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BOREL STRUCTURES AND A TOPOLOGICAL ZERO-ONE LAW

JENS PETER REUS CHRISTENSEN

This paper deals with miscellaneous results about Borel structures and some applications. Among other things we shall see that a finitely additive Borel measure on a Polish space is countably additive if it does not behave very irregularly on the closed sets. Furthermore we prove that a sequence of probability measures on a complete separable metric space is weakly convergent if its limit exists for every bounded uniformly continuous function. (Of course, completeness of the metric is necessary for this statement to hold.)

First we prove a result which we consider as a topological analogue of the zero-one law in probability theory. Although a weaker statement would suffice for later purposes, the result may be of independent interest.

Let $(M_n, \mathcal{O}_n)$ be a sequence of Polish spaces. Consider the space

$$M = \prod_{n=1}^{\infty} M_n .$$

With the product topology M is a Polish space. We define the equivalence relation $\sim$ in M by

$$x \sim y \Leftrightarrow \{n \mid x(n) \neq y(n)\} \text{ is finite} .$$

The BP σ-field on M is the system of all sets $A \subseteq M$ with the Baire property. By definition, A has the Baire property if there exists an open set O such that $(A \setminus O) \cup (O \setminus A)$ is of the first category (contained in a countable union of closed sets with empty interior). A real-valued function f on M is BP measurable if and only if there exists a dense G_δ set $A \subseteq M$ such that the restriction of f to A is continuous on A. (For f to be continuous it is sufficient that $f^{-1}(O_p)$ is open for all $p \in \mathsf{N}$ where O_p is a countable base for the topology of R; therefore, if f is BP measurable, we can remove a countable union of first category sets such that f is continuous on the remaining set.) The function f respects $\sim$ if

$$x \sim y \Rightarrow f(x) = f(y)$$

is satisfied. Let $[x]$ denote the equivalence class of $x \in M$.

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THEOREM 1. *Let $A \subseteq M$ be a dense G_δ set. Then there exists an $x \in A$ such that $[x] \cap A$ is dense. Therefore, if a function f on M respects $\sim$ and is* BP *measurable, then f is constant on a dense G_δ set.*

PROOF. We consider the space

$$\bar{M} = \prod M_i^j ,$$

where $M_i^j = M_i$ for all $i,j \in \mathsf{N}$ (the product is taken over all i,j from 1 to infinity). With the product topology $\bar{M}$ is a Polish space, and $\sim$ can be defined in $\bar{M}$ similarly. We consider the projection $P: \bar{M} \to M$ defined by

$$P(x)(n) = x(n,1) .$$

The set $\bar{A} = P^{-1}(A)$ is a dense G_δ set in $\bar{M}$. Let T be the group of all homeomorphisms of $\bar{M}$ which are determined by permutations of the indexes (i,j) such that only finitely many indexes are permuted. The group T is countable. Therefore

$$B = \bigcap_{t \in T} t(\bar{A})$$

is a dense G_δ set contained in $\bar{A}$ and T invariant. Let $C \subseteq \bar{M}$ be the set defined by

$$C = \{x \in \bar{M} \mid T(x) \text{ is dense in } \bar{M}\} .$$

We easily see that

$$C = \bigcap_p \bigcup_{t \in T} t(O_p) ,$$

where $\{O_p\}$ is a countable base for the topology of $\bar{M}$.

From this it follows that C is a dense G_δ set. Hence $B \cap C$ is non empty. Let $y \in B \cap C$. For $x = P(y)$ we see that $[x] \cap A$ is dense. This finishes the proof of theorem 1.

In the sequel, measurable (without specification) means measurable with respect to the Borel field of the topology under consideration.

Let $K = \{0,1\}^{\mathsf{N}}$. With the usual product topology and group structure K is a compact metrizable abelian group. An element $x \in K$ may be considered as the characteristic function of a subset of N and conversely. Let φ be a real-valued finitely additive set function defined on the subsets of the natural numbers N.

THEOREM 2. *Consider φ above as a function on K, and suppose one of the following conditions is fulfilled:*

i) φ *is non-negative and* BP *measurable.*
ii) φ *is measurable.*
Then φ *is countably additive.*

PROOF. This requires some steps. First we prove the case i). By subtraction we can assume that $\varphi(\{n\})=0$ for every $n \in \mathsf{N}$. The zero-one law implies that there exist an $\alpha \in \mathsf{R}$ and a dense G_δ set $A \subseteq K$ such that $\varphi(x)=\alpha$ for every $x \in A$ (φ is considered as a function on K). Let $u \in K$ be the sequence $u=(1,1,1,\ldots)$. The set $u+A=\{u+x \mid x \in A\}$ is a dense G_δ set, and hence $A \cap (u+A)$ is non empty. Therefore there exist $x,y \in A$ such that $x\cdot y=0$ (the sequences are "disjoint") and $x+y=1$. This implies

$$\varphi(u) = \varphi(\mathsf{N}) = \varphi(x+y) = \varphi(x)+\varphi(y) = 2\alpha ,$$
$$\alpha = \tfrac{1}{2}\varphi(\mathsf{N}) .$$

Now it only remains to show that $\alpha=0$. This is an immediate consequence of the following

LEMMA. *Let* $A \subseteq K$ *be a dense* G_δ *set. Then there are* $x,y,z \in A$ *such that* $y\cdot z=0$ *and* $x=y+z$.

To see this we consider the mappings $g,h\colon K^2 \to K$ defined by

$$g((a,b)) = a \setminus b = a-(a\cdot b) ,$$
$$h((a,b)) = a\cdot b .$$

These mappings are surjective, open, and continuous. Hence $g^{-1}(A)$ and $h^{-1}(A)$ are dense G_δ sets in K^2. We choose

$$(a,b) \in g^{-1}(A) \cap h^{-1}(A) \cap (A \times K) ,$$

and put $x=a$, $y=g((a,b))$, and $z=h((a,b))$.

In the case ii) we first show that φ is uniformly bounded. Suppose this is not true. Then there is a sequence $x_n \in K$ of disjoint elements ($x_n\cdot x_m=0$ if $n \neq m$) such that $|\varphi(x_n)|>1$ for all $n \in \mathsf{N}$. The mapping $\theta\colon K \to K$ defined by

$$\theta(y) = \textstyle\sum_k y(k)x_k$$

is continuous. Therefore $\varphi\circ\theta$ is a measurable function on K, in particular BP measurable. Let $A \subseteq K$ be a dense G_δ set such that $\varphi\circ\theta$ is continuous on A. We may assume that A is invariant under translations by elements in

$$K_0 = \{x \in K \mid x(n)=1 \text{ for at most finitely many } n\} .$$

We choose $y \in A$. Let y_λ be the element in K which differs from y only at the λth place. Then $y_\lambda \in A$ and $y_\lambda \to y$ but

$$|\varphi \circ \theta(y) - \varphi \circ \theta(y_\lambda)| = \varphi(x_\lambda) > 1 .$$

This contradiction shows that φ is uniformly bounded. Let us also consider the positive part φ^+ of φ as a function on K. Then we get

$$\varphi^+(x) = \sup\{\varphi(y) \mid y \in K \cap y \cdot x = y\} .$$

Let $D \subseteq K$ be the set

$$D = \{x \in K \mid \varphi^+(x) > \alpha\} ,$$

where α is a fixed number. D is the projection on the second coordinate of the set

$$\{(x,y) \in K^2 \mid y \cdot x = y\} \cap \{(x,y) \mid \varphi(y) > \alpha\} .$$

Now this set is measurable in K^2, hence D is analytic in K. Therefore D is BP measurable (see [4, Chap. 1, § 11, p. 62–63]. An application of the first case finishes the proof. (This argument showing that φ^+ is BP measurable is due to J. Hoffmann-Jørgensen.)

Let X be an arbitrary set equipped with a σ-field $\mathscr{F}$. Let φ_n be a sequence of countably additive set functions defined on $\mathscr{F}$. Suppose $\lim \varphi_n(A) = \varphi(A)$ exists for every $A \in \mathscr{F}$. Then φ is countably additive. This well-known result of Nikodym can now be proved as an application of theorem 2. It is an immediate consequence of the following

Theorem 3. *Let $\mathscr{C}$ be the Borel field on $\mathscr{F}$ generated by the topology on $\mathscr{F}$ induced by all countably additive real-valued set functions. Let φ be a $\mathscr{C}$ measurable finitely additive realvalued set function on $\mathscr{F}$. Then φ is countably additive.*

Proof. Let F_n be a sequence of disjoint $\mathscr{F}$ sets. We define the mapping $\theta : K \to \mathscr{F}$ by

$$\theta(x) = \textstyle\sum_k x(k) F_k \quad (1 \cdot F_k = F_k \text{ and } 0 \cdot F_k = \emptyset) .$$

This mapping is continuous with respect to the above mentioned topology (use Lebesgue's dominated convergence theorem). Hence $\varphi \circ \theta$ is a measurable function on K. An application of theorem 2 finishes the proof of theorem 3.

The following application of theorem 2 was pointed out to the author by J. Hoffmann-Jørgensen.

THEOREM 4. *Let (S,Σ,μ) be a positive measure space with μ σ-finite. Consider $L_\infty(S,\Sigma,\mu)$ with the weak topology $\sigma(L_\infty,L_1)$. Let L be a linear functional on L_∞ which is measurable in the Borel structure generated by the weak topology. Then L is induced by a L_1-function.*

PROOF. We define a finitely additive signed measure l on Σ by

$$l(A) = L(\chi_A) .$$

Of course $l(N)=0$ if $\mu(N)=0$. An argument similar to the proof of theorem 3 shows that l is countably additive. An application of the Radon–Nikodym theorem finishes the proof of theorem 4.

Let $(M,\mathcal{O})$ be a Polish space. We shall always tacitly assume that "metrics on M" generate the topology of M. Let M^* be the set of closed subsets of M. For each bounded metric d on M we define the metric d^* on M^* by

$$d^*(A,B) = \sup\{d(a,B),d(A,b) \mid a\in A \wedge b\in B\} .$$

Since M is separable, one can choose a precompact metric d on it. If d is such a metric, (M^*,d^*) is a precompact metric space and the d^*-topology is Polish. Moreover, the Borel structure of the d^*-topology is independent of the metric d (which is assumed to be precompact). These results seem to be due to Edward G. Effros (see [3]). From now on measurable on M^* or Effros measurable means measurable with respect to this canonical Borel structure.

THEOREM 5. *Let μ be a finitely additive Borel probability measure on the Polish space $(M,\mathcal{O})$. Suppose μ is measurable as a function on M^*. Then μ is countably additive.*

PROOF. Let $B(M)$ be the Borel field and $A(M)\subseteq B(M)$ be the field generated by the open sets. First we show that μ is regular on $A(M)$ with respect to the paving of closed sets. It is enough to show that each open set G can be approximated in μ measure from within by closed sets.

Let $A=M\setminus G$ and let d be a precompact metric on M. For each $\varepsilon>0$ we define $S_\varepsilon=\{x\in M \mid d(x,A)=\varepsilon\}$. Let ε_n be a strictly decreasing sequence tending to zero, and suppose that $\mu(S_{\varepsilon_n})=0$ for all n. Put $S=\bigcup_n S_{\varepsilon_n}$. The mapping $\varphi\colon K \to M^*$ defined by

$$\varphi(x) = (\textstyle\bigcup_n x(n)\, S_{\varepsilon_n}) \cup A$$

is continuous with respect to the d^* topology, thus measurable. The function θ on K defined by

$$\theta(x) = \mu(\varphi(x)) - \mu(A),$$

is measurable and $\theta(x+y) = \theta(x) + \theta(y)$ if $x \cdot y = 0$. Furthermore $\theta(x) = 0$ if $x(n) = 1$ only for finitely many n. Therefore $\theta = 0$ identically (use theorem 2). This gives $\mu(S) = 0$. Let now

$$R(n) = \{x \in M \mid \varepsilon_{n-1} \geqq d(x,A) \geqq \varepsilon_n\} \quad \text{for } n \geqq 2,$$
$$R(1) = \{x \in M \mid d(x,A) \geqq \varepsilon_1\}.$$

In a similar way as before we define

$$\varphi_1(x) = (\bigcup_n x(n) R(n)) \cup A$$

and

$$\theta_1(x) = \mu(\varphi_1(x)) - \mu(A).$$

An application of theorem 2 and the preceding result gives

$$\sum_n \mu(R(n)) = 1 - \mu(A) = \mu(G).$$

This means that the open set G can be approximated from within in μ measure by closed sets whose distance from A is non-zero. (In the discussion above we assumed tacitly $A \neq \emptyset$.) It was sufficient to assume that μ is measurable with respect to the σ-field spanned by all sets which are analytic with respect to the Effros Borel structure. (This ensures that θ and θ_1 are BP measurable.)

Let $\bar{M}$ denote the completion of the space (M,d). It is a compact metric space. The measure μ is extended to a Borel measure on $\bar{M}$ by $\mu(B) = \mu(M \cap B)$ for a Borel set B in $\bar{M}$. The mapping $T\colon A \to A \cap M$ is a mapping from $\bar{M}^*$ onto M^*. Since M is G_δ in $\bar{M}$, any open set $O \subseteq M$ is G_δ in $\bar{M}$. Therefore the set of $A \in \bar{M}^*$ not intersecting O is coanalytic in $\bar{M}^*$. This shows that T is measurable if M^* is considered with the Effros structure and $\bar{M}^*$ with the σ-field spanned by sets analytic with respect to the Effros structure. (Note that the Effros structure is spanned by sets of sets not intersecting open sets.)

The above arguments now yield the fact that the extended measure μ is regular with respect to the closed (compact) subsets of $\bar{M}$. Therefore μ is countably additive. This finishes the proof of theorem 5.

Let μ_n be a sequence of countably additive Borel probability measures on M such that

$$\mu(A) = \lim \mu_n(A)$$

exists for every $A \in M^*$. Every μ_n is upper semicontinuous on M^*

(equipped with the d^* topology where d is a precompact metric on M). Hence μ is measurable on M^*. By theorem 5 we now easily show that $\lim \mu_n(B) = \mu(B)$ exists for every Borel set B in M and that μ is a measure. This result is due to Dieudonné [2] and has been generalized by Wells [5].

We call a subset $I \subseteq M^*$ an ideal if it satisfies

i) $M \notin I \wedge I \neq \emptyset$,

ii) $A \in I \wedge B \in M^* \Rightarrow A \cap B \in I$,

iii) $A, B \in I \Rightarrow A \cup B \in I$.

An ideal I is maximal if I is not contained in a strictly bigger ideal. This is the case if and only if I satisfies

$$\text{C)} \qquad A \in M^* \wedge A \notin I \wedge A \neq M \Rightarrow \operatorname{cl}(A^c) \in I .$$

THEOREM 6. *A maximal ideal I is measurable as a subset of M^* if and only if there exists an isolated point $m \in M$ such that $I = \{A \in M^* \mid m \notin A\}$. An ultrafilter $\mathscr{U}$ intersects M^* in a measurable set if and only if it is trivial.*

PROOF. Let I be a maximal ideal. The set $\{A^c \mid A \in I\}$ is a filter base consisting of open sets. Let $\mathscr{U}$ be an ultrafilter extending it. Put $\mu(B) = 1$ if $B \in \mathscr{U}$ and zero otherwise. If $A \in M^*$, but $A \notin I$, then $\operatorname{cl}(A^c) \in I$, hence $A \in \mathscr{U}$ and therefore $\mu(A) = 1$ (of course $\mu(A) = 0$ if $A \in I$). Now μ is a finitely additive Borel probability measure and the above remarks show that μ is a measurable function on M^* (because I is measurable). Therefore there exists an $m \in M$ such that $\mu(B) = 1$ if and only if $m \in B$. In particular,

$$I = \{A \in M \mid m \notin A\} .$$

Because I is maximal, m is isolated. The remaining part of theorem 6 is proved similarly.

We do not use the preceding results in the sequel, but our use of $K = \{0,1\}^{\mathsf{N}}$ will be analogous.

Let now (M,d) be a separable metric space. Let $S(M)$ be a space of bounded continuous functions which satisfy:

1) $S(M)$ is a linear space and an algebra.
2) If A, B are disjoint closed sets in M, there exists an $f \in S(M)$ with values in $[0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.
3) If g is a bounded continuous function such that for all $x \in M$ there exists a neighbourhood U of x and an $f \in S(M)$ with $g \mid U = f \mid U$, then $g \in S(M)$.

In the following measures on M are assumed to be non-negative countably additive bounded Borel measures if not explicitly stated otherwise.

THEOREM 7. *Let $\mathscr{B}$ be a uniformly bounded set of measures on M. Then $\mathscr{B}$ is conditionally compact in the space of measures (equipped with the weak topology) if and only if $S(M)$ equipped with the seminorm*

$$\|f\|_{\mathscr{B}} = \sup\{|\mu(f)| \mid \mu \in \mathscr{B}\}$$

is separable.

PROOF. The only if part follows from the fact that a weakly compact set is a compact metric space with the weak topology.

Let now A_n be a strictly decreasing sequence of sets with empty intersection. Let f_n be a function in $S(M)$ with values in the interval $[0,1]$ and assuming the value 1 on A_n and 0 on the set

$$\{x \in M \mid d(x, A_n) \geqq 2^{-n}\}.$$

Then $g_n = f_1 \ldots f_n$ is a decreasing sequence of functions in $S(M)$, and for each particular $x \in M$, g_n is zero in a neighbourhood of x from a certain stage. The mapping $\theta\colon K \to S(M)$ defined by

$$\theta(x) = \textstyle\sum_n s_n(x)\, x(n) g_n ,$$

where $s_n(x)$ is -1 to the power $\sum_1^n x(\tau)$, is continuous with respect to the topology on $S(M)$ induced by all measures (and the usual product topology on K).

Let $B_\varepsilon = \{f \in S(M) \mid \|f\|_{\mathscr{B}} \leqq \varepsilon\}$ and choose a countable covering

$$S(M) = \textstyle\bigcup_p (f_p + B_\varepsilon).$$

The sets $\theta^{-1}(f_p + B_\varepsilon)$ form a countable covering of K by closed sets. Hence there exists p such that $\theta^{-1}(f_p + B_\varepsilon)$ has non-empty interior. This shows that $g_n \in f_p + B_{3\varepsilon}$ for n sufficiently great (and p fixed). Because this is true for every particular $\varepsilon > 0$, we see that g_n is a Cauchy sequence with respect to the $\|\ \|_{\mathscr{B}}$ metric. But $\mu(g_n)$ tends to zero pointwise on $\mathscr{B}$, and hence it tends to zero uniformly on $\mathscr{B}$. Therefore $\mu(A_n)$ tends to zero uniformly on $\mathscr{B}$. Since this is true for every decreasing sequence of closed sets with empty intersection, $\mathscr{B}$ is weakly conditionally compact. This finishes the proof of theorem 7.

COROLLARY. *Let μ_n be a sequence of measures and suppose that for each $f \in S(M)$ there exists a number $\mu(f)$ such that $\lim \mu_n(f) = \mu(f)$. Then μ is a measure and $\mu_n \to \mu$ weakly.*

PROOF. The criterion in theorem 7 shows that $\{\mu_n\}$ is conditionally compact in the space of measures with the weak topology. Therefore the sequence has exactly one weak accumulation point. This proves the corollary.

It is a well-known result due to A. D. Alexandroff that convergence on the bounded continuous functions implies that the limit is a measure. In the special case where the metric is complete the above results can be improved substantially.

THEOREM 8. *Let* (M,d) *be a complete separable metric space. Let* $\mathrm{BU}(M)$ *be the space of bounded uniformly continuous functions on* M. *A uniformly bounded set of measures* $\mathscr{B}$ *is conditionally compact in the space of measures if and only if* $\mathrm{BU}(M)$ *is a separable space equipped with the seminorm* $\|\cdot\|_{\mathscr{B}}$.

PROOF. The only if part is identical with that of theorem 7. Let $\varepsilon > 0$ be fixed. Suppose $A_n \subseteq M$ is a sequence of closed sets with

$$\inf\{d(x,y) \mid x \in A_n, y \in A_m\} \geqq \varepsilon \quad \text{for } n \neq m .$$

We define f_n by

$$f_n(x) = d(x,B_n)/(d(x,A_n)+d(x,B_n)) ,$$

where

$$B_n = \{x \in M \mid d(x,A_n) \geqq \tfrac{1}{2}\varepsilon\};$$

(note that B_n cannot be empty). Further we define the mapping $\theta\colon K \to \mathrm{BU}(M)$ by $\theta(x) = \sum_n x(n) f_n$.

Let $\mathrm{BU}(M) = \bigcup_p (h_p + B_\delta)$ (where B_δ is defined as in the proof of theorem 7). Each of these sets is closed in the topology induced by all measures. An argument similar to the proof of theorem 7 yields that $f_n \in B_{2\delta}$ for n sufficiently great. This shows (since δ was arbitrary) that $\mu(f_n) \to 0$ uniformly as $n \to \infty$. Hence $\mu(A_n) \to 0$ uniformly as $n \to \infty$. Because this is true for every $\varepsilon > 0$, we easily see that $\mathscr{B}$ is uniformly tight. (The completeness of the metric is essential at this point of the proof.)

COROLLARY. *Let* μ_n *be a sequence of measures on* M *and suppose that for each* $f \in \mathrm{BU}(M)$ *there is a number* $\mu(f)$ *such that* $\lim \mu_n(f) = \mu(f)$. *Then* μ *is a measure and the limit relation holds also for each bounded continuous function.*

PROOF. The criterion in theorem 8 shows that $\{\mu_n\}$ is weakly conditionally compact in the space of measures.

Some problems in connection with the preceding results remain open. It seems rather probable that most of the results are also valid with Borel measurability replaced by measurability with respect to the universal completion of the Borel field. For analogous results see [1]. For the conclusion in theorem 2 to hold it is not enough to require measurability with respect to the Haar measure on K because there exists a finitely additive probability measure on N which equals the density of a set whenever this exists. It seems probable that some of our assumptions about positivity of the measures can be removed, but non trivial complications seem to arise.

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We solve a problem proposed above. Let M be a Polish space and f_n a uniformly bounded sequence of universally measurable functions on M. Let G_n be defined by $G_n = \mathrm{cl}\,\mathrm{conv}\{f_n, f_{n+1}, \ldots\}$ and $T = \bigcap_n G_n$ (the closure being taken with respect to the topology of pointwise convergence).

THEOREM. *The continuum hypothesis implies that T contains at least one universally measurable function.*

PROOF. We only indicate the idea of the proof. For each countable ordinal number ω we choose a probability u_ω such that each probability on M is chosen at least once. By transfinite induction we choose, for each countable ordinal ω, a sequence c_n^{ω} of finite convex combinations of f_n's such that $c_n^{\omega} \in G_n$ and $\lim c_n^{\omega}(x)$ exists for u_ω almost every $x \in M$ and such that the existence of $\lim c_n^{\nu}(x)$ implies that $\lim c_n^{\nu}(x) = \lim c_n^{\omega}(x)$ if ν is an ordinal number less than ω. The net $c_n^{\omega}(x)$ then tends pointwise to a universally measurable function in T. This finishes the proof.

Let $K = \{0,1\}^{\mathsf{N}}$. Define f_n on K by $f_n(x) = n^{-1} \sum_{\nu=1}^{n} x(\nu)$. If we apply the theorem to this sequence we obtain

COROLLARY. *There exists a finitely additive universally measurable probability measure defined for all subsets of* N *which equals the arithmetic density whenever this exists.*

We do not know whether or not the above results can be proved independently of the continuum hypothesis.

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REFERENCES

1. J. P. Reus Christensen, *Borel structures in groups and semigroups*, Math. Scand. 28 (1971), 124–128.
2. J. Dieudonné, *Sur la convergence des suites de mesures de Radon*, An. Acad. Brasil Ci. 23 (1951), 21–38, 277–282.
3. E. G. Effros, *Convergence of closed subsets in a topological space*, Proc. Amer. Math. Soc. 16 (1965), 929–931.
4. C. Kuratowski, *Topologie* I (Monografie Matematyczne 20), Warszawa · Wrocław, 1948.
5. B. B. Wells, jr., *Weak compactness of measures*, Proc. Amer. Math. Soc. 20 (1969), 124–130.

UNIVERSITY OF COPENHAGEN, DENMARK