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ON FINITELY GENERATED FLAT MODULES III

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0. Introduction.

Let us first recall some definitions from [3]. A ring R is said to be a left-n-FGFP ring (right-n-FGFP ring) if any n-generated flat left (right) R-module is projective. R is called an n-FGFP ring if R is a left- and right-n-FGFP ring. If R is a right-n-FGFP for all n, then we say that R is a right-FGFP ring. A right- and left-FGFP ring is called an FGFP ring. We always assume that n is finite. For n equal to $\aleph_0$ a left-n-FGFP ring is left perfect [1].

It is well-known that a semiperfect ring is an FGFP ring, cf. J. Lambek [4, § 5.4, exercise 10]. In [3] it is proved that a subring of a right noetherian ring is an FGFP ring and it is proved that subrings of left-n-FGFP rings are left-n-FGFP rings, too. This last theorem was also proved by I. I. Sahaev [6].

In section 1 we shall prove that a ring R, with w.gl.dim.$R \leqq 1$, is a left-n-FGFP ring if and only if R is a right-n-FGFP ring. As a corollary of the proof we get a new and simple proof of the fact that any n-fir is an n-FGFP ring.

In section 2 we construct a ring R with the following properties:

(i) There exists a cyclic flat non-projective left R-module.

(ii) R is a right-1-FGFP ring.

All rings considered in this note are associative, with 1, all ring homomorphisms preserve 1 and subrings have the same 1.

1. Left- and right-n-FGFP rings.

In the study of left-n-FGFP rings the following lemma is useful:

LEMMA 1.1. *Let R be any ring. Then the following conditions are equivalent*:

(i) *Any cyclic flat left R-module is projective.*

(ii) *Any ascending chain of principal left ideals*

$$(a_1) \subseteqq \ldots \subseteqq (a_m) \subseteqq \ldots, \text{ where } a_m a_{m+1} = a_m ,$$

terminates.

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(iii) *Any descending chain of principal right ideals*

$$(a_1) \supseteqq \dots \supseteqq (a_m) \supseteqq \dots, \textit{ where } a_m a_{m+1} = a_{m+1},$$

terminates.

The lemma is proved by I. I. Sahaev in [5].

Furthermore we need the following well-known lemma:

LEMMA 1.2. *The following statements are equivalent for the ring R:*

(i) *R has no infinite set of non-zero orthogonal idempotents.*

(ii) *R satisfies the ascending chain condition on ideals eR (or ideals Re), where e denotes an idempotent in R.*

(iii) *R satisfies the descending chain condition on ideals eR (or ideals Re), where e denotes an idempotent in R.*

LEMMA 1.3. *Suppose R satisfies the ascending chain condition on left point annulets, that is, ideals of the form $l\{a\} = \{x \mid xa = 0\}$, then any cyclic flat left R-module is projective.*

PROOF. It is easily checked that R satisfies the equivalent conditions in lemma 1.2.

Suppose we are given an ascending chain of principal left ideals

$$(1) \qquad (a_1) \subseteqq \dots \subseteqq (a_m) \subseteqq \dots, \quad \text{with} \quad a_m a_{m+1} = a_m .$$

We have to prove that (1) becomes stationary. We claim that $l\{(1-a_m)\} \subseteqq l\{(1-a_{m+1})\}$ for all m. If x is an element in $l\{(1-a_m)\}$, then $x - xa_m = 0$. Hence $xa_{m+1} - xa_m a_{m+1} = 0$ and consequently

$$xa_{m+1} = xa_m = x, \quad \text{that is,} \quad x \in l\{(1-a_{m+1})\} .$$

It follows now that there exists an m_0 such that

$$l\{(1-a_m)\} = l\{(1-a_{m_0})\}$$

for all $m \geqq m_0$. In particular if $m \geqq m_0 + 1$, then

$$l\{(1-a_m)\} = l\{(1-a_{m-1})\} .$$

Now $a_{m-1} \in l\{(1-a_m)\}$, hence $a_{m-1} \in l\{(1-a_{m-1})\}$, that is, a_{m-1} is an idempotent. Thus, a_m is idempotent for all $m \geqq m_0$, and the proof of lemma 1.3 is completed.

A similar argument will show that the cyclic flat left modules of a ring with descending chain condition on right point annulets are projective.

For a later purpose we need the left-right symmetric of lemma 1.3.

Lemma 1.4. *If the ring R satisfies the ascending chain condition on right point annulets or if R satisfies the descending chain condition on left point annulets, then any cyclic flat right module is projective.*

Proposition 1.5. *Let R be a ring with no infinite set of orthogonal idempotents. If the principal left ideals in R are projective, then R is a* 1-FGFP *ring.*

Proof. Since all left point annulets are generated by idempotents, it follows from lemma 1.2 that R satisfies the ascending and descending chain conditions on left point annulets. Thus, by lemma 1.3 and lemma 1.4, R is a 1-FGFP ring.

Morita technique gives the next results.

Corollary 1. *If all n-generated left ideals of the ring R are projective and if R_n (the ring of $(n \times n)$-matrices over R) has no infinite set of non-zero orthogonal idempotents, then R is an n-*FGFP *ring.*

Corollary 2. *Assume* w.gl.dim.$R \leqq 1$. *Then R is a left-n-*FGFP *ring if and only if R is a right-n-*FGFP *ring.*

Proof. It suffices to prove corollary 2 for $n=1$. If R is a left-1-FGFP ring, then the principal left ideals of R are projective. It follows from lemma 1.1 and lemma 1.2 that R has no infinite set of non-zero orthogonal idempotents. Corollary 2 follows now immediately.

Corollary 3 (cf. [2] and [3]). *Any n-*fir *is an n-*FGFP *ring.*

Proof. It is well known or readily checked that the ring of $(n \times n)$-matrices over an n-fir satisfies the ascending chain condition on left and right point annulets.

Remark. As a consequence of corollary 2 let us note that a left semi-hereditary ring for which the ring of $(n \times n)$-matrices for all n has no infinite set of non-zero orthogonal idempotens is right semihereditary (cf. L. W. Small [7]).

Proposition 1.6. *If any left or right zero divisor in R is nilpotent, then R is a* 1-FGFP *ring.*

Proof. Assume we have a chain of principal left ideals

$$(2) \qquad \ldots \subseteqq (a_{-n}) \subseteqq \ldots \subseteqq (a_1) \subseteqq \ldots \subseteqq (a_n) \subseteqq \ldots, \qquad \text{where} \quad a_m = a_m a_{m+1}$$

for all $m \in \mathbf{Z}$. It suffices to prove that (2) becomes stationary. We can assume that

$$(0) \neq (a_m) \neq R \quad \text{for all } m .$$

The equation $a_m(1-a_{m+1})=0$ shows that a_m or $1-a_m$ is nilpotent for all m. If a_m is nilpotent, then

$$a_{m-1} = a_{m-1}a_m = \ldots = a_{m-1}a_m{}^s = 0$$

for s suitably large, and we are done in this case. If $(1-a_m)$ is nilpotent for all m, then the equation $(1-a_m)(1-a_{m+1})=(1-a_{m+1})$ shows that $a_{m+1}=1$. The proof of the proposition is now completed.

2. An example.

An example of a ring, R, with any cyclic flat right module projective and with a cyclic flat non-projective left module is given as follows.

Let K be any commutative field. We take R to be the K-algebra on the generators X_i, $i=1,2,\ldots$, and defining relations

$$X_i X_{i+1} = X_i, \quad i=1,2,\ldots .$$

From lemma 1.1 it follows that there exists a cyclic flat non-projective left R-module. The left-right symmetric to lemma 1.1 shows that R is a right-1-FGFP ring if and only if any ascending chain of principal right ideals

$$(3) \qquad (a_1) \subseteqq \ldots \subseteqq (a_m) \subseteqq \ldots, \qquad \text{with} \quad a_m a_{m-1} = a_{m-1} ,$$

terminates. It is not hard to check that any element in R can be written as $k+a$, $k \in K$ and a of the form $\sum k_{\gamma_r\ldots\gamma_1} X_r{}^{\gamma_r}\ldots X_1{}^{\gamma_1}$, where in no term all the γ_j are zero, the r-tuples $(\gamma_r,\ldots,\gamma_1)$ are all different, and all $k_{\gamma_r\ldots\gamma_1}$ are different from zero. This representation is unique. Each term $X_r{}^{\gamma_r}\ldots X_1{}^{\gamma_1}$ is called a monomial in the representation of a.

Let us first assume that all the k_i which appear as constant terms in the expressions of the a_i in (3) are zero. We can furthermore assume that a_1 is non-zero. Choose p maximal with respect to the existence of a monomial of the form $X_r{}^{\gamma_r}\ldots X_p{}^{\gamma_p}$ in a_1. Consider all these monomials in the representation of a_1 and fix one with $(\gamma_p,\ldots,\gamma_r,0,\ldots)$ minimal in lexicographical order. From the minimality of $(\gamma_p,\ldots,\gamma_r,0,\ldots)$ it follows that the corresponding minimal monomial never appears as a monomial in an expression of the form a_2a_1.

We can now assume that infinitely many of the k_i in (3) are non-zero. It suffices to consider the case, where all the k_i are non-zero.

If $a_i = k_i + b_i$ with $k_i \in K$ and b_i a K-linear combination of monomials, then the relations (3) become

$$k_i k_{i-1} = k_{i-1} \quad \text{and} \quad b_i k_{i-1} + b_i b_{i-1} + k_i b_{i-1} = b_{i-1} .$$

Thus, we conclude that $k_i = 1$ for all $i > 1$. Hence for $i \geqq 2$

$$b_{i+1} b_i = -b_{i+1} .$$

Let us assume that these equations hold for all i. In b_1, which is non-zero, let us choose a monomial $X_r^{\gamma_r} \ldots X_1^{\gamma_1}$ such that $(\gamma_1, \ldots, \gamma_r, 0, \ldots)$ is minimal in lexicographical order. Suppose

$$\gamma_1 = \ldots = \gamma_p = 0 .$$

The equation $b_3 b_2 = -b_3$ shows that there exists a monomial of the form $X_t^{\gamma_t} \ldots X_1^{\gamma_1}$ with $\gamma_1 = 0$ in the canonical representation of b_2. So it follows that all the b_i have a monomial with "$\gamma_1 = 0$". Again the equation $b_3 b_2 = -b_3$ shows that b_2 "has a monomial with $\gamma_1 = \gamma_2 = 0$" (look at the terms with "$\gamma_1 = 0$"). Thus, there exists a monomial in b_2 of the form $X_t^{\gamma_t} \ldots X_n^{\gamma_n}$ for all n. In particular b_2 has a monomial of the form $X_s^{\gamma_s} \ldots X_{p+1}^{\gamma_{p+1}}$, and this contradicts the equation $b_2 b_1 = -b_2$.

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