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A CONVERGENCE THEOREM FOR MEASURES IN REGULAR HAUSDORFF SPACES

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0. Summary.

The purpose of this paper is to extend a recent theorem of Wells, jr. [3, Theorem 3, p. 125], concerning converging and bounding classes for the family of all regular signed measures on the Borel sets of a compact Hausdorff space, from compact spaces to regular Hausdorff spaces (Theorem 3.1). As a special case we obtain that in a regular Hausdorff space $(X,\mathscr{T})$ any sequence $(\mu_n)_{n\in\mathsf{N}}$ of $\mathscr{K}$-regular measures (inner regular with respect to the paving $\mathscr{K}$ of compact subsets of X) has the property that $(\mu_n(A))_{n\in\mathsf{N}}$ is a convergent sequence for every Borel set $A \subseteqq X$ if $(\mu_n(U))_{n\in\mathsf{N}}$ is a convergent sequence for every regular open set $U \subseteqq X$.

Among other convergence theorems this result was at first obtained in [2, Theorem 4.10] for normal Hausdorff spaces; it is a further refinement of the techniques employed by Wells which yield the present theorem using compactness results of [2] for $\mathscr{K}$-regular measures in (regular) Hausdorff spaces.

1. Introduction.

The notions used in the following are those from [2]. Throughout X denotes a regular Hausdorff space and $\mathscr{F}$ the Borel field in X. A *measure* $\mu|\mathscr{F}$ is a real-valued σ-additive set function defined on $\mathscr{F}$. We denote by $\mathscr{K}$ the paving of compact subsets of X and $\mu|\mathscr{F}$ is called *$\mathscr{K}$-regular* if: For every $U \in \mathscr{T}$ and every $\varepsilon > 0$ there exists $K \in \mathscr{K}$, $K \subset U$, such that $|\mu(A)| < \varepsilon$ for all $A \in \mathscr{F}$ with $A \subset U \setminus K$ or equivalently:

For every $U \in \mathscr{T}$ and every $\varepsilon > 0$ there exists $K \in \mathscr{K}$, $K \subset U$, such that the total variation of μ on $U \setminus K$, $|\mu|(U \setminus K) < \varepsilon$.

We remark that this implies that for every $A \in \mathscr{F}$ and every $\varepsilon > 0$ there exists $K \in \mathscr{K}$, $K \subset A$, such that $|\mu|(A \setminus K) < \varepsilon$.

Let $\mathscr{K}$-rca$(X,\mathscr{F})$ [$\mathscr{K}$-rca$_+(X,\mathscr{F})$] denote the space of all [non-negative] $\mathscr{K}$-regular measures.

A subsystem $\mathscr{C} \subset \mathscr{F}$ is called a *converging class* (for $\mathscr{K}$-rca$(X,\mathscr{F})$) provided every sequence $\mu_n \in \mathscr{K}$-rca$(X,\mathscr{F})$, $n \in \mathsf{N}$, which converges on

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$\mathscr{C}$ (that is $\lim \mu_n(C)$ exists in R for every $C \in \mathscr{C}$) converges on $\mathscr{F}$. In that case it follows (see [2, 1.10 and 3.8]) that for every sequence $\mu_n \in \mathscr{K}\text{-rca}(X,\mathscr{F})$, $n \in \mathsf{N}$, which converges on $\mathscr{C}$ there exists $\mu_0 \in \mathscr{K}\text{-rca}(X,\mathscr{F})$ such that $\lim_{n\to\infty}\mu_n(A) = \mu_0(A)$ for every $A \in \mathscr{F}$, that is, if $\mathscr{C}$ is a converging class, then for every sequence $\mu_n \in \mathscr{K}\text{-rca}(X,\mathscr{F})$, $n \in \mathsf{N}$, which converges on $\mathscr{C}$, the set function

$$\mu_0(C) := \lim \mu_n(C), \quad C \in \mathscr{C},$$

is the trace on $\mathscr{C}$ of a $\mathscr{K}$-regular measure $\mu_0 | \mathscr{F}$ to which $(\mu_n)_{n\in\mathsf{N}}$ converges on every $A \in \mathscr{F}$.

We call $\mathscr{C}$ a *bounding class* (for $\mathscr{K}\text{-rca}(X,\mathscr{F})$) provided every sequence $\mu_n \in \mathscr{K}\text{-rca}(X,\mathscr{F})$, $n \in \mathsf{N}$, for which $\sup_{n\in\mathsf{N}} |\mu_n(C)| < \infty$ for every $C \in \mathscr{C}$ is bounded, that is, $\sup_{n\in\mathsf{N}} |\mu_n|(X) < \infty$.

We write A^c and $\operatorname{int} A$ for the closure resp. interior of $A \subseteq X$. An open set U is called *regular* if $U = \operatorname{int}(U^c)$. The system $\mathscr{T}_r$ of all regular open sets is a complete Boolean algebra which coincides with the system of all $\operatorname{int} F$, F closed in X, and $\mathscr{T}_r$ is strictly smaller than $\mathscr{T}$, even for $X = \mathsf{R}$: The set $U := (-1,1) \setminus \{0\}$ belongs to $\mathscr{T}$ but not to $\mathscr{T}_r$. The supremum [infimum] of a family $(U_\beta)_{\beta\in B}$ of regular open sets is defined to be

$$\operatorname{int}((\textstyle\bigcup_{\beta\in B} U_\beta)^c) \ [\operatorname{int}((\textstyle\bigcap_{\beta\in B} U_\beta)^c)] .$$

The intersection of two regular open sets is regular. However, $\mathscr{T}_r$ is not closed under the formation of countable unions (even the union of two regular open sets need not be regular) and this fact presents an essential difficulty in proving the main result below.

2. Two lemmata.

2.1 LEMMA. *Let $\mathscr{C}$ be a family of open sets in a regular Hausdorff space $(X,\mathscr{T})$ fulfilling the condition:*

(1) *If $K \in \mathscr{K}$, $U \in \mathscr{T}$, $K \subset U$, then there exists $C \in \mathscr{C}$ such that $K \subset C \subset U$.*

Assume further that $\mu_n \in \mathscr{K}\text{-rca}(X,\mathscr{F})$, $n \in \mathsf{N}$, is a sequence of measures satisfying

(2) $\sup_{n\in\mathsf{N}} |\mu_n(C)| < \infty$ *for every $C \in \mathscr{C}$, and*

(3) $\lim_{j\to\infty} \mu_n(U_j) = 0$ *uniformly in $n \in \mathsf{N}$, whenever $U_j \in \mathscr{T}$, $j \in \mathsf{N}$, is a sequence of pairwise disjoint sets.*

Then the following is true:

(i) $(\mu_n)_{n\in\mathsf{N}}$ *is bounded, that is,* $\sup_{n\in\mathsf{N}} |\mu_n|(X) < \infty$.

(ii) *If (2) is replaced by the condition that $(\mu_n)_{n\in\mathsf{N}}$ converges on $\mathscr{C}$, then $(\mu_n)_{n\in\mathsf{N}}$ converges on $\mathscr{F}$.*

Proof. (i): According to [2, Corollary 4.8], it suffices to show that
(a) $\sup_{n\in\mathsf{N}}|\mu_n(U)|<\infty$ for all $U\in\mathscr{T}$,
(b) $\{|\mu_n| : n\in\mathsf{N}\}$ is uniformly tight, that is, for every $\varepsilon>0$ there exists $K\in\mathscr{K}$ such that $\sup_{n\in\mathsf{N}}|\mu_n|(X\setminus K)<\varepsilon$.

Now it was shown in [2, Theorem 3.11], that in a regular Hausdorff space uniform tightness is implied by (3). Furthermore (3) implies that for every $U\in\mathscr{T}$ and $\varepsilon>0$ there exists $K\in\mathscr{K}$ such that

$$\sup_{n\in\mathsf{N}}|\mu_n|(U\setminus K) < \varepsilon$$

(see [2, Theorem 3.11 and 3.13]). It follows by (1) that there exists $C\in\mathscr{C}$ with $K\subset C\subset U$, whence, together with (2),

$$\sup_{n\in\mathsf{N}}|\mu_n(U)| \leqq \sup_{n\in\mathsf{N}}|\mu_n(C)|+\sup_{n\in\mathsf{N}}|\mu_n|(U\setminus K) < \infty$$

for every $U\in\mathscr{T}$.

(ii): Follows using (i) from [2, Lemma 4.3].

Besides Lemma 2.1 the following result, an analogon to Lemma 1 in [3], plays an essential role in proving the main theorem below. It was proposed by F. Topsøe to use (6) below as the appropriate separating condition.

2.2 Lemma. *Let $\mathscr{C}$ be a family of open sets in a regular Hausdorff space $(X,\mathscr{T})$ fulfilling the following conditions*:

(4) *If $C_1,C_2\in\mathscr{C}$, then $C_1\cap C_2\in\mathscr{C}$.*

(5) *If $C_1,C_2\in\mathscr{C}$ and $C_1^c\cap C_2^c=\emptyset$, then $C_1\cup C_2\in\mathscr{C}$.*

(6) *If $K\in\mathscr{K}$, $U\in\mathscr{T}$, $K\subset U$, then there exist $C',C''\in\mathscr{C}$ such that $K\subset C'\subset X\setminus C''\subset U$.*

(7) *If C_n' and C_n'', $n\in\mathsf{N}$, are sequences from $\mathscr{C}$ such that $C_1''\subset C_2''\subset\ldots\subset C_n''\subset\ldots\subset C_n'\subset C_{n-1}'\subset\ldots\subset C_2'\subset C_1'$, then there exists $C_0\in\mathscr{C}$ interpolating the given sequence, that is, $C_n''\subset C_0\subset C_n'$ for every $n\in\mathsf{N}$.*

Let $U_j\in\mathscr{T}$, $j\in\mathsf{N}$, be a sequence of pairwise disjoint sets and $K_j\in\mathscr{K}$, $K_j\subset U_j$, $j\in\mathsf{N}$. If we fix, according to (6), *$C_j', C_j''\in\mathscr{C}$ such that $K_j\subset C_j'\subset X\setminus C_j''\subset U_j$, then for every $\delta>0$ and $\lambda\in\mathscr{K}\text{-rca}_+(X,\mathscr{F})$ there exists an infinite subset $\mathsf{N}'\subset\mathsf{N}$ and a $C_{\mathsf{N}'}'\in\mathscr{C}$ such that $C_{\mathsf{N}'}'\supset\bigcup_{j\in\mathsf{N}'}C_j'$ and $\lambda(C_{\mathsf{N}'}')<\delta$.*

Proof. Let $\delta>0$ be given. If we choose $U:=X\setminus(\bigcup_{i\in\mathsf{N}}C_{2i-1}')^c$, then $U\in\mathscr{T}$ and $U\supset K_{2m}$ for every $m\in\mathsf{N}$, hence (6) allows us to pick for every $m\in\mathsf{N}$ a set $D_{2m}''\in\mathscr{C}$ such that $K_{2m}\subset X\setminus D_{2m}''\subset U$ and where, by (4), D_{2m}'' can be choosen in such a way that $D_{2m}''\subset C_{2m}''$ which implies $D_{2m}''\subset X\setminus C_{2m}'$ for every $m\in\mathsf{N}$. We obtain the following sequence:

$$C_1' \subset (C_1' \cup C_3') \subset (C_1' \cup C_3' \cup C_5') \subset \ldots \subset (D_6'' \cap D_4'' \cap D_2'') \subset (D_4'' \cap D_2'') \subset D_2'' ,$$

where, according to (5) and (4), each member occurring within the brackets belongs to $\mathscr{C}$. Hence by the interpolating condition (7) there exists $E_1 \in \mathscr{C}$ such that

$$\bigcup_{i \in \mathsf{N}} C_{2i-1}' \subset E_1 \subset \bigcap_{m \in \mathsf{N}} D_{2m}'' \subset \bigcap_{m \in \mathsf{N}} (X \setminus C_{2m}') ,$$

hence $E_1 \cap \bigcup_{m \in \mathsf{N}} C_{2m}' = \emptyset$. If $\lambda(E_1) < \delta$, the proof would be concluded. If $\lambda(X \setminus E_1) < \frac{1}{2}\delta$, $\mathscr{K}$ regularity of λ implies the existence of $K^{(1)} \in \mathscr{K}$, $K^{(1)} \subset E_1$, such that $\lambda(E_1 \setminus K^{(1)}) < \frac{1}{2}\delta$. By (6) there exists a $E_1'' \in \mathscr{C}$ such that $K^{(1)} \subset X \setminus E_1'' \subset E_1$, hence $E_1'' \supset X \setminus E_1 \supset \bigcup_{m \in \mathsf{N}} C_{2m}'$, $E_1'' \subset X \setminus K^{(1)}$ and $\lambda(E_1'') < \delta$, whence the proof would be concluded.

If neither $\lambda(E_1) < \delta$ nor $\lambda(X \setminus E_1) < \frac{1}{2}\delta$, then one may repeat the process to find disjoint infinite subsets N_1 and N_2 of $\{2i-1 : i \in \mathsf{N}\}$ and an $E_2 \in \mathscr{C}$, $E_2 \subset E_1$ (by (4)), such that

$$\bigcup_{i \in \mathsf{N}_1} C_i' \subset E_2 \subset \bigcap_{j \in \mathsf{N}_2} (X \setminus C_j') .$$

If $\lambda(E_2) < \delta$ or if $\lambda(E_1 \cap (X \setminus E_2)) < \frac{1}{2}\delta$, the proof would be concluded as before. Otherwise, continuing in this splitting procedure, one would find $E_n \in \mathscr{C}$, $E_n \downarrow$, $n \in \mathsf{N}$, such that $\lambda(E_1) \geqq \delta$, $\lambda(X \setminus E_1) \geqq \frac{1}{2}\delta$, $\lambda(E_2) \geqq \delta$, $\lambda(E_1 \cap (X \setminus E_2)) \geqq \frac{1}{2}\delta, \ldots, \lambda(E_n) \geqq \delta$, $\lambda(E_{n-1} \cap (X \setminus E_n)) \geqq \frac{1}{2}\delta, \ldots,$ which would contradict the boundedness of λ if the procedure would not terminate. This concludes the proof of Lemma 2.2.

3. The Main Theorem.

The results of section 2 enable us to prove the following theorem in rather the same way as it was done by Wells proving Theorem 3 in [3, p. 125], in the case of a compact basic space X. In order to point out the additional refinements which are necessary for the present case we give a complete proof being fully aware that we contribute only a modest amount to the ideas occurring already in the proof of Wells' Theorem.

3.1 THEOREM. *Let $\mathscr{C}$ be a family of open sets in a regular Hausdorff space $(X, \mathscr{T})$ fulfilling the conditions* (4)–(7) *of Lemma* 2.2. *Then $\mathscr{C}$ is both a converging class and a bounding class. In particular: $\mathscr{C} = \mathscr{T}_r$ is both a converging class and a bounding class.*

PROOF. Let $(\mu_n)_{n \in \mathsf{N}}$ be a sequence of $\mathscr{K}$-regular measures which converges on $\mathscr{C}$. Assume that, for some $A_0 \in \mathscr{F}$, $(\mu_n(A_0))_{n \in \mathsf{N}}$ does not converge. Then there exists $\varepsilon_0 > 0$ and for every $n \in \mathsf{N}$ an $m_n \in \mathsf{N}$ such that

$$|\mu_n(A_0) - \mu_{n+m_n}(A_0)| > \varepsilon_0 .$$

Hence $\tilde{\nu}_n := \mu_n - \mu_{n+m_n}$, $n \in \mathsf{N}$, defines a sequence of $\mathscr{K}$-regular measures which converges to zero on $\mathscr{C}$, but does not converge to zero on $\mathscr{F}$. The proof will be concluded by showing that every sequence of $\mathscr{K}$-regular measures, say $(\nu_n)_{n \in \mathsf{N}}$, which converges to zero on $\mathscr{C}$ converges to zero on $\mathscr{F}$. We remark that if $(\nu_n)_{n \in \mathsf{N}}$ converges on $\mathscr{F}$, it follows that $\nu_0(A) := \lim_{n \to \infty} \nu_n(A)$, $A \in \mathscr{F}$, is a $\mathscr{K}$-regular measure [2, 3.8] which must be identically zero according to (6).

Now assume, on the contrary, that $(\nu_n)_{n \in \mathsf{N}}$ does not converge on $\mathscr{F}$. Then, by Lemma 2.1, (3) must be wrong, whence there exists a sequence of pairwise disjoint sets $U_j \in \mathscr{T}$, $j \in \mathsf{N}$, so that there exist $\varepsilon_0 > 0$, an infinite subset $\mathsf{N}_0 \subset \mathsf{N}$ and a subsequence $(\nu_{n_j})_{j \in \mathsf{N}_0}$ of $(\nu_n)_{n \in \mathsf{N}}$ with $\inf_{j \in \mathsf{N}} |\nu_{n_j}(U_j)| > \varepsilon_0$. Without loss of generality we may assume that $\inf_{j \in \mathsf{N}} |\nu_j(U_j)| > \varepsilon_0$. Since $\nu_j \in \mathscr{K}\text{-rca}(X, \mathscr{F})$, there exist $K_j \in \mathscr{K}$, $K_j \subset U_j$, $j \in \mathsf{N}$, such that

$$|\nu_j|(U_j \setminus K_j) < |\nu_j(U_j)| - \varepsilon_0 ,$$

hence $\inf_{j \in \mathsf{N}} |\nu_j(C_j)| > \varepsilon_0$ whenever

$$K_j \subset C_j \subset U_j, \quad j \in \mathsf{N} .$$

By (6) for every $j \in \mathsf{N}$ there exist C_j', $C_j'' \in \mathscr{C}$ such that

$$K_j \subset C_j' \subset X \setminus C_j'' \subset U_j .$$

Now we apply Lemma 2.2 with $\lambda := |\nu_1|$, $\delta := \frac{1}{3}\varepsilon_0$ and $\mathsf{N}_1 := \mathsf{N} \setminus \{1\}$ instead of N to obtain an infinite subset N_1' of N_1 and a $C_{\mathsf{N}_1'}' \in \mathscr{C}$ such that $C_{\mathsf{N}_1'}' \supset \bigcup_{j \in \mathsf{N}_1'} C_j'$ and $|\nu_1|(C_{\mathsf{N}_1'}') < \frac{1}{3}\varepsilon_0$. Put $C_{\mathsf{N}_1'} := C_{\mathsf{N}_1'}' \cap C_1''$ and choose, according to (6), D_1', D_1'' such that

$$K_1 \subset D_1' \subset X \setminus D_1'' \subset C_1' .$$

Then, if we put $C_1 := D_1'$, we have $C_1^c \cap C_{\mathsf{N}_1'}^c = \emptyset$, hence, by (5),

$$C_1 \cup C_{\mathsf{N}_1'} \in \mathscr{C}$$

and furthermore,

$$|\nu_1|(C_{\mathsf{N}_1'}) < \tfrac{1}{3}\varepsilon_0, \quad |\nu_1(C_1)| > \varepsilon_0 .$$

Next, choose $n_0 := 1$ and pick $n_1 \in \mathsf{N}_1'$ so large that $|\nu_n(C_1)| < \frac{1}{3}\varepsilon_0$ for all $n \geqq n_1$ and, applying Lemma 2.2 again with $\lambda := |\nu_{n_1}|$, $\delta := \frac{1}{3}\varepsilon_0$ and the sequences

$$(K_j \subset C_j' \subset X \setminus C_j'' \subset U_j)_{j \in \mathsf{N}_2}, \quad \text{with} \quad \mathsf{N}_2 := \mathsf{N}_1' \cap \{n \in \mathsf{N} : n > n_1\} ,$$

we obtain an infinite subset N_2' of N_2 and a $C_{\mathsf{N}_2'}' \in \mathscr{C}$ such that $C_{\mathsf{N}_2'}' \supset \bigcup_{j\in\mathsf{N}_2'} C_j'$ and $|\nu_{n_1}|(C_{\mathsf{N}_2'}') < \frac{1}{3}\varepsilon_0$. Put

$$C_{\mathsf{N}_2'} := C_{\mathsf{N}_2'}' \cap C_{\mathsf{N}_1'} \cap C_{n_1}''$$

and choose, according to (6), $D_{n_1}', D_{n_1}'' \in \mathscr{C}$ such that

$$K_{n_1} \subset D_{n_1}' \subset X \setminus D_{n_1}'' \subset C_{n_1}' .$$

Then, if we choose $C_{n_1} := D_{n_1}'$, we have $C_{\mathsf{N}_2'}{}^c \cap C_{n_1}{}^c = \emptyset$, hence, by (5) $C_{\mathsf{N}_2'} \cup C_{n_1} \in \mathscr{C}$ and also $C_{\mathsf{N}_2'} \cup C_{n_1} \cup C_1 \in \mathscr{C}$ and, furthermore,

$$|\nu_{n_1}|(C_{\mathsf{N}_2'}) < \tfrac{1}{3}\varepsilon_0, \quad |\nu_{n_1}(C_{n_1})| > \varepsilon_0 .$$

Now pick $n_2 > n_1$, $n_2 \in \mathsf{N}_2'$ so large that

$$|\nu_n(C_1)| + |\nu_n(C_{n_1})| < \tfrac{1}{3}\varepsilon_0$$

for all $n \geqq n_2$. Continuing in this way we obtain a sequence of integers $\{n_0 = 1, n_1, n_2, \ldots\}$, a sequence $(C_{n_i})_{i\in\mathsf{N}}$ and a decreasing sequence $(C_{\mathsf{N}_i'})_{i\in\mathsf{N}}$ of sets in $\mathscr{C}$ such that

$$(*) \qquad |\nu_{n_i}|(C_{\mathsf{N}_{i+1}'}) < \tfrac{1}{3}\varepsilon_0 \quad \text{and} \quad |\nu_{n_i}(C_{n_i})| > \varepsilon_0 \qquad \text{for all } i \in \mathsf{N} ,$$

$$(**) \qquad \textstyle\sum_{i=0}^{j-1} |\nu_n(C_{n_i})| < \tfrac{1}{3}\varepsilon_0 \qquad \text{for all } n \geqq n_j ,$$

$$(***) \quad C_1 \subset (C_1 \cup C_{n_1}) \subset (C_1 \cup C_{n_1} \cup C_{n_2}) \subset \ldots \subset (C_{\mathsf{N}_3'} \cup C_{n_2} \cup C_{n_1} \cup C_1) \subset (C_{\mathsf{N}_2'} \cup C_{n_1} \cup C_1) \subset (C_{\mathsf{N}_1'} \cup C_1) ,$$

where each member occurring in (***) within the brackets belongs to $\mathscr{C}$. Hence by (7) there exists $C_0 \in \mathscr{C}$ which interpolates (***). Since, by (*) and (**),

$$|\nu_{n_j}(C_0)| \geqq |\nu_{n_j}(C_{n_j})| - \textstyle\sum_{i=0}^{j-1} |\nu_{n_j}(C_{n_i})| - |\nu_{n_j}|(C_{\mathsf{N}_{j+1}'}) \geqq \tfrac{1}{3}\varepsilon_0$$

for every $j \in \mathsf{N}$, $(\nu_n)_{n\in\mathsf{N}}$ does not converge to zero on C_0; this is a contradiction and we have thus proved that $\mathscr{C}$ is a converging class.

The proof that $\mathscr{C}$ is also a bounding class follows exactly the patterns of Wells [3, Corollary, p. 128].

Finally, $\mathscr{C} = \mathscr{T}_r$ satisfies (4), (5) and (7) (cf. [2, 4.6]) and also (6): Since $(X, \mathscr{T})$ is assumed to be a regular Hausdorff space, for every $K \in \mathscr{K}$, $U \in \mathscr{T}$, $K \subset U$, there exists $V \in \mathscr{T}$ such that $K \subset V \subset V^c \subset U$. Taking $C' := \operatorname{int}(V^c)$ and $C'' := X \setminus V^c$ we obtain $C', C'' \in \mathscr{C}$ with

$$K \subset C' \subset X \setminus C'' \subset U .$$

This concludes the proof of the theorem.

3.2 Remark. *We obtain equivalent conditions if we replace* $\mathscr{T}$ *in* (1), (3) *and* (6) *by* $\mathscr{T}'$, *where* $\mathscr{T}'$ *is a base for* $\mathscr{T}$ *which is closed under the forma-*

tion of finite unions. This is due to the fact that given $K \in \mathscr{K}$, $U \in \mathscr{T}$ with $K \subset U$, there exists $U' \in \mathscr{T}$ such that $K \subset U' \subset U$.

Furthermore we claim that (3) *is equivalent to the condition*

(3′) $\lim_{j\to\infty}\mu_n(C_j)=0$ *uniformly in* $n \in \mathsf{N}$, *whenever* C_j, $j \in \mathsf{N}$, *is a sequence of regular open sets with* $(\bigcup_{j\neq i}C_j)^c \cap C_i^c = \emptyset$ *for every* $i \in \mathsf{N}$.

For assume (3′) holds and let $U_j \in \mathscr{T}$, $j \in \mathsf{N}$, be a sequence of pairwise disjoint sets with

$$\inf_{j\in\mathsf{N}}|\mu_j(U_j)| > \varepsilon_0 > 0;$$

$\mathscr{K}$-regularity of μ_j implies the existence of $K_j \in \mathscr{K}$, $K_j \subset U_j$ such that

$$|\mu_j|(U_j \setminus K_j) < |\mu_j(U_j)| - \varepsilon_0 .$$

According to (6) (which holds especially with $\mathscr{C} = \mathscr{T}_r$) we obtain $C_j \in \mathscr{T}_r$ with

$$K_j \subset C_j \subset U_j, \quad j \in \mathsf{N},$$

and so that

$$(\bigcup_{j\neq i}C_j)^c \cap C_i^c = \emptyset \quad \text{for every } i \in \mathsf{N}$$

(cf. [2, Proof of 4.5 (b)]); furthermore $\inf_{j\in\mathsf{N}}|\mu_j(C_j)| > \varepsilon_0$, a contradiction.

3.3 REMARK. The proof of Theorem 3.1 for the special case $\mathscr{C} = \mathscr{T}_r$ can be substantially simplified applying, instead of Lemma 2.2, the following Lemma which was essentially pointed out to me by D. Fremlin:

3.4 LEMMA. *Let* $(\mu_j)_{j\in\mathsf{N}}$ *be a sequence of* $\mathscr{K}$*-regular measures which converges to zero on* $\mathscr{T}_r$ *and let* C_j, $j \in \mathsf{N}$, *be a sequence of regular open sets in* X *with*

$$((\bigcup_{j\neq i}C_j)^c \cap C_i^c = \emptyset \quad \textit{for every } i \in \mathsf{N} .$$

Then $\lim_{j\to\infty}\mu_j(C_j) = 0$.

PROOF. For any open set U, let $\alpha(U) := \operatorname{int}(U^c) \setminus U$. [This is a kind of boundary of U for the closure operation $U \to \operatorname{int}(U^c)$.] Let N_0 be any subset of N and $i \notin \mathsf{N}_0$, then it follows at once that

$$C_i \cap \alpha(\bigcup_{j\in\mathsf{N}_0}C_j) = \emptyset; \tag{8}$$

on the other hand, if $i \in \mathsf{N}_0$, then $C_i \cap (X \setminus \bigcup_{j\in\mathsf{N}_0}C_j) = \emptyset$, hence (8). Thus $\alpha(\bigcup_{j\in\mathsf{N}_0}C_j)$ does not meet any C_i, $i \in \mathsf{N}$. Consequently, if N_1 and N_2 are subsets of N such that $\mathsf{N}_1 \cap \mathsf{N}_2$ is finite, $\alpha(\bigcup_{j\in\mathsf{N}_1}C_j)$ does not meet $\alpha(\bigcup_{j\in\mathsf{N}_2}C_j)$. [Note that $\alpha(\bigcup_{j\in\Gamma}C_j) = \emptyset$ for finite subsets Γ of N (cf. [2, 4.6 (2)]).] At once there must be an infinite subset N' of N such that

$$|\mu_n|(\alpha(\bigcup_{j\in\mathsf{N}'}C_j)) = 0 \quad \text{for every } n \in \mathsf{N} .$$

Now, let Δ be an arbitrary subset of N'. Then

$$\bigcup_{j\in\Delta} C_j = \mathrm{int}((\bigcup_{j\in\Delta} C_j)^c) \setminus \alpha(\bigcup_{j\in\Delta} C_j) ,$$

where $\mathrm{int}((\bigcup_{j\in\Delta} C_j)^c)$ belongs to $\mathscr{T}_r$. Hence convergence of μ_n on $\mathscr{T}_r$ to zero implies that $\nu_n(\Delta) := \mu_n(\bigcup_{j\in\Delta} C_j)$ tends to zero as $n \to \infty$ for every subset Δ of N'. This implies $\lim_{n\in\mathsf{N}'} \mu_n(C_n) = 0$ (cf. Nikodým's Theorem in [1, III.7.4]) and therefore the assertion of the Lemma.

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