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HOMOLOGY OF DELETED PRODUCT SPACES

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1. Introduction and notation.

The deleted product space X^* of a space X is $X \times X - \Delta$. In this paper we continue the investigations begun in [7] of describing those finite, contractible, n-dimensional polyhedra whose deleted products have the homotopy type of the n-sphere and of describing the same such polyhedra whose deleted products have trivial homology groups in dimension greater than p, where $p \geqq n-1$. In [2], the author showed that if X is a finite, n-dimensional polyhedron such that $H_n(X)=0$, then $H_{2n}(X^*)=0$. In [4], the author computed the homology groups of the deleted product of a polyhedron in a subcollection $\mathfrak{B}$ of the finite, contractible, 2-dimensional polyhedra, and, in [5], the author described a member C of $\mathfrak{B}$ with the property that if $X \in \mathfrak{B}$, then C can be imbedded in X if and only if $H_2(X^*) \neq 0$. In the same paper, he used this result to show that an element X of $\mathfrak{B}$ can be imbedded in the plane if and only if $H_2(X^*)=0$, and he also described a member CC of $\mathfrak{B}$ with the property that if $X \in \mathfrak{B}$ then CC can be imbedded in X if and only if $H_3(X^*) \neq 0$. In the present paper, we show that if $n \geqq 2$ and $n < p < 2n$, then there is a finite, contractible, n-dimensional polyhedron X such $H_p(X^*) \neq 0$. (This result is known if $n=p$. See [6].) If $n > 2$ and $n \leqq p < 2n$, this paper is a first step in describing

$$\{X \mid X \text{ is a finite, contractible, } n\text{-dimensional polyhedron and } H_p(X^*)=0\}.$$

In particular we are interested in the relation between imbeddings and the vanishing of the p-dimensional homology group of the deleted product, and this investigation is to be continued in a forthcoming paper.

We let $S^n=\{x \in E^{n+1} \mid |x|=1\}$ and $B^n=\{x \in E^n \mid |x| \leqq 1\}$. If v is a vertex of a polyhedron A, we let $\mathrm{St}(v,A)$ denote the open star of v in A, and if $v_1,v_2,\ldots,v_n$ are the vertices of a simplex σ, we denote σ by $\langle v_1,v_2,\ldots,v_n\rangle$. We use the circumflex $\hat{v}_i$ to denote that v_i has been

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omitted, and if $w_1, w_2, \ldots, w_n$ are points, j is an integer, $1 \leqq j \leqq n$, and $k = 1, 2, \ldots, j$; we let $w_1, w_2, \ldots, \{\hat{w}_{i_k}\}, \ldots, w_n$ denote the subset of $w_1, w_2, \ldots, w_n$ obtained by omitting $w_{i_1}, w_{i_2}, \ldots, w_{i_j}$. Thus

$$\bigcup\nolimits'^{j}_{k=1} \langle w_1, w_2, \ldots, \{\hat{w}_{i_k}\}, \ldots, w_n \rangle$$

denotes the simplex whose vertices are $w_1, w_2, \ldots, w_n$ with $w_{i_1}, w_{i_2}, \ldots, w_{i_j}$ omitted. Also we shall denote

$$A - \bigcup_{i=1}^{n} \mathrm{St}(u_i, A) \quad \text{by} \quad A[u_1, \ldots, u_n]\,,$$

$$\overline{\mathrm{St}(u_m, A)} - \bigcup_{i=1}^{n} \mathrm{St}(u_i, A) \quad \text{by} \quad A[u_m \,|\, u_1, \ldots, u_n]\,,$$

$$A - \bigcup_{q=0}^{n} \{\mathrm{St}(u_q, A) \mid q \neq i_k \text{ for any } k\} \quad \text{by} \quad A[u_0, \ldots, \{\hat{u}_{i_k}\}, \ldots, u_n]\,,$$

$$\overline{\mathrm{St}(u_m, A)} - \bigcup_{q=0}^{n} \{\mathrm{St}(u_q, A) \mid q \neq i_k \text{ for any } k\} \quad \text{by} \quad A[u_m \,|\, u_0, \ldots, \{\hat{u}_{i_k}\}, \ldots, u_n]\,.$$

The homology groups used throughout this paper will be the reduced homology groups with integral coefficients, and we let $\mathbf{Z}$ denote the group of integers. If X is a finite polyhedron and A and B are subpolyhedra of X, let

$$P(A \times B - \Delta) = \bigcup \{\sigma \times \tau \mid \sigma \text{ is a simplex of } A, \ \tau \text{ is a simplex of } B, \text{ and } \sigma \cap \tau = \emptyset\}\,.$$

Hu [1] has shown that X^* and $P(X^*)$ are homotopically equivalent. If X and Y are finite polyhedra and $f\colon X \to Y$ is a simplicial map, let X_f^* denote the inverse image of Y^* in the map $f \times f\colon X \times X \to Y \times Y$ and let

$$P(X_f^*) = \bigcup \{\sigma \times \tau \mid \sigma \text{ and } \tau \text{ are simplexes of } X \text{ and } f(\sigma) \cap f(\tau) = \emptyset\}\,.$$

The author [3] has observed that X_f^* and $P(X_f^*)$ are homotopically equivalent.

2. Deleted products which are not homology spheres.

Throughout this section, we let A denote a finite, contractible, n-dimensional polyhedron, and we let B denote an m-simplex with vertices $v_0, v_1, \ldots, v_m$. In Theorem 1, we assume that $2 \leqq m \leqq n$, and, in Theorems 2 and 3, we assume that $3 \leqq m \leqq n$.

THEOREM 1. *If $A \cap B = \langle v_0 \rangle$, where v_0 is a vertex of A, and $X = A \cup B$, then $H_r(X^*)$ is isomorphic to $H_r(A^*)$ for all $r > n$ and X^* does not have the homotopy type of a sphere. Furthermore if $A = B^n$, v_0 is the center of B^n, and $m > 2$, then $H_n(X^*)$, $H_{m-1}(X^*)$, and $H_1(X^*)$ are isomorphic to $\mathbf{Z}$, and $H_r(X^*) = 0$ if r is neither n, $m-1$, nor 1; and if $A = B^n$, v_0 is the center* *of* B^n, *and* $m=2$, *then* $H_n(X^*)$ *is isomorphic to* Z, $H_1(X^*)$ *is isomorphic to the direct sum* $\mathsf{Z}+\mathsf{Z}$, *and* $H_r(X^*)=0$ *if* $n\neq r\neq 1$.

PROOF. First observe that

$$P(X^*) = P(A^*) \cup (B\times A[v_0]) \cup (\langle v_1,v_2,\ldots,v_m\rangle \times \overline{\mathrm{St}(v_0,A)}) \cup \cup (A[v_0]\times B) \cup (\overline{\mathrm{St}(v_0,A)} \times \langle v_1,v_2,\ldots,v_m\rangle) \cup P(B^*)\ .$$

Now $P(A^*)\cap(B\times A[v_0])=\langle v_0\rangle\times A[v_0]$, and hence

$$X_1 = P(A^*)\cup(B\times A[v_0])$$

has the homotopy type of $P(A^*)$. Also

$$X_1\cap(\langle v_1,v_2,\ldots,v_m\rangle\times\overline{\mathrm{St}(v_0,A)}) = \langle v_1,v_2,\ldots,v_m\rangle\times\partial(\mathrm{St}(v_0,A))\ ,$$

and thus if $X_2=X_1\cup(\langle v_1,v_2,\ldots,v_m\rangle\times\overline{\mathrm{St}(v_0,A)})$, then $H_r(X_2)$ is isomorphic to $H_r(A^*)$ for all $r>n$. If $A=B^n$, we may assume that A is an n-simplex which has been triangulated so that the vertices are v_0 and the vertices of a minimal triangulation of $\partial A=S^{n-1}$. In the proof of Theorem 1 of [6], the author defined a homeomorphism f of S^{n-1} onto $P(S^{n-1}\times S^{n-1}-\Delta)$. If $x\in S^{n-1}$, then there may not be a cell in $P(B^n\times B^n-\Delta)$ which contains both (x,v_0) and $f(x)$. However if there is no such cell, then there are two cells σ and τ such that $(x,v_0)\in\sigma$, $f(x)\in\tau$ and $\sigma\cap\tau\neq\emptyset$. Therefore for the purpose of defining a map whose range is $P(B^n\times B^n-\Delta)$, we may assume that for each $x\in S^{n-1}$ the line segment joining (x,v_0) and $f(x)$ is contained in $P(B^n\times B^n-\Delta)$. [We "straighten" $P(B^n\times B^n-\Delta)$ locally, define the map, and then "bend" it back.] Then the function $g\colon S^{n-1}\times I\to P(B^n\times B^n-\Delta)$ defined by

$$\begin{aligned} g(x,t) &= (1-2t)(x,v_0)+2tf(x), && 0\leqq t\leqq \tfrac{1}{2}\ ,\\ &= (2t-1)(v_0,x)+(2-2t)f(x), && \tfrac{1}{2}\leqq t\leqq 1\ ,\end{aligned}$$

is a homeomorphism of $S^{n-1}\times I$ onto $P(B^n\times B^n-\Delta)$ which maps $S^{n-1}\times\{0\}$ onto $S^{n-1}\times\{v_0\}$ and $S^{n-1}\times\{1\}$ onto $\{v_0\}\times S^{n-1}$. Therefore if $A=B^n$,

$$(1)\qquad \langle v_1,v_2,\ldots,v_m\rangle\times\partial(\mathrm{St}(v_0,A))$$

has the homotopy type of S^{n-1}, each $(n-1)$-cycle in (1) bounds in

$$\langle v_1,v_2,\ldots,v_m\rangle\times\overline{\mathrm{St}(v_0,A)}\ ,$$

but no nontrivial cycle in (1) bounds in X_1. Thus if $A=B^n$, then $H_r(X_2)=0$ for all r. Continuing,

$$X_2\cap(A[v_0]\times B) = A[v_0]\times\langle v_0\rangle\ ,$$

and hence $X_3 = X_2 \cup (A[v_0] \times B)$ has the homotopy type of X_2. Also

$$X_3 \cap (\overline{\mathrm{St}(v_0, A)} \times \langle v_1, v_2, \ldots, v_m \rangle) = \partial(\mathrm{St}(v_0, A)) \times \langle v_1, v_2, \ldots, v_m \rangle .$$

Therefore if $X_4 = X_3 \cup (\overline{\mathrm{St}(v_0, A)} \times \langle v_1, v_2, \ldots, v_m \rangle)$, then $H_r(X_4)$ is isomorphic to $H_r(A^*)$ for all $r > n$. If $A = B^n$, then, because of the homeomorphism g, each $(n-1)$-cycle in $\partial(\mathrm{St}(v_0, A)) \times \langle v_1, v_2, \ldots, v_m \rangle$ bounds in both X_3 and $\overline{\mathrm{St}(v_0, A)} \times \langle v_1, v_2, \ldots, v_m \rangle$. Hence, in this case, $H_n(X_4)$ is isomorphic to $\mathbf{Z}$ and $H_r(X_4) = 0$ for all $r \neq n$. Finally

$$X_4 \cap P(B^*) = (\langle v_1, v_2, \ldots, v_m \rangle \times \langle v_0 \rangle) \cup (\langle v_0 \rangle \times \langle v_1, v_2, \ldots, v_m \rangle) .$$

By Corollary 1 of [6], $P(B^*)$ has the homotopy type of S^{m-1}. Thus it follows immediately that $H_r(X^*)$ is isomorphic to $H_r(A^*)$ for all $r > n$. Since A is n-dimensional and $n \geqq 2$, there is a 2-simplex $\langle u_0, u_1, u_2 \rangle$ in A. Since A is contractible, there is an arc R in A from v_0 to $\langle u_0, u_1, u_2 \rangle$ such that $R \cap \langle u_0, u_1, u_2 \rangle$ is a vertex. We may assume that this vertex is u_0. Also there is an arc T in $P(B^*)$ from (v_0, v_1) to (v_1, v_0). Then

$$\begin{gathered}(R \times \langle v_1 \rangle) \cup (\langle u_0, u_1 \rangle \times \langle v_1 \rangle) \cup (\langle u_1 \rangle \times \langle v_1, v_0 \rangle) \cup (\langle u_1 \rangle \times R) \cup \\ \cup (\langle u_1 \rangle \times \langle u_0, u_2 \rangle) \cup (\langle u_1, u_0 \rangle \times \langle u_2 \rangle) \cup (R \times \langle u_2 \rangle) \cup \\ \cup (\langle v_0, v_1 \rangle \times \langle u_2 \rangle) \cup (\langle v_1 \rangle \times \langle u_2, u_0 \rangle) \cup (\langle v_1 \rangle \times R) \cup T\end{gathered}$$

is a circle in $P(X^*)$ and no nontrivial 1-cycle associated with this circle bounds in $P(X^*)$. Therefore if $m > 2$, then $H_{m-1}(X^*)$ and $H_1(X^*)$ are two nontrivial homology groups, and if $m = 2$, then the direct sum $\mathbf{Z} + \mathbf{Z}$ is a direct summand in $H_1(X^*)$. Thus, in either case, X^* does not have the homotopy type of a sphere. It also follows immediately from the above that if $A = B^n$ and $m > 2$, then $H_n(X^*)$, $H_{m-1}(X^*)$, and $H_1(X^*)$ are isomorphic to $\mathbf{Z}$ and $H_r(X^*) = 0$ if r is neither n, $m-1$, nor 1. Similarly if $A = B^n$ and $m = 2$, then $H_n(X^*)$ is isomorphic to $\mathbf{Z}$, $H_1(X^*)$ is isomorphic to the direct sum $\mathbf{Z} + \mathbf{Z}$, and $H_r(X^*) = 0$ if $n \neq r \neq 1$.

Theorem 2. *Suppose $1 \leqq p < m$ and $A \cap B = \bigcup_{\lambda=1}^{p} \{\langle v_{\lambda-1}, v_\lambda \rangle\}$, where, for each λ, $\langle v_{\lambda-1}, v_\lambda \rangle$ is a simplex of A. If $X = A \cup B$, then $H_r(X^*)$ is isomorphic to $H_r(A^*)$ for all $r > n$ and X^* does not have the homotopy type of an n-sphere.*

Proof. Now $P(X^*)$ can be constructed by starting with $P(A^*)$ and adding cells. We express $P(X^*)$ as the union of $P(A^*)$ and these cells, and, after this expression, we explain the order in which we are going to add cells to $P(A^*)$ in order to get $P(X^*)$. If $i_0 = -1$, then

$$P(X^*) = P(A^*) \cup (B \times A[v_0, \ldots, v_p]) \cup$$
$$\cup \bigcup_{j=1}^{p+1} \bigcup_{k=1}^{j} \bigcup_{i_k=i_{k-1}+1}^{p-j+k} (\langle v_0, v_1, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m\rangle \times A[v_0, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_p]) \cup$$
$$\cup (A[v_0, \ldots, v_p] \times B) \cup$$
$$\cup \bigcup_{j=1}^{p+1} \bigcup_{k=1}^{j} \bigcup_{i_k=i_{k-1}+1}^{p-j+k} (A[v_0, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_p] \times \langle v_0, v_1, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m\rangle) \cup$$
$$\cup P(B^*) .$$

We add the above unions to $P(A^*)$ in the order in which we have listed them. We introduce some notation in order to explain the order in which we add the cells in the second union. With each cell

$$\bigcup_{k=1}^{j} (\langle v_0, v_1, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m\rangle \times A[v_0, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_p]) ,$$

associate a $(p+1)$-tuple $(\alpha_0, \alpha_1, \ldots, \alpha_p)$ as follows: $\alpha_\mu = 1$ if v_μ is omitted in the simplex $\bigcup_{k=1}^{j} \langle v_0, v_1, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m\rangle$ and $\alpha_\mu = 0$ otherwise. If $(\alpha_0, \alpha_1, \ldots, \alpha_p)$ and $(\beta_0, \beta_1, \ldots, \beta_p)$ are distinct $(p+1)$-tuples obtained in this manner, we define $(\alpha_0, \alpha_1, \ldots, \alpha_p) < (\beta_0, \beta_1, \ldots, \beta_p)$ if and only if either

$$(1) \qquad \sum_{\mu=0}^{p} \alpha_\mu < \sum_{\mu=0}^{p} \beta_\mu$$

or

$$(2) \qquad \sum_{\mu=0}^{p} \alpha_\mu = \sum_{\mu=0}^{p} \beta_\mu \text{ and, if } r = \min\{s \mid \alpha_s \neq \beta_s\}, \text{ then } \alpha_r > \beta_r .$$

Then if $(\alpha_0, \alpha_1, \ldots, \alpha_p) < (\beta_0, \beta_1, \ldots, \beta_p)$ we add the cell associated with $(\alpha_0, \alpha_1, \ldots, \alpha_p)$ before we add the cell associated with $(\beta_0, \beta_1, \ldots, \beta_p)$.

Now if $\sigma_1 \times \tau_1$ and $\sigma_2 \times \tau_2$ are two cells in the fourth union, then $\tau_1 \times \sigma_1$ and $\tau_2 \times \sigma_2$ are cells in the second union, and we add $\sigma_1 \times \tau_1$ before $\sigma_2 \times \tau_2$ if and only if we added $\tau_1 \times \sigma_1$ before $\tau_2 \times \sigma_2$.

From $P(A^*) \cap (B \times A[v_0, \ldots, v_p]) = (A \cap B) \times A[v_0, \ldots, v_p])$, it follows that $P(A^*) \cup P(B \times A[v_0, \ldots, v_p])$ is homotopically equivalent to $P(A^*)$. Now suppose $1 \leqq \alpha \leqq p$, and let X_1 be the union of $P(A^*)$ with all those cells which have been added before

$$E_1 = \bigcup_{k=1}^{\alpha} (\langle v_0, v_1, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m\rangle \times A[v_0, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_p] .$$

Then

$$X \cap E_1 = [A \cap \bigcup_{k=1}^{\alpha} \langle v_0, v_1, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m\rangle] \times A[v_0, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_p] \cup$$
$$\cup \bigcup_{\beta=1}^{\alpha} \{\bigcup_{k=1}^{\alpha} \langle v_0, v_1, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m\rangle \times$$
$$\times [A - (\bigcup_{q=0}^{p} \{\mathrm{St}(v_q, A) \mid q \neq i_k \text{ for any } k\} \cup \mathrm{St}(v_{i_\beta}, A))]\} ,$$

and hence $H_r(X_1 \cup E_1)$ is isomorphic to $H_r(X_1)$ for all $r > n$. Therefore, if

$$X_2 = P(A^*) \cup (B \times A[v_0, \ldots, v_p]) \cup$$
$$\cup \bigcup_{j=1}^{p} \bigcup_{k=1}^{j} \bigcup_{i_k=i_{k-1}+1}^{p-j+k} (\langle v_0, v_1, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m\rangle \times A[v_0, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_p]) ,$$

then $H_r(X_2)$ is isomorphic to $H_r(A^*)$ for all $r>n$. Now

$$X_2 \cap (\langle v_{p+1}, v_{p+2}, \ldots, v_m\rangle \times A) = \bigcup_{q=0}^{p} (\langle v_{p+1}, v_{p+2}, \ldots, v_m\rangle \times A[v_q]) .$$

Hence if $X_3 = X_2 \cup (\langle v_{p+1}, v_{p+2}, \ldots, v_m\rangle \times A)$, then $H_r(X_3)$ is isomorphic to $H_r(A^*)$ for all $r>n$. By essentially repeating the above argument, we can show that if

$$X_4 = X_3 \cup (A[v_0, \ldots, v_p] \times B) \cup$$
$$\cup \bigcup_{j=1}^{p+1} \bigcup_{k=1}^{j} \bigcup_{i_k=i_{k-1}+1}^{p-j+k} (A[v_0, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_p] \times \langle v_0, v_1, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m\rangle) ,$$

then $H_r(X_4)$ is isomorphic to $H_r(A^*)$ for all $r>n$. Finally

$$X_4 \cap P(B^*) = \bigcup_{q=1}^{p} (\langle v_{q-1}, v_q\rangle \times \langle v_0, v_1, \ldots, v_{q-2}, v_{q+1}, \ldots, v_m\rangle) \cup$$
$$\cup \bigcup_{q=0}^{p} (\langle v_q\rangle \times \langle v_0, v_1, \ldots, \hat{v}_q, \ldots, v_m\rangle) \cup \bigcup_{q=0}^{p} (\langle v_0, v_1, \ldots, \hat{v}_q, \ldots, v_m\rangle \times \langle v_q\rangle) \cup$$
$$\cup \bigcup_{q=1}^{p} (\langle v_0, v_1, \ldots, v_{q-2}, v_{q+1}, \ldots, v_m\rangle \times \langle v_{q-1}, v_q\rangle) .$$

For each $\alpha = 2, 3, \ldots, p$,

$$\bigcup_{\beta=1}^{\alpha-1} (\langle v_{\beta-1}, v_\beta\rangle \times \langle v_0, v_1, \ldots, v_{\beta-2}, v_{\beta+1}, \ldots, v_m\rangle) \cap$$
$$\cap (\langle v_{\alpha-1}, v_\alpha\rangle \times \langle v_0, v_1, \ldots, v_{\alpha-2}, v_{\alpha+1}, \ldots, v_m\rangle)$$
$$= (\langle v_{\alpha-1}\rangle \times \langle v_0, v_1, \ldots, v_{\alpha-3}, v_{\alpha+1}, \ldots, v_m\rangle) .$$

Therefore $\bigcup_{q=1}^{p} (\langle v_{q-1}, v_q\rangle \times \langle v_0, v_1, \ldots, v_{q-2}, v_{q+1}, \ldots, v_m\rangle)$ is contractible. Also, for each $\alpha = 0, 1, \ldots, p$,

$$[\bigcup_{q=1}^{p} (\langle v_{q-1}, v_q\rangle \times \langle v_0, v_1, \ldots, v_{q-2}, v_{q+1}, \ldots, v_m\rangle) \cup$$
$$\cup \bigcup_{\beta=0}^{\alpha-1} (\langle v_\beta\rangle \times \langle v_0, v_1, \ldots, \hat{v}_\beta, \ldots, v_m\rangle)] \cap (\langle v_\alpha\rangle \times \langle v_0, v_1, \ldots, \hat{v}_\alpha, \ldots, v_m\rangle)$$
$$= (\langle v_\alpha\rangle \times \langle v_0, v_1, \ldots, v_{\alpha-2}, v_{\alpha+1}, \ldots, v_m\rangle) \cup$$
$$\cup (\langle v_\alpha\rangle \times \langle v_0, v_1, \ldots, v_{\alpha-1}, v_{\alpha+2}, \ldots, v_m\rangle) .$$

Hence

$$X_5 = \bigcup_{q=1}^{p} (\langle v_{q-1}, v_q\rangle \times \langle v_0, v_1, \ldots, v_{q-2}, v_{q+1}, \ldots, v_m\rangle) \cup$$
$$\cup \bigcup_{q=0}^{p} (\langle v_q\rangle \times \langle v_0, v_1, \ldots, \hat{v}_q, \ldots, v_m\rangle)$$

is contractible. Similarly

$$X_6 = \bigcup_{q=0}^{p} (\langle v_0, v_1, \ldots, \hat{v}_q, \ldots, v_m\rangle \times \langle v_q\rangle) \cup$$
$$\cup \bigcup_{q=1}^{p} \langle v_0, v_1, \ldots, v_{q-2}, v_{q+1}, \ldots, v_m\rangle \times \langle v_{q-1}, v_q\rangle)$$

is contractible. Since

$$X_5 = P[(A \cap B) \times B - \Delta], \ X_6 = P[B \times (A \cap B) - \Delta] ,$$
$$X_5 \cap X_6 = P[(A \cap B) \times (A \cap B) - \Delta] ,$$

and $A \cap B$ is an arc; $X_5 \cap X_6$ has two components, each of which is contractible. Therefore $X_4 \cap P(B^*)$ has the homotopy type of a circle. By

Corollary 1 of [6], $P(B^*)$ has the homotopy of S^{m-1}. Therefore $H_r(X^*)$ is isomorphic to $H_r(A^*)$ for all $r>n$, and $\mathbf{Z}$ is a direct summand in $H_{m-1}(X^*)$. Since $m-1<n$, X^* does not have the homotopy type of an n-sphere.

THEOREM 3. *If* $A\cap B=\bigcup_{\lambda=1}^{m}\{v_{\lambda-1},v_\lambda\}$, *where, for each* λ, $\langle v_{\lambda-1},v_\lambda\rangle$ *is a simplex of* A, *and* $X=A\cup B$, *then* $H_r(X^*)$ *is isomorphic to* $H_r(A^*)$ *for all* $r>n$ *and* X^* *does not have the homotopy type of an* n*-sphere.*

PROOF. The proof is very similar to the proof of Theorem 2. However if $i_0=-1$, then

$$\begin{gathered}P(X^*) = P(A^*)\cup(B\times A[v_0,\dots,v_m])\cup\\ \cup\bigcup_{j=1}^{m-2}\bigcup_{k=1}^{j}\bigcup_{i_k=i_{k-1}+1}^{m-j+k}(\langle v_0,v_1,\dots,\{\hat v_{i_k}\},\dots,v_m\rangle\times A[v_0,\dots,\{\hat v_{i_k}\},\dots,v_m])\cup\\ \cup\bigcup_{a=0}^{m-2}\bigcup_{b=a+2}^{m}(\langle v_a,v_b\rangle\times A[v_a,v_b])\cup(A[v_0,\dots,v_m]\times B)\cup\\ \cup\bigcup_{j=1}^{m-2}\bigcup_{k=1}^{j}\bigcup_{i_k=i_{k-1}+1}^{m-j+k}(A[v_0,\dots,\{\hat v_{i_k}\},\dots,v_m]\times\langle v_0,v_1,\dots,\{\hat v_{i_k}\},\dots,v_m\rangle)\cup\\ \cup\bigcup_{a=0}^{m-2}\bigcup_{b=a+2}^{m}(A[v_a,v_b]\times\langle v_a,v_b\rangle)\cup P(B^*)\,.\end{gathered}$$

We add the above unions to $P(A^*)$ in the order in which we have listed them. We order the cells within a given union in exactly the same way that we ordered the cells within the corresponding union in the proof of Theorem 2, and then we add the cells according to this ordering. Also, as in the proof of Theorem 2,

$$P(A^*)\cup(B\times A[v_0,\dots,v_m])$$

is homotopically equivalent to $P(A^*)$. By making the obvious modifications to the proof of Theorem 2, we can show that if

$$\begin{gathered}X_1 = P(A^*)\cup(B\times A[v_0,\dots,v_m])\cup\\ \cup\bigcup_{j=1}^{m-2}\bigcup_{k=1}^{j}\bigcup_{i_k=i_{k-1}+1}^{m-j+k}(\langle v_0,v_1,\dots,\{\hat v_{i_k}\},\dots,v_m\rangle\times A[v_0,\dots,\{\hat v_{i_k}\},\dots,v_m])\cup\\ \cup\bigcup_{a=0}^{m-2}\bigcup_{b=a+2}^{m}(\langle v_a,v_b\rangle\times A[v_a,v_b])\cup(A[v_0,\dots,v_m]\times B)\cup\\ \cup\bigcup_{j=1}^{m-2}\bigcup_{k=1}^{j}\bigcup_{i_k=i_{k-1}+1}^{m-j+k}(A[v_0,\dots,\{\hat v_{i_k}\},\dots,v_m]\times\langle v_0,v_1,\dots,\{\hat v_{i_k}\},\dots,v_m\rangle)\cup\\ \cup\bigcup_{a=0}^{m-2}\bigcup_{b=a+2}^{m}(A[v_a,v_b]\times\langle v_a,v_b\rangle)\,,\end{gathered}$$

then $H_r(X_1)$ is isomorphic to $H_r(A^*)$ for all $r>n$. Also

$$\begin{gathered}X_1\cap P(B^*) = \bigcup_{q=1}^{m}(\langle v_{q-1},v_q\rangle\times\langle v_0,v_1,\dots,v_{q-2},v_{q+1},\dots,v_m\rangle)\cup\\ \cup\bigcup_{q=0}^{m}(\langle v_q\rangle\times\langle v_0,v_1,\dots,\hat v_q,\dots,v_m\rangle)\cup\bigcup_{q=0}^{m}(\langle v_0,v_1,\dots,\hat v_q,\dots,v_m\rangle\times\langle v_q\rangle)\cup\\ \cup\bigcup_{q=1}^{m}(\langle v_0,v_1,\dots,v_{q-2},v_{q+1},\dots,v_m\rangle\times\langle v_{q-1},v_q\rangle)\,,\end{gathered}$$

and hence we can use the proof of Theorem 2 again to show that $H_r(X^*)$ is isomorphic to $H_r(A^*)$ for all $r>n$ and X^* does not have the homotopy type of an n-sphere.

3. Trivial homology groups of deleted products.

THEOREM 4. *Let A be a finite, n-dimensional polyhedron, and let B be an m-simplex, $2 \leqq m \leqq n$, with vertices $v_0, v_1, \ldots, v_m$. If $A \cap B = \langle v_0, v_2, \ldots, v_m \rangle$, where $\langle v_0, v_2, \ldots, v_m \rangle$ is a simplex of A, and $X = A \cup B$; then $H_r(X^*)$ is isomorphic to $H_r(A^*)$ for all $r > n$.*

PROOF. Let $f\colon X \to A$ be the simplicial map defined by $f(w) = w$ for each vertex w of A and $f(v_1) = v_0$. Then, by Theorem 2 of [7], $P(X_f^*)$ is homotopically equivalent to $P(A^*)$. If $i_0 = -1$, then

$$\begin{aligned} P(X^*) = {} & P(X_f^*) \cup (\langle v_1, v_2, \ldots, v_m \rangle \times A[v_0 \mid v_2, \ldots, v_m]) \cup \\ & \cup \textstyle\bigcup_{j=1}^{m-1} \bigcup_{k=1}^{j} \bigcup_{i_k = i_{k-1}+1}^{m-j+k} (\langle v_1, v_2, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m \rangle \times \\ & \times A[v_0 \mid v_2, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m]) \cup (A[v_0 \mid v_2, \ldots, v_m] \times \langle v_1, v_2, \ldots, v_m \rangle) \cup \\ & \cup \textstyle\bigcup_{j=1}^{m-1} \bigcup_{k=1}^{j} \bigcup_{i_k = i_{k-1}+1}^{m-j+k} (A[v_0 \mid v_2, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m] \times \\ & \times \langle v_1, v_2, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m \rangle) . \end{aligned}$$

We add the above unions to $P(X_f^*)$ in the order in which we have listed them. We order the cells within a given union in the same manner that we ordered the cells within a given union in the proof of Theorem 2, and then we add the cells according to this ordering. Since

$$\begin{aligned} & P(X_f^*) \cap (\langle v_1, v_2, \ldots, v_m \rangle \times A[v_0 \mid v_2, \ldots, v_m]) \\ & \quad = (\langle v_2, v_3, \ldots, v_m \rangle \times A[v_0 \mid v_2, \ldots, v_m]) \cup \\ & \qquad \cup (\langle v_1, v_2, \ldots, v_m \rangle \times [\partial(\mathrm{St}(v_0, A)) - \textstyle\bigcup_{q=2}^{m} \mathrm{St}(v_q, A)]) , \end{aligned}$$

$$H_k(P(X_f^*) \cup (\langle v_1, v_2, \ldots, v_m \rangle \times A[v_0 \mid v_2, \ldots, v_m]))$$

is isomorphic to $H_k(P(X_f^*))$ for each k. Now suppose $1 \leqq \alpha \leqq m-2$, and let X_1 be the union of $P(X_f^*)$ with all those cells which have been added before

$$E_1 = \textstyle\bigcup_{k=1}^{\alpha} (\langle v_1, v_2, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m \rangle \times A[v_0 \mid v_2, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m]) .$$

Then

$$\begin{aligned} X_1 \cap E_1 = {} & \textstyle\bigcup_{k=1}^{\alpha} (\langle v_2, v_3, \ldots, \{\hat{v}_{i_k}\}, \ldots \times, v_m \rangle \times A[v_0 \mid v_2, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m]) \cup \\ & \cup \textstyle\bigcup_{k=1}^{\alpha} (\langle v_1, v_2, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m \rangle \times \\ & \times [\partial(\mathrm{St}(v_0, A)) - \textstyle\bigcup_{q=2}^{m} \{\mathrm{St}(v_q, A) \mid q \neq i_k \text{ for any } k\}]) \cup \\ & \cup \underline{\textstyle\bigcup_{\beta=1}^{\alpha}} \{(\textstyle\bigcup_{k=1}^{\alpha} \langle v_1, v_2, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m \rangle) \times \\ & \times \overline{[\mathrm{St}(v_0, A) - (\textstyle\bigcup_{q=2}^{m} \{\mathrm{St}(v_q, A) \mid q \neq i_k \text{ for any } k\} \cup \mathrm{St}(v_{i_\beta}, A))]} \} . \end{aligned}$$

Therefore, if

$$X_2 = P(X_f{}^*) \cup (\langle v_1, v_2, \ldots, v_m\rangle \times A[v_0 \mid v_2, \ldots, v_m]) \cup \\ \cup \bigcup_{j=1}^{m-2} \bigcup_{k=1}^{j} \bigcup_{i_k=i_{k-1}+1}^{m-j+k} (\langle v_1, v_2, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m\rangle \times \\ \times A[v_0 \mid v_2, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m]) ,$$

then $H_k(X_2)$ is isomorphic to $H_k(P(X_f{}^*))$ for each k. Now

$$X_2 \cap (\langle v_1\rangle \times \overline{\mathrm{St}(v_0, A)}) = (\langle v_1\rangle \times \partial(\mathrm{St}(v_0, A))) \cup \bigcup_{q=2}^{m} (\langle v_1\rangle \times A[v_0 | v_q]) ,$$

and hence $H_k(X_2 \cup (\langle v_1\rangle \times \overline{\mathrm{St}(v_0, A)}))$ is isomorphic to $H_k(A^*)$ for each $k > n$. By essentially repeating the above argument, we can show that $H_k(X^*)$ is isomorphic to $H_k(A^*)$ for each $k > n$.

THEOREM 5. *If X is a finite, contractible, n-dimensional, $n \geqq 3$, polyhedron with the property that a homeomorph of X can be constructed out of an n-simplex A by appending m-simplexes, $1 \leqq m \leqq n$, in such a way that the construction may be factored*

$$A = X_1 \to X_2 \to \ldots \to X_p = X$$

so that X_i is obtained from X_{i-1} by

(1) *adding an m-simplex, $1 \leqq m \leqq n$, which meets X_{i-1} in just one of its vertices,*

(2) *adding a 2-simplex which meets X_{i-1} in just one of its 1-faces,*

(3) *adding an m-simplex, $3 \leqq m \leqq n$, which meets X_{i-1} in exactly p of its 1-faces, $1 \leqq p \leqq m$, where the intersection of X_{i-1} and the appended simplex is an arc, or*

(4) *adding an m-simplex, $3 \leqq m \leqq n$, which meets X_{i-1} in exactly one of its $(m-1)$-faces,*

then $H_r(X^*) = 0$ for all $r > n$.

PROOF. By Corollary 1 of [6], A^* has the homotopy type of S^{n-1}. Suppose $2 \leqq i \leqq p$ and $H_r((X_{i-1})^*) = 0$ for all $r > n$. If X_i is obtained from X_{i-1} by (1), where $m = 1$, then, by Theorem 4 of [7], $H_r((X_i)^*) = 0$ for all $r > n$. If X_i is obtained from X_{i-1} by (1) and $2 \leqq m \leqq n$, then, by Theorem 1, $H_r((X_i)^*) = 0$ for all $r > n$. If X_i is obtained from X_{i-1} by (2), then, by Theorem 4, $H_r((X_i)^*) = 0$ for all $r > n$. If X_i is obtained from X_{i-1} by (3) and $1 \leqq p < m$, then $H_r((X_i)^*) = 0$ for all $r > n$ by Theorem 2. If X_i is obtained from X_{i-1} by (3) and $p = m$, then, by Theorem 3, $H_r((X_i)^*) = 0$ for all $r > n$. If X_i is obtained from X_{i-1} by (4), then $H_r((X_i)^*) = 0$ for all $r > n$ by Theorem 4. Hence $H_r(X^*) = 0$ for all $r > n$.

THEOREM 6. *Suppose $2 \leqq m \leqq n$ and let $D_m{}^n$ be the polyhedron whose vertices are $\{u_0, u_1, \ldots, u_n, v_0, v_1, \ldots, v_m, v\}$ and whose simplexes are*

$$\{\langle u_0, u_1, \ldots, \hat{u}_i, \ldots, u_n, v\rangle \mid i=0,1,\ldots,n\}\,,$$
$$\square\{\langle v_0, v_1, \ldots, \hat{v}_j, \ldots, v_m, v\rangle \mid j=0,1,\ldots,m\}\,.$$

If $m \neq n$, $H_{n+m-1}((D_m{}^n)^*)$ *is isomorphic to the direct sum* $\mathsf{Z}+\mathsf{Z}$,

$$H_n((D_m{}^n)^*), \quad H_m((D_m{}^n)^*), \quad H_1((D_m{}^n)^*)$$

are isomorphic to Z, *and* $H_r((D_m{}^n)^*)=0$ *if* r *is neither* $n+m-1$, n, m, *nor* 1. *If* $m=n$, $H_{2n-1}((D_m{}^n)^*)$ *and* $H_n((D_m{}^n)^*)$ *are isomorphic to the direct sum* $\mathsf{Z}+\mathsf{Z}$, $H_1((D_m{}^n)^*)$ *is isomorphic to* Z, *and* $H_r((D_m{}^n)^*)=0$ *if* r *is neither* $2n-1$, n, *nor* 1.

PROOF. If $A=\bigcup_{i=0}^n \langle u_0, u_1, \ldots, \hat{u}_i, \ldots, u_n, v\rangle \cup \langle v_1, v_2, v\rangle$, then, by Theorem 1, $H_n(A^*)$ is isomorphic to Z, $H_1(A^*)$ is isomorphic to the direct sum $\mathsf{Z}+\mathsf{Z}$, and $H_r(A^*)=0$ if $n \neq r \neq 1$. Also if

$$A = \bigcup_{i=0}^n \langle u_0, u_1, \ldots, \hat{u}_i, \ldots, u_n, v\rangle \times \langle v_1, v_2, \ldots, v_m, v\rangle\,,$$

where $m>2$, then, by Theorem 1,

$$H_n(A^*), \quad H_{m-1}(A^*), \quad H_1(A^*)$$

are isomorphic to Z and $H_r(A^*)=0$ if r is neither n, $m-1$, nor 1. It is clear that if

$$B = \bigcup_{i=0}^n \langle u_0, u_1, \ldots, \hat{u}_i, \ldots, u_n, v\rangle \cup \bigcup_{j=0}^{m-1} \langle v_0, v_1, \ldots, \hat{v}_j, \ldots, v_m, v\rangle\,,$$

then $H_k(B^*)$ is isomorphic to $H_k(A^*)$ for each k. If $\sigma=\langle v_0, v_1, \ldots, v_{m-1}, v\rangle$, then

$$\begin{aligned} P((D_m{}^n)^*) = {} & P(B^*) \cup \bigcup_{i=0}^n (\sigma \times \langle u_0, u_1, \ldots, \hat{u}_i, \ldots, u_n\rangle) \cup (\sigma \times \langle v_m\rangle) \cup \\ & \cup \bigcup_{i=0}^n (\langle v_0, v_1, \ldots, v_{m-1}\rangle \times \langle u_0, u_1, \ldots, \hat{u}_i, \ldots, u_n, v\rangle) \cup \\ & \cup (\langle v_0, v_1, \ldots, v_{m-1}\rangle \times \langle v_m, v\rangle) \cup \\ & \cup \bigcup_{i=0}^n (\langle u_0, u_1, \ldots, \hat{u}_i, \ldots, u_n\rangle \times \sigma) \cup (\langle v_m\rangle \times \sigma) \cup \\ & \cup \bigcup_{i=0}^n (\langle u_0, u_1, \ldots, \hat{u}_i, \ldots, u_n, v\rangle \times \langle v_0, v_1, \ldots, v_{m-1}\rangle) \cup \\ & \cup (\langle v_m, v\rangle \times \langle v_0, v_1, \ldots, v_{m-1}\rangle)\,. \end{aligned}$$

For each $k=0,1,\ldots,n$,

$$\begin{aligned} & [P(B^*) \cup \bigcup_{i=0}^{k-1} (\sigma \times \langle u_0, u_1, \ldots, \hat{u}_i, \ldots, u_n\rangle)] \cap (\sigma \times \langle u_0, u_1, \ldots, \hat{u}_k, \ldots, u_n\rangle) \\ & \quad = \bigcup_{j=0}^{m-1} (\langle v_0, v_1, \ldots, \hat{v}_j, \ldots, v_{m-1}, v\rangle \times \langle u_0, u_1, \ldots, \hat{u}_k, \ldots, u_n\rangle) \cup \\ & \qquad \cup \bigcup_{i=0}^{k-1} (\sigma \times \langle u_0, u_1, \ldots, \hat{u}_i, \ldots, \hat{u}_k, \ldots, u_n\rangle)\,. \end{aligned}$$

Therefore $X_1=P(B^*) \cup \bigcup_{i=0}^n (\sigma \times \langle u_0, u_1, \ldots, \hat{u}_i, \ldots, u_n\rangle)$ is homotopically equivalent to $P(B^*)$. Also

$$X_1 \cap (\sigma \times \langle v_m\rangle) = \bigcup_{j=0}^{m-1} (\langle v_0, v_1, \ldots, \hat{v}_j, \ldots, v_{m-1}, v\rangle \times \langle v_m\rangle)\,,$$

and hence $X_2 = X_1 \cup (\sigma \times \langle v_m \rangle)$ is homotopically equivalent to $P(B^*)$. Now, for each $k = 0, 1, \ldots, n$,

$$\begin{aligned}
&[X_2 \cup \textstyle\bigcup_{i=0}^{k-1} (\langle v_0, v_1, \ldots, v_{m-1} \rangle \times \langle u_0, u_1, \ldots, \hat{u}_i, \ldots, u_n, v \rangle)] \cap \\
&\quad \cap (\langle v_0, v_1, \ldots, v_{m-1} \rangle \times \langle u_0, u_1, \ldots, \hat{u}_k, \ldots, u_n, v \rangle) \\
&= \textstyle\bigcup_{j=0}^{m-1} (\langle v_0, v_1, \ldots, \hat{v}_j, \ldots, v_{m-1} \rangle \times \langle u_0, u_1, \ldots, \hat{u}_k, \ldots, u_n, v \rangle) \cup \\
&\quad \cup (\langle v_0, v_1, \ldots, v_{m-1} \rangle \times \langle u_0, u_1, \ldots, \hat{u}_k, \ldots, u_n \rangle) \cup \\
&\quad \cup \textstyle\bigcup_{i=0}^{k-1} (\langle v_0, v_1, \ldots, v_{m-1} \rangle \times \langle u_0, u_1, \ldots, \hat{u}_i, \ldots, \hat{u}_k, \ldots, u_n, v \rangle) ,
\end{aligned}$$

and therefore

$$X_3 = X_2 \cup \textstyle\bigcup_{i=0}^{n-1} (\langle v_0, v_1, \ldots, v_{m-1} \rangle \times \langle u_0, u_1, \ldots, \hat{u}_i, \ldots, u_n, v \rangle)$$

is homotopically equivalent to $P(B^*)$. However

$$X_3 \cap (\langle v_0, v_1, \ldots, v_{m-1} \rangle \times \langle u_0, u_1, \ldots, u_{n-1}, v \rangle)$$

has the homotopy type of S^{n+m-2}. It is clear that each cycle associated with

$$X_3 \cap (\langle v_0, v_1, \ldots, v_{m-1} \rangle \times \langle u_0, u_1, \ldots, u_{n-1}, v \rangle)$$

bounds in

$$\langle v_0, v_1, \ldots, v_{m-1} \rangle \times \langle u_0, u_1, \ldots, u_{n-1}, v \rangle ,$$

and it is also true that each cycle associated with

$$X_3 \cap (\langle v_0, v_1, \ldots, v_{m-1} \rangle \times \langle u_0, u_1, \ldots, u_{n-1}, v \rangle)$$

bounds in the subset

$$\begin{aligned}
&\textstyle\bigcup_{i=0}^{n} (\sigma \times \langle u_0, u_1, \ldots, \hat{u}_i, \ldots, u_n \rangle) \cup \\
&\cup \textstyle\bigcup_{i=0}^{n-1} (\langle v_0, v_1, \ldots, v_{m-1} \rangle \times \langle u_0, u_1, \ldots, \hat{u}_i, \ldots, u_n, v \rangle) \cup \\
&\cup \textstyle\bigcup_{j=0}^{m-1} \bigcup_{i=0}^{n} (\langle v_0, v_1, \ldots, \hat{v}_j, \ldots, v_m, v \rangle \times \langle u_0, u_1, \ldots, \hat{u}_i, \ldots, u_n \rangle) \cup \\
&\cup \textstyle\bigcup_{j=0}^{m-1} \bigcup_{i=0}^{n} (\langle v_0, v_1, \ldots, \hat{v}_j, \ldots, v_m \rangle \times \langle u_0, u_1, \ldots, \hat{u}_i, \ldots, u_n, v \rangle)
\end{aligned}$$

of X_3. Therefore if

$$X_4 = X_3 \cup (\langle v_0, v_1, \ldots, v_{m-1} \rangle \times \langle u_0, u_1, \ldots, u_{n-1}, v \rangle) ,$$

then, if $m > 2$,

$$H_{n+m-1}(X_4), \quad H_n(X_4), \quad H_{m-1}(X_4), \quad H_1(X_4)$$

are isomorphic to $\mathbf{Z}$ and $H_r(X_4) = 0$ if r is neither $n+m-1$, n, $m-1$, nor 1 and, if $m = 2$, $H_{n+1}(X_4)$ and $H_n(X_4)$ are isomorphic to $\mathbf{Z}$, $H_1(X_4)$ is isomorphic to the direct sum $\mathbf{Z}+\mathbf{Z}$, and $H_r(X_4) = 0$ if r is neither $n+1$, n, nor 1. Now

$$\begin{aligned} X_4 \cap (\langle v_0, v_1, \ldots, v_{m-1}\rangle \times \langle v_m, v\rangle) \\ = \bigcup_{j=0}^{m-1} (\langle v_0, v_1, \ldots, \hat{v}_j, \ldots, v_{m-1}\rangle \times \langle v_m, v\rangle) \cup \\ \cup (\langle v_0, v_1, \ldots, v_{m-1}\rangle \times \langle v_m\rangle) \cup (\langle v_0, v_1, \ldots, v_{m-1}\rangle \times \langle v\rangle) , \end{aligned}$$

and hence $X_4 \cap (\langle v_0, v_1, \ldots, v_{m-1}\rangle \times \langle v_m, v\rangle)$ has the homotopy type of S^{m-1}. Let

$$\begin{aligned} G &= \bigcup_{i=0}^{n} \langle u_0, u_1, \ldots, \hat{u}_i, \ldots, u_n, v\rangle , \\ H &= \bigcup_{j=0}^{m-1} \langle v_0, v_1, \ldots, \hat{v}_j, \ldots, v_m, v\rangle , \\ Y_1 &= P(G^*) \cup P(G \times H - \Delta) \cup P(H \times G - \Delta) \cup \\ &\quad \cup \bigcup_{i=0}^{n} (\sigma \times \langle u_0, u_1, \ldots, \hat{u}_i, \ldots, u_n\rangle) \cup \\ &\quad \cup \bigcup_{i=0}^{n} (\langle v_0, v_1, \ldots, v_{m-1}\rangle \times \langle u_0, u_1, \ldots, \hat{u}_i, \ldots, u_n, v\rangle) , \\ Y_2 &= P(H^*) \cup (\sigma \times \langle v_m\rangle) . \end{aligned}$$

Then

$$\begin{aligned} X_4 &= Y_1 \cup Y_2 , \\ Y_1 \cap Y_2 &= \bigcup_{j=0}^{m-1} (\langle v\rangle \times \langle v_0, v_1, \ldots, \hat{v}_j, \ldots, v_m\rangle) \cup \\ &\quad \cup \bigcup_{j=0}^{m-1} (\langle v_0, v_1, \ldots, \hat{v}_j, \ldots, v_m\rangle \times \langle v\rangle) , \end{aligned}$$

$$\bigcup_{j=0}^{m-1} (\langle v_0, v_1, \ldots, \hat{v}_j, \ldots, v_{m-1}\rangle \times \langle v_m, v\rangle) \cup (\langle v_0, v_1, \ldots, v_{m-1}\rangle \times \langle v_m\rangle) \subset Y_2 ,$$

and

$$(\langle v_0, v_1, \ldots, v_{m-1}\rangle \times \langle v\rangle) \subset Y_1 .$$

Therefore if any nontrivial $(m-1)$-cycle associated with

$$X_4 \cap (\langle v_0, v_1, \ldots, v_{m-1}\rangle \times \langle v_m, v\rangle)$$

bounds in X_4, then there exists an m-cycle in Y_1 whose boundary is $\langle v_0, v_1, \ldots, v_{m-1}\rangle \times \langle v\rangle$. But this is impossible since $\langle v_0, v_1, \ldots, v_{m-1}\rangle \times \langle v\rangle$ is not an $(m-1)$-cycle. Hence no nontrivial $(m-1)$-cycle associated with

$$X_4 \cap (\langle v_0, v_1, \ldots, v_{m-1}\rangle \times \langle v_m, v\rangle)$$

bounds in X_4. Now observe that if $m > 2$, then $P(H^*)$ carries the $(m-1)$-dimensional homology of X_4, and, if $m = 2$, then $P(H^*)$ carries a subgroup of $H_1(X_4)$ isomorphic to $\mathbf{Z}$. Also observe that if z_1 is a nontrivial $(m-1)$-cycle in $P(H^*)$. then there is an m-chain associated with the subset

$$\begin{aligned} (\sigma \times \langle v_m\rangle) \cup \bigcup_{j=0}^{m} (\langle v_0, v_1, \ldots, \hat{v}_j, \ldots, v_m, v\rangle \times \langle u_0\rangle) \cup \\ \cup \bigcup_{j=0}^{m} (\langle v_0, v_1, \ldots, \hat{v}_j, \ldots, v_m\rangle \times \langle u_0, v\rangle) \end{aligned}$$

of X_4 whose boundary is $z_1 - z_2$, where z_2 is an $(m-1)$-cycle in

$$X_4 \cap (\langle v_0, v_1, \ldots, v_{m-1}\rangle \times \langle v_m, v\rangle) .$$

Thus it follows that if

$$X_5 = X_4 \cup (\langle v_0, v_1, \ldots, v_{m-1}\rangle \times \langle v_m, v\rangle),$$

then $H_{n+m-1}(X_5)$, $H_n(X_5)$, and $H_1(X_5)$ are isomorphic to Z and $H_r(X_5) = 0$ if r is neither $n+m-1$, n, nor 1. For each $k = 0, 1, \ldots, n$,

$$\begin{aligned}[X_5 \cup \textstyle\bigcup_{i=0}^{k-1}(\langle u_0, u_1, \ldots, \hat{u}_i, \ldots, u_n\rangle \times \sigma)] \cap (\langle u_0, u_1, \ldots, \hat{u}_k, \ldots, u_n\rangle \times \sigma) \\ = \textstyle\bigcup_{j=0}^{m-1}(\langle u_0, u_1, \ldots, \hat{u}_k, \ldots, u_n\rangle \times \langle v_0, v_1, \ldots, \hat{v}_j, \ldots, v_{m-1}, v\rangle) \cup \\ \cup \textstyle\bigcup_{i=0}^{k-1}(\langle u_0, u_1, \ldots, \hat{u}_i, \ldots, \hat{u}_k, \ldots, u_n\rangle \times \sigma).\end{aligned}$$

Therefore if $X_6 = X_5 \cup \bigcup_{i=0}^{n}(\langle u_0, u_1, \ldots, \hat{u}_i, \ldots, u_n\rangle \times \sigma)$, then $H_{n+m-1}(X_6)$, $H_n(X_6)$, and $H_1(X_6)$ are isomorphic to Z and $H_r(X_6) = 0$ if r is neither $n+m-1$, n, nor 1. Also

$$X_6 \cap (\langle v_m\rangle \times \sigma) = \textstyle\bigcup_{j=0}^{m-1}(\langle v_m\rangle \times \langle v_0, v_1, \ldots, \hat{v}_j, \ldots, v_{m-1}, v\rangle),$$

and hence $X_7 = X_6 \cup (\langle v_m\rangle \times \sigma)$ is homotopically equivalent to X_6. Now for each $k = 0, 1, \ldots, n$,

$$\begin{aligned}[X_7 \cup \textstyle\bigcup_{i=0}^{k-1}(\langle u_0, u_1, \ldots, \hat{u}_i, \ldots, u_n, v\rangle \times \langle v_0, v_1, \ldots, v_{m-1}\rangle)] \cap \\ \cap (\langle u_0, u_1, \ldots, \hat{u}_k, \ldots, u_n, v\rangle \times \langle v_0, v_1, \ldots, v_{m-1}\rangle) \\ = \textstyle\bigcup_{j=0}^{m-1}(\langle u_0, u_1, \ldots, \hat{u}_k, \ldots, u_n, v\rangle \times \langle v_0, v_1, \ldots, \hat{v}_j, \ldots, v_{m-1}\rangle) \cup \\ \cup (\langle u_0, u_1, \ldots, \hat{u}_k, \ldots, u_n\rangle \times \langle v_0, v_1, \ldots, v_{m-1}\rangle) \cup \\ \cup \textstyle\bigcup_{i=0}^{k-1}(\langle u_0, u_1, \ldots, \hat{u}_i, \ldots, \hat{u}_k, \ldots, u_n, v\rangle \times \langle v_0, v_1, \ldots, v_{m-1}\rangle),\end{aligned}$$

and hence

$$X_8 = X_7 \cup \textstyle\bigcup_{i=0}^{n-1}(\langle u_0, u_1, \ldots, \hat{u}_i, \ldots, u_n, v\rangle \times \langle v_0, v_1, \ldots, v_{m-1}\rangle)$$

is homotopically equivalent to X_7. However

$$X_8 \cap (\langle u_0, u_1, \ldots, u_{n-1}, v\rangle \times \langle v_0, v_1, \ldots, v_{m-1}\rangle)$$

has the homotopy type of S^{n+m-2}. Again each cycle associated with

$$X_8 \cap (\langle u_0, u_1, \ldots, u_{n-1}, v\rangle \times \langle v_0, v_1, \ldots, v_{m-1}\rangle)$$

bounds in $\langle u_0, u_1, \ldots, u_{n-1}, v\rangle \times \langle v_0, v_1, \ldots, v_{m-1}\rangle$ and in the subset

$$\begin{aligned}\textstyle\bigcup_{i=0}^{n}(\langle u_0, u_1, \ldots, \hat{u}_i, \ldots, u_n\rangle \times \sigma) \cup \\ \cup \textstyle\bigcup_{i=0}^{n-1}(\langle u_0, u_1, \ldots, \hat{u}_i, \ldots, u_n, v\rangle \times \langle v_0, v_1, \ldots, v_{m-1}\rangle) \cup \\ \cup \textstyle\bigcup_{i=0}^{n}\bigcup_{j=0}^{m-1}(\langle u_0, u_1, \ldots, \hat{u}_i, \ldots, u_n\rangle \times \langle v_0, v_1, \ldots, \hat{v}_j, \ldots, v_m, v\rangle) \cup \\ \cup \textstyle\bigcup_{i=0}^{n}\bigcup_{j=0}^{m-1}(\langle u_0, u_1, \ldots, \hat{u}_i, \ldots, u_n, v\rangle \times \langle v_0, v_1, \ldots, \hat{v}_j, \ldots, v_m\rangle)\end{aligned}$$

of X_8. Therefore if $X_9 = X_8 \cup (\langle u_0, u_1, \ldots, u_{n-1}, v\rangle \times \langle v_0, v_1, \ldots, v_{m-1}\rangle)$, then $H_{n+m-1}(X_9)$ is isomorphic to the direct sum $\mathsf{Z} + \mathsf{Z}$, $H_n(X_9)$ and $H_1(X_9)$

are isomorphic to $\mathbf{Z}$ and $H_r(X_9)=0$ if r is neither $n+m-1$, n, nor 1. Finally

$$X_9 \cap (\langle v_m, v\rangle \times \langle v_0, v_1, \ldots, v_{m-1}\rangle) = (\langle v\rangle \times \langle v_0, v_1, \ldots, v_{m-1}\rangle) \cup$$
$$\cup (\langle v_m\rangle \times \langle v_0, v_1, \ldots, v_{m-1}\rangle) \cup \textstyle\bigcup_{j=0}^{m-1} (\langle v_m, v\rangle \times \langle v_0, v_1, \ldots, \hat{v}_j, \ldots, v_{m-1}\rangle) ,$$

and hence $X_9 \cap (\langle v_m, v\rangle \times \langle v_0, v_1, \ldots, v_{m-1}\rangle)$ has the homotopy type of S^{m-1}. Now if z_1 is a cycle in $X_9 \cap (\langle v_m, v\rangle \times \langle v_0, v_1, \ldots, v_{m-1}\rangle)$, there is a chain c in

$$P(H^*) \cup (\sigma \times \langle v_m\rangle) \cup (\langle v_m\rangle \times \sigma) \cup$$
$$\cup \textstyle\bigcup_{j=0}^{m} (\langle u_0, v\rangle \times \langle v_0, v_1, \ldots, \hat{v}_j, \ldots, v_m\rangle) \cup$$
$$\cup \textstyle\bigcup_{j=0}^{m} (\langle u_0\rangle \times \langle v_0, v_1, \ldots, \hat{v}_j, \ldots, v_m, v\rangle) \cup$$
$$\cup \textstyle\bigcup_{j=0}^{m} (\langle v_0, v_1, \ldots, \hat{v}_j, \ldots, v_m\rangle \times \langle u_0, v\rangle) \cup$$
$$\cup \textstyle\bigcup_{j=0}^{m} (\langle v_0, v_1, \ldots, \hat{v}_j, \ldots, v_m, v\rangle \times \langle u_0\rangle)$$

such that $\partial c = z_1 - z_2$, where z_2 is a cycle in

$$X_4 \cap (\langle v_0, v_1, \ldots, v_{m-1}\rangle \times \langle v_m, v\rangle) .$$

Therefore each cycle associated with $X_9 \cap (\langle v_m, v\rangle \times \langle v_0, v_1, \ldots, v_{m-1}\rangle)$ bounds in X_9 and in $(\langle v_m, v\rangle \times \langle v_0, v_1, \ldots, v_{m-1}\rangle)$. Hence if $m \neq n$, $H_{n+m-1}\big(P((D_m{}^n)^*)\big)$ is isomorphic to the direct sum $\mathbf{Z}+\mathbf{Z}$, $H_n\big(P((D_m{}^n)^*)\big)$, $H_m\big(P((D_m{}^n)^*)\big)$, and $H_1\big(P((D_m{}^n))\big)$ are isomorphic to $\mathbf{Z}$, and $H_r\big(P((D_m{}^n)^*)\big) = 0$ if r is neither $n+m-1$, n, m, nor 1. If $m=n$, $H_{2n-1}\big(P((D_m{}^n)^*)\big)$ and $H_n\big(P((D_m{}^n)^*)\big)$ are isomorphic to the direct sum $\mathbf{Z}+\mathbf{Z}$, $H_1\big(P((D_m{}^n)^*)\big)$ is isomorphic to $\mathbf{Z}$, and $H_r\big(P((D_m{}^n)^*)\big)=0$ if r is neither $2n-1$, n, nor 1.

THEOREM 7. *Let X be a finite, contractible, n-dimensional polyhedron, and let $B=\langle v_0, v_1, \ldots, v_m\rangle$ be an m-simplex $(3 \leqq m \leqq n)$. If*

$$X \cap B = \textstyle\bigcup_{\lambda=1}^{m} \langle v_0, v_\lambda\rangle ,$$

where, for each λ, $\langle v_0, v_\lambda\rangle$ is a simplex of X, and $Y = X \cup B$; then $H_r(Y^*)$ is isomorphic to $H_r(X^*)$ for all $r > n+1$. Moreover if a homeomorph of X can be constructed out of an n-simplex A by appending t-simplexes $(1 \leqq t \leqq n)$ in such a way that the construction may be factored

$$A = X_1 \to X_2 \to \ldots \to X_p = X$$

so that X_i is obtained from X_{i-1} by (1), (2), (3), *or* (4) *of Theorem* 5, *then $H_r(Y^*)=0$ for all $r>n$.*

PROOF. If the additional hypothesis is satisfied, then, by Theorem 5, $H_r(X^*)=0$ for all $r>n$. Thus if X can be constructed in this manner, we need to show that $H_r(Y^*)$ is isomorphic to $H_r(X^*)$ for all $r>n$. In any case, if $i_0=-1$, then

$$P(Y^*) = P(X^*) \cup (B \times X[v_0,\ldots,v_m]) \cup$$
$$\cup \bigcup_{j=1}^{m-2} \bigcup_{k=1}^{j} \bigcup_{i_k=i_{k-1}+1}^{m-j+k} (\langle v_0,v_1,\ldots,\{\hat{v}_{i_k}\},\ldots,v_m\rangle \times X[v_0,\ldots,\{\hat{v}_{i_k}\},\ldots,v_m]) \cup$$
$$\cup \bigcup_{i_1=1}^{3} \bigcup_{k=2}^{m-2} \bigcup_{i_k=i_{k-1}+1}^{k+2} (\langle v_1,v_2,\ldots,\{\hat{v}_{i_k}\},\ldots,v_m\rangle \times X[v_1,\ldots,\{\hat{v}_{i_k}\},\ldots,v_m]) \cup$$
$$\cup (X[v_0,\ldots,v_m] \times B) \cup$$
$$\cup \bigcup_{j=1}^{m-2} \bigcup_{k=1}^{j} \bigcup_{i_k=i_{k-1}+1}^{m-j+k} (X[v_0,\ldots,\{\hat{v}_{i_k}\},\ldots,v_m] \times \langle v_0,v_1,\ldots,\{\hat{v}_{i_k}\},\ldots,v_m\rangle) \cup$$
$$\cup \bigcup_{i_1=1}^{3} \bigcup_{k=2}^{m-2} \bigcup_{i_k=i_{k-1}+1}^{k+2} (X[v_1,\ldots,\{\hat{v}_{i_k}\},\ldots,v_m] \times \langle v_1,v_2,\ldots,\{\hat{v}_{i_k}\},\ldots,v_m\rangle) \cup$$
$$\cup P(B^*) .$$

We add the above unions to $P(X^*)$ in the order in which we have listed them. We order the cells within a given union in the same manner that we ordered the cells within a given union in the proof of Theorem 2, and then we add the cells according to this ordering. Since

$$P(X^*) \cap (B \times X[v_0,\ldots,v_m]) = (X \cap B) \times X[v_0,\ldots,v_m]) ,$$

$P(X^*) \cup (B \times X[v_0,\ldots,v_m])$ is homotopically equivalent to $P(X^*)$. Now suppose $1 \leqq \alpha \leqq m-2$, and let X' be the union of $P(X^*)$ with all those cells which have been added before

$$E_1 = \bigcup_{k=1}^{\alpha} (\langle v_0,v_1,\ldots,\{\hat{v}_{i_k}\},\ldots,v_m\rangle \times X[v_0,\ldots,\{\hat{v}_{i_k}\},\ldots,v_m]) .$$

Then

$$X' \cap E_1 =$$
$$[X \cap \bigcup_{k=1}^{\alpha} (\langle v_0,v_1,\ldots,\{\hat{v}_{i_k}\},\ldots,v_m\rangle)] \times X[v_0,\ldots,\{\hat{v}_{ik}\},\ldots,v_m] \cup$$
$$\cup \bigcup_{\beta=1}^{\alpha} \{(\bigcup_{k=1}^{\alpha} (\langle v_0,v_1,\ldots,\{\hat{v}_{ik}\},\ldots,v_m\rangle)) \times$$
$$\times [X - (\bigcup_{q=0}^{m} \{\mathrm{St}(v_q,X) \mid q \neq i_k \text{ for any } k\} \cup \mathrm{St}(v_{i_\beta},X))]\} .$$

If $i_1 > 0$, $X' \cup E_1$ is homotopically equivalent to X'. However if $i_1 = 0$, we may change the homotopy type of X' by adding E_1. But, in this case, if $r > n$, $H_r(X' \cup E_1)$ is isomorphic to $H_r(X')$ unless $H_{r-1}(X' \cap E_1) \neq 0$. Since X is a contractible, n-dimensional polyhedron, $H_s(X' \cap E_1) = 0$ for all $s > n$. Therefore $H_r(X' \cup E_1)$ is isomorphic to $H_r(X')$ for all $r > n+1$. If $H_n(X' \cap E_1) \neq 0$, then in the construction

$$A = X_1 \to X_2 \to \ldots \to X_p = X$$

of X, there must exist i, $2 \leqq i \leqq p$, such that X_i is obtained from X_{i-1} by adding an n-simplex which meets X_{i-1} in at least two of its $(n-1)$-faces. Thus if the additional hypothesis is satisfied, then $H_r(X' \cup E_1)$ is isomorphic to $H_r(X')$ for all $r > n$. If $1 \leqq a \leqq m-1$, $a < b \leqq m$, and X'' is the union of $P(X^*)$ with all those cells which have been added before $E_2 = \langle v_a, v_b\rangle \times X[v_a,v_b]$, then

$$X'' \cap E_2 = (\langle v_a\rangle \times X[v_a, v_b]) \cup (\langle v_b\rangle \times X[v_a, v_b]) \cup \cup \bigcup_{\substack{q=0\\ a\neq q\neq b}}^{m} (\langle v_a, v_b\rangle \times X[v_q, v_a, v_b]) .$$

Therefore by an argument similar to the one above, $H_r(X''\cup E_2)$ is isomorphic to $H_r(X'')$ for all $r>n+1$, and, if the additional hypothesis is satisfied, then $H_{n+1}(X''\cup E_2)$ is isomorphic to $H_{n+1}(X'')$. Thus it follows that if $i_0=-1$ and

$$X^3 = P(X^*) \cup (B\times X[v_0,\ldots,v_m]) \cup$$
$$\cup \textstyle\bigcup_{j=1}^{m-2} \bigcup_{k=1}^{j} \bigcup_{i_k=i_{k-1}+1}^{m-j+k} (\langle v_0, v_1,\ldots,\{\hat v_{i_k}\},\ldots,v_m\rangle \times X[v_0,\ldots,\{\hat v_{i_k}\},\ldots,v_m]) \cup$$
$$\cup \textstyle\bigcup_{i_1=1}^{3} \bigcup_{k=2}^{m-2} \bigcup_{i_k=i_{k-1}+1}^{k+2} (\langle v_1, v_2,\ldots,\{\hat v_{i_k}\},\ldots,v_m\rangle \times X[v_1,\ldots,\{\hat v_{i_k}\},\ldots,v_m]) ,$$

then $H_r(X^3)$ is isomorphic to $H_r(X^*)$ for all $r>n+1$, and, if the additional hypothesis is satisfied, then $H_r(X^3)=0$ for all $r>n$. By essentially repeating the above argument, we can show that if

$$X^4 = X^3 \cup (X[v_0,\ldots,v_m]\times B) \cup$$
$$\cup \textstyle\bigcup_{j=1}^{m-2} \bigcup_{k=1}^{j} \bigcup_{i_k=i_{k-1}+1}^{m-j+k} (X[v_0,\ldots,\{\hat v_{i_k}\},\ldots,v_m] \times \langle v_0, v_1,\ldots,\{\hat v_{i_k}\},\ldots,v_m\rangle) \cup$$
$$\cup \textstyle\bigcup_{i_1=1}^{3} \bigcup_{k=2}^{m-2} \bigcup_{i_k=i_{k-1}+1}^{k+2} (X[v_1,\ldots,\{\hat v_{i_k}\},\ldots,v_m] \times \langle v_1, v_2,\ldots,\{\hat v_{i_k}\},\ldots,v_m\rangle) ,$$

then $H_r(X^4)$ is isomorphic to $H_r(X^*)$ for all $r>n+1$, and, if the additional hypothesis is satisfied, then $H_r(X^4)=0$ for all $r>n$. Since $P(B^*)$ has the homotopy type of S^{m-1} (see [6], Corollary 1) and $m\leqq n$, $H_r(X^4\cup P(B^*))$ is isomorphic to $H_r(X^4)$ for all $r>n$ unless there exists a nontrivial s-cycle, $s\geqq n$, in $X^4\cap P(B^*)$ which bounds in $P(B^*)$. But this cannot happen since $P(B^*)$ is $(m-1)$-dimensional. Therefore $H_r(Y^*)$ is isomorphic to $H_r(X^4)$ for all $r>n$.

The following theorem gives an example to show that adding an m-simplex to a finite, contractible, n-dimensional polyhedron $(3\leqq m\leqq n)$ at an m-odd may add $(n+1)$-dimensional homology to the deleted product.

THEOREM 8. *If $3\leqq m\leqq n$ and A is the polyhedron whose vertices are $\{v_0, v_1,\ldots,v_m, w_1, w_2,\ldots,w_{n+1}, u_1, u_2,\ldots,u_m\}$ and whose simplexes are*

$$\{\langle v_0, w_1, w_2,\ldots,\hat w_i,\ldots,w_{n+1}\rangle \mid i=1,2,\ldots,n+1\} \cup$$
$$\cup \{\langle v_0, v_j, u_j\rangle \mid j=1,2,\ldots,m\} \cup \langle v_0, u_1, u_2\rangle ,$$

then $H_n(A^*)$ is the free abelian group on $2m-3$ generators, $H_1(A^*)$ is the free abelian group on m^2-m generators, and $H_r(A^*)=0$ if $n\neq r\neq 1$. Furthermore if $B=\langle v_0, v_1,\ldots,v_m\rangle$ is an m-simplex and $X=A\cup B$, then $H_{n+1}(X^*)$ and $H_2(X^*)$ are isomorphic to the direct sum $\mathbf{Z}+\mathbf{Z}$, $H_n(X^*)$,

$H_{m-1}(X^*)$, *and* $H_1(X^*)$ *are isomorphic to* Z, *and* $H_r(X^*)=0$ *if* r *is neither* $n+1$, n, $m-1$, 2, *nor* 1.

PROOF. If $K_1=(\bigcup_{i=1}^{n+1}\langle v_0,w_1,w_2,\ldots,\hat{w}_i,\ldots,w_{n+1}\rangle)\cup\langle v_0,u_1,u_2\rangle$, then by Theorem 1, $H_n(K_1{}^*)$ is isomorphic to Z, $H_1(K_1{}^*)$ is isomorphic to the direct sum $\mathsf{Z}+\mathsf{Z}$, and $H_r(K_1{}^*)=0$ if $n\neq r\neq 1$. Since

$$K_2 = K_1 \cup \langle v_0,v_1,u_1\rangle \cup \langle v_0,v_2,u_2\rangle$$

is homeomorphic to K_1, $H_r(K_2{}^*)$ is isomorphic to $H_r(K_1{}^*)$ for each r. Now for each $k=3,4,\ldots,m$, let

$$K_k = K_2 \cup \textstyle\bigcup_{j=3}^{k}\langle v_0,v_j,u_j\rangle .$$

Then, for each $k=3,4,\ldots,m$,

$$\begin{aligned}P(K_k{}^*) = P(K_{k-1}^*) \cup P(K_{k-1}\times\langle v_0,v_k,u_k\rangle-\Delta)\cup\\ \cup P(\langle v_0,v_k,u_k\rangle\times K_{k-1}-\Delta)\cup P((\langle v_0,v_k,u_k\rangle)^*) .\end{aligned}$$

Now

$$\begin{aligned}P(K_{k-1}\times\langle v_0,v_k,u_k\rangle-\Delta) = (K_{k-1}\times\langle v_k,u_k\rangle)\cup\\ \cup(\textstyle\bigcup_{i=1}^{n+1}\langle w_1,w_2,\ldots,\hat{w}_i,\ldots,w_{n+1}\rangle\times\langle v_0,v_k,u_k\rangle)\cup(\langle u_1,u_2\rangle\times\langle v_0,v_k,u_k\rangle)\cup\\ \cup\textstyle\bigcup(_{j=1}^{k-1}\langle v_j,u_j\rangle\times\langle v_0,v_k,u_k\rangle) ,\end{aligned}$$

and hence $P(K_{k-1}\times\langle v_0,v_k,u_k-\Delta)$ is contractible. Since

$$\begin{aligned}P(K_{k-1}^*)\cap P(K_{k-1}\times\langle v_0,v_k,u_k\rangle-\Delta) = (\langle u_1,u_2\rangle\times\langle v_0\rangle)\cup\\ \cup(\textstyle\bigcup_{i=1}^{n+1}\langle w_1,w_2,\ldots,\hat{w}_i,\ldots,w_{n+1}\rangle\times\langle v_0\rangle)\cup(\textstyle\bigcup_{j=1}^{k-1}\langle v_j,u_j\rangle\times\langle v_0\rangle) ,\end{aligned}$$

it follows that

$$H_n(P(K_{k-1}^*)\cup P(K_{k-1}\times\langle v_0,v_k,u_k\rangle-\Delta))$$

is isomorphic to the direct sum $H_n(P(K_{k-1}^*))+\mathsf{Z}$,

$$H_1(P(K_{k-1}^*)\cup P(K_{k-1}\times\langle v_0,v_k,u_k\rangle-\Delta))$$

is isomorphic to the direct sum of $H_1(P(K_{k-1}^*))$, and the free abelian group on $k-2$ generators, and

$$H_r(P(K_{k-1}^*)\cup P(K_{k-1}\times\langle v_0,v_k,u_k\rangle-\Delta))$$

is isomorphic to $H_r(P(K_{k-1}^*))$ if $n\neq r\neq 1$. Also $P(\langle v_0,v_k,u_k\rangle\times K_{k-1}-\Delta)$ is contractible and

$$\begin{aligned}[P(K_{k-1}^*)\cup P(K_{k-1}\times\langle v_0,v_k,u_k\rangle-\Delta)]\cap P(\langle v_0,v_k,u_k\rangle\times K_{k-1}-\Delta)\\ = (\langle v_0\rangle\times\langle u_1,u_2\rangle)\cup\\ \cup(\textstyle\bigcup_{j=1}^{k-1}(\langle v_0\rangle\times\langle v_j,u_j\rangle))\cup(\textstyle\bigcup_{i=1}^{n+1}(\langle v_0\rangle\times\langle w_1,w_2,\ldots,\hat{w}_i,\ldots,w_{n+1}\rangle)) .\end{aligned}$$

Therefore, if

$$M = P(K_{k-1}^*) \cup P(K_{k-1} \times \langle v_0, v_k, u_k \rangle - \Delta) \cup P(\langle v_0, v_k, u_k \rangle \times K_{k-1} - \Delta) ,$$

then $H_n(M)$ is isomorphic to $H_n(P(K_{k-1}^*)) + \mathbf{Z} + \mathbf{Z}$, $H_1(M)$ is isomorphic to the direct sum of $H_1(P(K_{k-1}^*))$ and the free abelian group on $2k-4$ generators, and $H_r(M)$ is isomorphic to $H_r(P(K_{k-1}^*))$ if $n \neq r \neq 1$. Since $P((\langle v_0, v_k, u_k \rangle)^*)$ has the homotopy type of S^1 and

$$M \cap P((\langle v_0, v_k, u_k \rangle)^*) = (\langle v_0 \rangle \times \langle v_k, u_k \rangle) \cup (\langle v_k, u_k \rangle \times \langle v_0 \rangle) ,$$

$H_n(P(K_k{}^*))$ is isomorphic to $H_n(P(K_{k-1}^*)) + \mathbf{Z} + \mathbf{Z}$, $H_1(P(K_k{}^*))$ is isomorphic to the direct sum of $H_1(P(K_{k-1}^*))$ and the free abelian group on $2k-2$ generators, and $H_r(P(K_k{}^*))$ is isomorphic to $H_r(P(K_{k-1}^*))$ if $n \neq r \neq 1$. Therefore it follows that $H_n(A^*)$ is the free abelian group on $2m-3$ generators, $H_1(A^*)$ is the free abelian group on $m^2 - m$ generators, and $H_r(A^*) = 0$ if $n \neq r \neq 1$.

Now let $G = \bigcup_{i=1}^{n+1} \langle v_0, w_1, w_2, \ldots, \hat{w}_i, \ldots, w_{n+1} \rangle$ and $N = G \cup B$. Then, by Theorem 1,

$$H_n(N^*), \quad H_{m-1}(N^*), \quad H_1(N^*)$$

are isomorphic to $\mathbf{Z}$ and $H_r(N^*) = 0$ if r is neither n, $m-1$, nor 1. For each $j = 1, 2, \ldots, m$, let $Q_j = N \cup \bigcup_{k=1}^{j} \langle v_0, v_k, u_k \rangle$. Then, if $Q_0 = N$,

$$P(Q_j{}^*) = P(Q_{j-1}^*) \cup P(Q_{j-1} \times \langle v_0, v_j, u_j \rangle - \Delta) \cup P(\langle v_0, v_j, u_j \rangle \times Q_{j-1} - \Delta) .$$

Now

$$\begin{aligned} P(Q_{j-1} \times \langle v_0, v_j, u_j \rangle - \Delta) &= (G \times \langle v_j, u_j \rangle) \cup \\ &\cup (\textstyle\bigcup_{i=1}^{n+1} \langle w_1, w_2, \ldots, \hat{w}_i, \ldots, w_{n+1} \rangle \times \langle v_0, v_j, u_j \rangle) \cup \\ &\cup (\langle v_0, v_1, \ldots, v_m \rangle \times \langle u_j \rangle \cup (\langle v_0, v_1, \ldots, \hat{v}_j, \ldots, v_m \rangle \times \langle v_j, u_j \rangle) \cup \\ &\cup (\langle v_1, v_2, \ldots, v_m \rangle \times \langle v_0, u_j \rangle) \cup (\langle v_1, v_2, \ldots, \hat{v}_j, \ldots, v_m \rangle \times \langle v_0, v_j, u_j \rangle) \cup \\ &\cup (\textstyle\bigcup_{k=1}^{j-1} \langle v_0, v_k, u_k \rangle \times \langle v_j, u_j \rangle) \cup (\textstyle\bigcup_{k=1}^{j-1} \langle v_k, u_k \rangle \times \langle v_0, v_j, u_j \rangle) . \end{aligned}$$

and

$$\begin{aligned} P(Q_{j-1}^*) \cap P(Q_{j-1} \times \langle v_0, v_j, u_j \rangle - \Delta) &= (G \times \langle v_j \rangle) \cup \\ \cup (\textstyle\bigcup_{i=1}^{n+1} \langle w_1, w_2, \ldots, \hat{w}_i, \ldots, w_{n+1} \rangle \times \langle v_0, v_j \rangle) &\cup (\langle v_0, v_1, \ldots, \hat{v}_j, \ldots, v_m \rangle \times \langle v_j \rangle) \cup \\ \cup (\langle v_1, v_2, \ldots, v_m \rangle \times \langle v_0 \rangle) &\cup (\langle v_1, v_2, \ldots, \hat{v}_j, \ldots, v_m \rangle \times \langle v_0, v_j \rangle) \cup \\ \cup (\textstyle\bigcup_{k=1}^{j-1} \langle v_0, v_k, u_k \rangle \times \langle v_j \rangle) &\cup (\textstyle\bigcup_{k=1}^{j-1} \langle v_k, u_k \rangle \times \langle v_0, v_j \rangle) \end{aligned}$$

are contractible. Therefore $P(Q_{j-1}^*) \cup P(Q_{j-1} \times \langle v_0, v_j, u_j \rangle - \Delta)$ is homotopically equivalent to $P(Q_{j-1}^*)$. Similarly it can be shown that $P(Q_j{}^*)$ is homotopically equivalent to $P(Q_{j-1}^*)$. Therefore if

$$Q = N \cup \textstyle\bigcup_{j=1}^{m} \langle v_0, v_j, u_j \rangle ,$$

then $P(Q^*)$ is homotopically equivalent to $P(N^*)$. Also

$$P(X^*) = P(Q^*) \cup P(Q \times \langle v_0, u_1, u_2\rangle - \Delta) \cup P(\langle v_0, u_1, u_2\rangle \times Q - \Delta) .$$

Now

$$\begin{aligned} P(Q \times \langle v_0, u_1, u_2\rangle - \Delta) = {} & (\textstyle\bigcup_{i=1}^{n+1} \langle w_1, w_2, \ldots, \hat{w}_i, \ldots, w_{n+1}\rangle \times \langle v_0, u_1, u_2\rangle) \cup \\ & \cup (G \times \langle u_1, u_2\rangle) \cup (B \times \langle u_1, u_2\rangle) \cup (\langle v_1, v_2, \ldots, v_m\rangle \times \langle v_0, u_1, u_2\rangle) \cup \\ & \cup (\textstyle\bigcup_{j=3}^{m} \langle v_0, v_j, u_j\rangle \times \langle u_1, u_2\rangle) \cup (\bigcup_{j=3}^{m} \langle v_j, u_j\rangle \times \langle v_0, u_1, u_2\rangle) \cup \\ & \cup (\langle v_0, v_1, u_1\rangle \times \langle u_2\rangle) \cup (\langle v_1, u_1\rangle \times \langle v_0, u_2\rangle) \end{aligned}$$

is contractible, and

$$\begin{aligned} P(Q^*) \cap P(Q \times \langle v_0, u_1, u_2\rangle - \Delta) = {} & (G \times \langle u_1\rangle) \cup (G \times \langle u_2\rangle) \cup \\ & \cup (\textstyle\bigcup_{i=1}^{n+1} \langle w_1, w_2, \ldots, \hat{w}_i, \ldots, w_{n+1}\rangle \times \langle v_0, u_1\rangle) \cup \\ & \cup (\textstyle\bigcup_{i=1}^{n+1} \langle w_1, w_2, \ldots, \hat{w}_i, \ldots, w_{n+1}\rangle \times \langle v_0, u_2\rangle) \cup (B \times \langle u_1\rangle) \cup (B \times \langle u_2\rangle) \\ & \cup (\langle v_1, v_2, \ldots, v_m\rangle \times \langle v_0, u_1\rangle) \cup (\langle v_1, v_2, \ldots, v_m\rangle \times \langle v_0, u_2\rangle) \cup \\ & \cup (\textstyle\bigcup_{j=3}^{m} \langle v_0, v_j, u_j\rangle \times \langle u_1\rangle) \cup (\bigcup_{j=3}^{m} \langle v_0, v_j, u_j\rangle \times \langle u_2\rangle) \cup \\ & \cup (\textstyle\bigcup_{j=3}^{m} \langle v_j, u_j\rangle \times \langle v_0, u_1\rangle) \cup (\bigcup_{j=3}^{m} \langle v_j, u_j\rangle \times \langle v_0, u_2\rangle) \cup \\ & \cup (\langle v_0, v_1, u_1\rangle \times \langle u_2\rangle) \cup (\langle v_0, v_2, u_2\rangle \times \langle u_1\rangle) \cup \\ & \cup (\langle v_1, u_1\rangle \times \langle v_0, u_2\rangle) \cup (\langle v_2, u_2\rangle \times \langle v_0, u_1\rangle) . \end{aligned}$$

Therefore

$$H_n(P(Q^*) \cap P(Q \times \langle v_0, u_1, u_2\rangle - \Delta))$$

and

$$H_1(P(Q^*) \cap P(Q \times \langle v_0, u_1, u_2\rangle - \Delta))$$

are isomorphic to $\mathbf{Z}$ and $H_r(P(Q^*) \cap P(Q \times \langle v_0, u_1, u_2\rangle - \Delta)) = 0$ if $n \neq r \neq 1$. In particular

$$\begin{aligned} \Gamma_1 = {} & ((G \times \langle u_1\rangle) \cup (\textstyle\bigcup_{i=1}^{n+1} \langle w_1, w_2, \ldots, \hat{w}_i, \ldots, w_{n+1}\rangle \times \langle v_0, u_1\rangle) \cup \\ & \cup (G \times \langle u_2\rangle) \cup (\textstyle\bigcup_{i=1}^{n+1} \langle w_1, w_2, \ldots, \hat{w}_i, \ldots, w_{n+1}\rangle \times \langle v_0, u_2\rangle) \end{aligned}$$

is an n-sphere in $P(Q^*) \cap P(Q \times \langle v_0, u_1, u_2\rangle - \Delta)$ such that no non-trivial n-cycle associated with this sphere bounds in

$$P(Q^*) \cap P(Q \times \langle v_0, u_1, u_2\rangle - \Delta) ,$$

and

$$\begin{aligned} \Gamma_2 = {} & (\langle v_0, w_1\rangle \times \langle u_1\rangle) \cup (\langle w_1\rangle \times \langle v_0, u_1\rangle) \cup (\langle w_1\rangle \times \langle v_0, u_2\rangle) \cup \\ & \cup (\langle v_0, w_1\rangle \times \langle u_2\rangle) \cup (\langle v_0, v_1\rangle \times \langle u_2\rangle) \cup (\langle v_1\rangle \times \langle v_0, u_2\rangle) \cup \\ & \cup (\langle v_1\rangle \times \langle v_0, u_1\rangle) \cup (\langle v_0, v_1\rangle \times \langle u_1\rangle) \end{aligned}$$

is a 1-sphere in $P(Q^*) \cap P(Q \times \langle v_0, u_1, u_2\rangle - \Delta)$ such that no non-trivial 1-cycle associated with this sphere bounds in

$$P(Q^*) \cap P(Q \times \langle v_0, u_1, u_2\rangle - \Delta) .$$

If z_n is an n-cycle on Γ_1, then there is an $(n+1)$-chain c_{n+1} on the subset

$$\begin{aligned}
&(\textstyle\bigcup_{i=1}^{n+1}\langle v_0,w_1,w_2,\ldots,\hat{w}_i,\ldots,w_{n+1}\rangle\times\langle v_1,u_1\rangle)\cup\\
&\cup(\textstyle\bigcup_{i=1}^{n+1}\langle v_0,w_1,w_2,\ldots,\hat{w}_i,\ldots,w_{n+1}\rangle\times\langle v_2,u_2\rangle)\cup\\
&\cup(\textstyle\bigcup_{i=1}^{n+1}\langle v_0,w_1,w_2,\ldots,\hat{w}_i,\ldots,w_{n+1}\rangle\times\langle v_1,v_2\rangle)\cup\\
&\cup(\textstyle\bigcup_{i=1}^{n+1}\langle w_1,w_2,\ldots,\hat{w}_i,\ldots,w_{n+1}\rangle\times\langle v_0,v_1,u_1\rangle)\cup\\
&\cup(\textstyle\bigcup_{i=1}^{n+1}\langle w_1,w_2,\ldots,\hat{w}_i,\ldots,w_{n+1}\rangle\times\langle v_0,v_2,u_2\rangle)\cup\\
&\cup(\textstyle\bigcup_{i=1}^{n+1}\langle w_1,w_2,\ldots,\hat{w}_i,\ldots,w_{n+1}\rangle\times\langle v_0,v_1,v_2\rangle)
\end{aligned}$$

of $P(Q^*)$ such that $\partial(c_{n+1})=z_n$. Also, if z_1 is a 1-cycle on Γ_2, then there is a 2-chain c_2 on the subset

$$\begin{aligned}
&(\langle w_1\rangle\times\langle v_0,v_1,u_1\rangle)\cup(\langle w_1\rangle\times\langle v_0,v_1,v_2\rangle)\cup(\langle w_1\rangle\times\langle v_0,v_2,u_2\rangle)\cup\\
&\cup(\langle v_0,w_1\rangle\times\langle v_1,u_1\rangle)\cup(\langle v_0,w_1\rangle\times\langle v_1,v_2\rangle)\cup(\langle v_0,w_1\rangle\times\langle v_2,u_2\rangle)\cup\\
&\cup(\langle v_0,v_3\rangle\times\langle v_1,u_1\rangle)\cup(\langle v_0,v_3\rangle\times\langle v_1,v_2\rangle)\cup(\langle v_0,v_3\rangle\times\langle v_2,u_2\rangle)\cup\\
&\cup(\langle v_0,v_1,v_3\rangle\times\langle u_2\rangle)\cup(\langle v_0,v_1,v_3\rangle\times\langle u_1\rangle)\cup(\langle v_1,v_3\rangle\times\langle v_0,u_1\rangle)\cup\\
&\cup(\langle v_1,v_3\rangle\times\langle v_0,u_2\rangle)\cup(\langle v_3\rangle\times\langle v_0,v_1,u_1\rangle)\cup(\langle v_3\rangle\times\langle v_0,v_1,v_2\rangle)\cup\\
&\cup(\langle v_3\rangle\times\langle v_0,v_2,u_2\rangle)
\end{aligned}$$

of $P(Q^*)$ such that $\partial(c_2)=z_1$. Therefore

$$H_s(P(Q^*)\cup P(Q\times\langle v_0,u_1,u_2\rangle-\Delta))$$

is isomorphic to $\mathbf{Z}$ if s is either $n+1$, n, $m-1$, 2, or 1, and it is zero if s is neither $n+1$, n, $m-1$, 2, nor 1. By a similar argument, it can be shown that $H_t(P(X^*))$ is isomorphic to the direct sum $\mathbf{Z}+\mathbf{Z}$ if t is either $n+1$ or 2, $H_s(P(X^*))$ is isomorphic to $\mathbf{Z}$ if s is either n, $m-1$, or 1, and $H_s(P(X^*))=0$ if s is neither $n+1$, n, $m-1$, 2, nor 1.

If we examine the last part of the proof of Theorem 8, we can observe that this theorem also gives an example to show that adding a 2-simplex to a finite, contractible, n-dimensional polyhedron, $n\geqq 3$, at two of its 1-faces may add $(n+1)$-dimensional homology to the deleted product. However we have the following theorem.

THEOREM 9. *If X is a finite, contractible, n-dimensional $(n\geqq 3)$ polyhedron with the property that a homeomorph of X can be constructed out of an n-simplex A by adding m-simplexes, $1\leqq m\leqq n$, in such a way that the construction may be factored*

$$A = X_1\to X_2\to\ldots\to X_p = X$$

so that X_i is obtained from X_{i-1} by (1), (2), (3), *or* (4) *of Theorem* 5 *or by* (5) *adding an m-simplex, $3\leqq m\leqq n$, which meets X_{i-1} in exactly m*

1-faces, where the intersection of X_{i-1} *and the appended simplex is an m-odd, or*

(6) *adding a 2-simplex which meets* X_{i-1} *in exactly two of its 1-faces; then* $H_r(X^*)=0$ *for all* $r>n+1$.

PROOF. By Corollary 1 of [6], A^* has the homotopy type of S^{n-1}. Suppose $2\leqq i\leqq p$ and $H_r((X_{i-1})^*)=0$ for all $r>n+1$. If X_i is obtained from X_{i-1} by (1), (2), (3), or (4) of Theorem 5, then we repeat the proof of Theorem 5 to show that $H_r((X_i)^*)=0$ for all $r>n+1$. If X_i is obtained from X_{i-1} by (5), then, by Theorem 7, $H_r((X_i)^*)=0$ for all $r>n+1$. If X_i is obtained from X_{i-1} by (6), then, by Theorem 7 of [7], $H_r((X_i)^*)=0$ for all $r>n+1$.

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