facets F in $\hat{\mathfrak{C}}(P)$ for which $g(F) > p$. Set $i_0 = \min\{i \mid g(F_i) > p\}$. By our assumption, $g_0 := g(F_{i_0})$ belongs to G_1, and we choose $t \in G_1$ with $g_0 < t < g(F_{i_0+1})$. We set $\mathfrak{A}' = \mathfrak{U}(P, t)$ and show, by nearly the same argument as in the first case,

$$(7) \qquad \mathfrak{A} = \mathfrak{A}' \cup \mathfrak{C}(F_{i_0}), \quad \mathfrak{C}(F_{i_0}) \cap \mathfrak{A}' = \mathfrak{U}(F_{i_0}, g_0) ,$$

which again implies that $\mathfrak{A}$ is shellable.

Corollary. *The boundary complex of every convex polytope P is shellable.*

Proof. Choose a facet $F_1 \in \hat{\mathfrak{C}}(P)$ and a point $p \in E^n$ which is beyond F_1 and beneath all the other facets of P. $\mathfrak{U}(P, p)$ is shellable, by proposition 2, and so is $\hat{\mathfrak{C}}(P) = \mathfrak{U}(P, p) \cup \{F_1\}$.

Notice that we can prescribe in advance the first and the last facet in a shelling of $\hat{\mathfrak{C}}(P)$. We only have to choose F_1 and the line G, used in the proof of proposition 2, accordingly. Since it has been made more than a century ago, let us finally recall Schläfli's computation.

5. The characteristic of Euler-Schläfli.

For each complex $\mathfrak{A}$ set

$$\chi(\mathfrak{A}) = \sum_{i=0}^{\dim \mathfrak{A}} (-1)^i f^i(\mathfrak{A}) .$$

Let α_k^n be the set of complexes introduced in the proof of proposition 2, set $\alpha^n = \bigcup_{k=1}^{\infty} \alpha_k^n$, and denote by β^n the set of boundary complexes $\hat{\mathfrak{C}}(P)$, P an $(n+1)$-polytope.

PROPOSITION 3.

$$X \in \alpha^n \Rightarrow \chi(X) = 1 \tag{8}$$

$$X \in \beta^n \Rightarrow \chi(X) = 1+(-1)^n . \tag{9}$$

PROOF. Both assertions are trivial for $n \leqq 1$. For $n \geqq 2$ the following remark will be useful. If X and Y are subcomplexes of the same complex, the definition of χ immediately implies that

$$\chi(X \cup Y) + \chi(X \cap Y) = \chi(X) + \chi(Y) .$$

Let us prove first, by induction on k, that (8) is true for each $X \in \alpha_k^n$. If $k=1$, we have $X = \mathfrak{C}(P)$, P an n-polytope. Clearly, $\chi(X) = \chi(\hat{\mathfrak{C}}(P)) + (-1)^n$. By the inductive assumption, $\chi(\hat{\mathfrak{C}}(P)) = 1 + (-1)^{n-1}$, therefore $\chi(X) = 1$. If $k > 1$, there are, by the relation (5) or (7), subcomplexes Y and Z of X such that

$$Y \in \alpha_{k-1}^n, \quad Z \in \alpha_1^n, \quad Y \cap Z \in \alpha^{n-1}, \quad Y \cup Z = X ,$$

which, together with the inductive hypothesis, means that $\chi(X) = \chi(Y) + \chi(Z) - \chi(Y \cap Z) = 1$. Finally, to prove (9), let P be an $(n+1)$-polytope. Choose a facet $F_1 \in \Delta^{n-1}(P)$ and a point $p \in E^{n+1}$ which is beyond F_1 and beneath all the other facets of P. We find $\hat{\mathfrak{C}}(P) = \mathfrak{C}(F_1) \cup \mathfrak{U}(P,p)$, while the intersection $\mathfrak{C}(F_1) \cap \mathfrak{U}(P,p)$ is the boundary complex $\hat{\mathfrak{C}}(F_1)$ and belongs to β^{n-1}. Therefore, by the inductive hypothesis, and by the additivity of χ mentioned at the beginning of this proof, we calculate

$$\chi(\hat{\mathfrak{C}}(P)) = 1 + 1 - (1 + (-1)^{n-1}) = 1 + (-1)^n ,$$

and proposition 3 follows.

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