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COALGEBRA EXTENSIONS IN TWO-STAGE POSTNIKOV SYSTEMS

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In this note we shall study the Hopf algebra structure of a large class of two-stage Postnikov systems. A two-stage Postnikov system is a diagram ξ

$$\begin{array}{ccc} \Omega B_0 & \longrightarrow & \Omega B_0 \\ {\scriptstyle i}\downarrow & & \downarrow \\ E & \longrightarrow & PB_0 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow{\ f\ } & B_0 \end{array}$$

where B and B_0 are generalized Eilenberg–Maclane spaces, π is the path fibration, and p is the induced fibration. Applying the loop functor Ω to each space and map, we obtain a new two-stage Postnikov system called $\Omega\xi$, which is *stable* in the notation of [15]. In particular, Ωf is a map of H-spaces.

We study systems satisfying the following two conditions:

1) The factors of B and B_0 are of the type $K(\pi,n)$ with π a finitely generated abelian group.

2) B and ΩB_0 are simply connected.

Let p be any fixed *odd* prime. All cohomology is with coefficients $\mathbf{Z}_p$. (The case $p=2$ has been studied in [9], [4], [7].) Let $R=H^*\Omega B//\mathrm{im}\,(\Omega f)^*$. Write $H^*(\Omega^k B_0)=U(Y_k)$, where Y_k is a free unstable module over the mod p Steenrod algebra A, and U is the free A-algebra functor of [16, p. 29]. The papers of Massey–Peterson [11], [12], L. Smith [15], and Barcus [2] yield the following theorem.

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THEOREM. a) *$H^*\Omega E = R \otimes U(M)$ as algebras over Z_p, where*

$$M = \sigma_{B_0}(\mathrm{Ker}(\Omega f)^*) \subset Y_2 .$$

b) *The sequence of Hopf algebras over Z_p*

$$Z_p \to R \xrightarrow{(\Omega p)^*} H^*\Omega E \xrightarrow{(\Omega i)^*} U(M) \to Z_p$$

is coexact; $\ker(\Omega i)^*$ *is the ideal generated by* $\mathrm{im}(\Omega_p)^*$ *in positive degrees.*

c) *Let $Y_1' = \ker(\Omega f)^* \mid Y_1$. Then the suspension restricts to an epimorphism α: $Y_1' \to M$ of degree -1. The kernel of α is $\theta Y_1'$, where θ is the map of Barcus* |2|:

$$\theta \mid (Y_1')_{2k} = P^k \quad \textit{and} \quad \theta \mid (Y_1')_{2k+1} = \beta P^k .$$

The subject of this note is the determination of the coproduct ψ on $H^*\Omega E$. From parts a) and b) it suffices to determine coproducts on (homogeneous) elements of $H^*\Omega E$ restricting to a Z_p-basis for M, since such elements form a simple system of generators for $H^*\Omega E$ as an algebra over R.

Consider the following diagrams:

$$M \xleftarrow{\alpha} Y_1' \xrightarrow{c} Y_1 \qquad\qquad \begin{array}{ccc} Y_0 & \xrightarrow{f^*} & H^*B \\ {\scriptstyle \sigma_{B_0}}\downarrow & & \downarrow{\scriptstyle \sigma_B} \\ Y_1 & \xrightarrow{\Omega f^*} & PH^*\Omega B \end{array}$$

where c is the inclusion. Let $x \in M$, and choose $y \in Y_1'$ such that $\alpha(y) = x$. Choose $v \in Y_0$ such that $\sigma_{B_0}(v) = y$. Since $\sigma_B(f^*v) = 0$, we have

$$f^*v = \textstyle\sum_i a_i b_i + \beta P^t m, \quad \deg m = 2t+1 \ (\deg a_i, \deg b_i > 0) .$$

(Note that m may be zero.)

Define

$$\alpha_i = (\Omega p)^* \sigma_B a_i, \quad \beta_i = (\Omega p)^* \sigma_B b_i \quad \mu = (\Omega p)^* \sigma_B m .$$

and define $r(x) \in H^*\Omega E \otimes H^*\Omega E$ by

$$r(x) = \textstyle\sum_i \alpha_i \otimes \beta_i + p^{-1} \displaystyle\sum_{i=1}^{p-1} \tbinom{p}{i} \mu^i \otimes \mu^{p-i} .$$

LEMMA 1. *$r(x)$ is a well defined function of x.*

The straightforward analysis of choices is left to the reader. We now can state the main theorem.

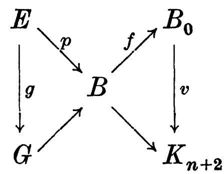
MAIN THEOREM. *Given $x \in M$, there exists an element $e \in H^*\Omega E$ such that $(\Omega i)^* e = x$, with coproduct*

$$\psi e = 1 \otimes e + e \otimes 1 + \lambda r(x), \quad 0 \neq \lambda \in Z_p .$$

PROOF. Fix some x and v as above, and represent v as a map

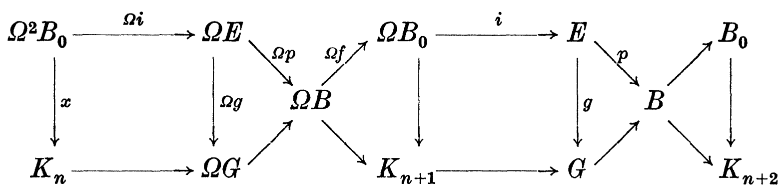
$$v : B_0 \to K_{n+2} = K(\mathbf{Z}_p, n+2) .$$

Writing the fundamental class as ι_{n+2} we have $f^* v^* \iota_{n+2} = \sum a_i b_i + \beta P^t m$, and $p^* f^* v^* \iota_{n+2} = 0$. Let G be the principal fiber space over B with fibre K_{n+1} induced by vf. We then have the commutative diagram



(g exists since $vfp \simeq 0$)

which we enlarge to the commutative diagram



Now ΩG is homotopy equivalent to $\Omega B \times K_n$ since $\sigma_B(f^* v) = 0$ (though *not* as H-spaces, in general). Let η_n be the class in $H^*\Omega G$ provided by Lemma 2 below. Define $e = (\Omega g)^* \lambda^{-1} \eta_n$. The result follows since $(\Omega g)^*$ is a map of Hopf algebras. It is thus sufficient to prove Lemma 2:

LEMMA 2. *Suppose $G \to B \xrightarrow{v} K_{n+2}$ is a fibration with*

$$v^* \iota_{n+2} = \sum a_i b_i + \beta P^t m, \quad \deg(m) = 2t+1 .$$

Then there is a class $\eta_n \in H^*\Omega G$ restricting to the fundamental class ι_n of the fiber K_n such that

$$\psi \eta_n = 1 \otimes \eta_n + \eta_n \otimes 1 + \lambda r(\iota_n) , \quad 0 \neq \lambda \in \mathbf{Z}_p .$$

To prove the lemma we must digress to recall some algebraic machinery. Suppose G is any topological space. Then there is a spectral sequence of algebras $\{E_r, d_r\}$ due to Eilenberg and Moore [14], [6], converging to H^*G, and with $E_2 = \mathrm{Ext}_{H_*\Omega G}(\mathbf{Z}_p, \mathbf{Z}_p)$. The E_1 term of the spectral sequence may be written $\mathscr{F}(H^*\Omega G)$ where $\mathscr{F}$ is the reduced cobar construction (see [1], [10], [13]). A monomial in $E_1{}^{k,*}$ is written $[a_1 | \ldots | a_k]$ with total degree $k + \sum \deg a_i$. Multiplication in E_1 is by juxtaposition. The differential d_1 is given on algebra generators by $d_1[a] = \sum [a' | a'']$, where $\psi a = a \otimes 1 + 1 \otimes a + \sum (a' \otimes a'')$, and insisting that d_1 be a derivation.

PROOF OF LEMMA 2. Reverting to the hypotheses of the lemma, we see that it suffices to prove that

$$d_1[\lambda' \iota_n] = \sum[\alpha_i|\beta_i] + p^{-1}\sum\binom{p}{i}[\mu^i|\mu^{p-i}] + d_1[b] ,$$

where $b \in p^*H^*\Omega B$. Each α_i and β_i is primitive, so $d_1[\alpha_i|\beta_i]=0$ for each i. If $p>2$ then the element $p^{-1}\sum\binom{p}{i}[\mu^i|\mu^{p-i}]$ is a cycle. (The proof is a short exercise in binomial coefficients.) In the universal example of $K_{n+1} \to PK_{n+2} \to K_{n+2}$ this element survives to E_∞ to create the element $-\beta P^t m$. (See [5] for the analogous statement in homology). On the other hand we know that $\sum a_i b_i + \beta P^t m = 0$ in H^*G, so the element $\sum[\alpha_i|\beta_i] + p^{-1}\sum\binom{p}{i}[\mu^i|\mu^{p-i}]$ must be a boundary. By dimension considerations and naturality it must be equal to $d_1([\lambda\eta_n + b])$. This proves the lemma, once we choose $\eta_n' = \eta_n - b$.

REMARK. $H^*\Omega E$ need not be co-commutative. For $p>2$, consider the two-stage system with $f^*(\iota_{2n+1}) = \iota_n \beta\iota_n$. Then the coproduct on η_{2n-1} is $1\otimes\eta + \eta\otimes 1 + \lambda(\iota_{n-1}\otimes\beta\iota_{n-1})$.

ACKNOWLEDGEMENT. The question of the coalgebra structure of $H^*\Omega E$ is closely related to the question of additivity of certain secondary cohomology operations. In this form the structure was obtained for $p=2$ by Kristensen [8], [9] using cochain methods. The work of Cheng [4] for $p=2$ indicates the usefulness of the Eilenberg–Moore spectral sequence in the general situation. The examples studied by Liulevicius [10] are helpful in identifying the main ingredients for the odd prime case. We have been informed that Ben Cooper (Benjamin G. Cooper, Coproducts in the mod p cohomology of stable two stage Postnikov systems, Thesis, Yale University, 1971) has independently obtained results overlapping with this work.

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ADDED IN PROOF. D. Kraines and the second author have extended Lemma 2 in a more general algebraic setting in »Differentials in the Eilenberg–Moore spectral sequence«, to appear in J. Pure and Applied Algebra.

REFERENCES

1. J. F. Adams, *On the cobar construction*, Proc. Nat. Acad. Sci. USA 42 (1956), 409–412.
2. W. D. Barcus, *On a theorem of Massey and Peterson*, Quart. J. Math. Oxford (2) 19 (1968), 33–41.

3. H. Cartan et al., *Algèbres d'Eilenberg–MacLane et homotopie,* Séminaire Henri Cartan, 7ième année, 1954–55, École Normale Superieure, Paris, 1956.
4. C. K. Cheng, *H-space and higher order structures on two-stage Postnikov systems,* Thesis, Notre Dame Univ. 1970.
5. A. Clark, *A homology transgression theorem,* Duke Math. J. 35 (1968), 169–173.
6. S. Eilenberg and J. Moore, *Homology and fibrations* I, *Coalgebras, cotensor product and its derived functors,* Comm. Math. Helv. 40 (1966), 199–236.
7. J. Harper, *On the cohomology of stable two-stage Postnikov systems,* Bull. Amer. Math. Soc. 76 (1970), 807–809, Trans. Amer. Math. Soc. 152 (1970) 375–388.
8. L. Kristensen, *On the cohomology of spaces with two non-vanishing homotopy groups,* Math. Scand. 12 (1963), 83–105.
9. L. Kristensen, *On secondary cohomology operations,* Math. Scand. 12 (1963), 57–82.
10. A. Liulevicius, *The factorization of cyclic reduced powers by secondary cohomology operations,* Mem. Amer. Math. Soc. 42, 1962.
11. W. S. Massey and F. P. Peterson, *The cohomology structure of certain fibre spaces,* Topology 4 (1964), 47–66.
12. W. S. Massey and F. P. Peterson, *The* mod 2 *cohomology structure of certain fibre spaces,* Mem. Amer. Math. Soc. 74, 1967.
13. J. P. May, *The cohomology of restricted Lie algebras and Hopf algebras,* J. Algebra 3 (1966), 123–146.
14. J. Moore, *Périodicité des groups d'homotopie stables des groupes classiques, d'apres Bott,* Séminaire H. Cartan, 12ième année, 1959–60, École Normale Superieure, 1961.
15. L. Smith, *Cohomology of stable two stage Postnikov systems,* Illinois J. Math. 11 (1967), 310–329.
16. N. E. Steenrod and D. B. A. Epstein, *Cohomology operations* (Annals of Math. Studies 50), Princeton University Press, 1962.

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