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ON A LIMIT THEOREM OF MEASURES

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Let $P(S)$ be the convolution semigroup of probability measures on the compact semigroup S. Given $\mu \in P(S)$, let $\mu_N = N^{-1} \sum_{k=1}^{N} \mu^k$ for each positive integer N and let $\Gamma(\mu) = \{\mu^n : n = 1, 2, \ldots\}^-$, where the bar denotes the closure of the set. It is known that the sequence $\{\mu_N\}$ is convergent to an idempotent measure $L(\mu)$ in $P(S)$ and that

$$\mu L(\mu) = L(\mu)\mu = L(\mu)$$

(see, for example, [2], [4] and [6]). The main purpose of this note is to give some necessary and sufficient conditions such that $L(\mu) \in \Gamma(\mu)$.

Let $K(S)$ be the kernel of the compact semigroup S and let $\operatorname{supp}\mu$ denote the support of μ for μ in $P(S)$. In a previous paper [1] we gave a sufficient condition such that $L(\mu\nu) = L(\mu)L(\nu)$ for μ, ν in $P(S)$ and showed that $L(\mu^2)$ might not be $L(\mu)$ even in the case of Abelian groups (as can be seen immediately by taking a measure μ on the additive group of positive integers modulo 2 with one point support different from the identity). However we have the next result in the case of compact groups.

THEOREM 1. *Let S be a compact group with identity e and let $\mu \in P(S)$ such that $e \in \operatorname{supp}\mu$. Then*

$$L(\mu^n) = L(\mu), \quad n = 1, 2, \ldots .$$

PROOF. Suppose $N^{-1} \sum_{k=1}^{N} (\mu^n)^k$ converges to an idempotent λ and $\operatorname{supp}\mu$ contains e. Then since

$$n[(nN)^{-1} \textstyle\sum_{k=n}^{nN+n-1} \mu^k] \to \lambda + \mu\lambda + \ldots + \mu^{n-1}\lambda ,$$

we have

$$nL(\mu) = \lambda + \mu\lambda + \ldots + \mu^{n-1}\lambda .$$

Hence

$$\operatorname{supp} L(\mu) = (\operatorname{supp}\lambda) \cup (\operatorname{supp}\mu\lambda) \cup \ldots \cup (\operatorname{supp}\mu^{n-1}\lambda) .$$

Let δ_x be the unit point mass at $x \in S$. Now since e is the identity of S,

$$\mu^p \lambda = \mu^p {\delta_e}^{n-p} \lambda$$

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for $1 \leqq p \leqq (n-1)$. Since $\mu^n \lambda = \lambda$, we have

$$\operatorname{supp}(\mu^p \lambda) = \operatorname{supp}(\mu^p \delta_e^{n-p} \lambda) \subset (\operatorname{supp}\mu)^n \operatorname{supp}\lambda = \operatorname{supp}\lambda$$

for each $1 \leqq p \leqq (n-1)$. It follows that

$$\operatorname{supp} L(\mu) = \operatorname{supp}\lambda .$$

Since an idempotent measure on a compact group is the Haar measure on its support, we have $L(\mu^n) = L(\mu)$.

It is well known that there is one and only one idempotent measure in $\Gamma(\mu)$ (see, for example, [3, pp. 98–105]).

THEOREM 2. *Let μ be in $P(S)$ and let ν be the idempotent in $\Gamma(\mu)$. Then the following conditions are equivalent:*

(a) $L(\mu) = \nu$;
(b) $\lim_{n\to\infty} \mu^n$ *exists*;
(c) $K(\Gamma(\mu)) = \{\nu\}$;
(d) $(\operatorname{supp}\nu)(\operatorname{supp}\mu) = \operatorname{supp}\nu$;
(e) $\operatorname{supp} L(\mu) = \operatorname{supp}\nu$.

PROOF. (a) implies (b). Suppose $L(\mu) = \nu$. Recall that the set of the cluster points of the set $\Gamma(\mu)$ is a closed subgroup and that the identity is the only idempotent in $\Gamma(\mu)$. Let $\{\mu^\alpha\}$ be a convergent subnet of $\{\mu^n\}$ and let $\{\mu^\alpha\}$ be convergent to τ (say). Then, since $\nu = L(\mu)$ is the identity of the set of the cluster points, $L(\mu)\tau = \tau$. On the other hand, since $L(\mu)\mu^\alpha = L(\mu)$ for every α, we see $L(\mu)\tau = L(\mu)$. Hence $\tau = L(\mu)$. That is, every convergent subnet of $\{\mu^n\}$ is convergent to $L(\mu)$. A routine verification shows that $\lim_{n\to\infty} \mu^n = L(\mu)$ and (a) implies (b).

(b) implies (c). Suppose $\lim_{n\to\infty} \mu^n$ exists. Let $\lim_{n\to\infty} \mu^n = \tau$ (say). Then, since

$$\mu^N \tau = \tau \mu^N = \tau, \quad N = 1,2,\ldots ,$$

τ is the zero element of $\Gamma(\mu)$. In particular τ is an idempotent in $\Gamma(\mu)$. Hence $\tau = \nu$ and $\{\nu\}$ is the kernel of $\Gamma(\mu)$.

(c) implies (d). Suppose $K(\Gamma(\mu)) = \{\nu\}$. Then $\nu\mu = \nu$ implies

$$(\operatorname{supp}\nu)(\operatorname{supp}\mu) = \operatorname{supp}\nu .$$

(d) implies (e). Suppose $(\operatorname{supp}\nu)(\operatorname{supp}\mu) = \operatorname{supp}\nu$. We note first that, since $\Gamma(\mu)$ is a compact commutative semigroup, $K(\Gamma(\mu))$ is a group. Therefore the only idempotent ν in $\Gamma(\mu)$ is in $K(\Gamma(\mu))$. Let $S(\mu)$ be the smallest closed subsemigroup of S containing $\operatorname{supp}\mu$. Then

$$\operatorname{supp}\nu \subset K(S(\mu)) = \operatorname{supp} L(\mu)$$

(see Lin [4, Theorem 3]). On the other hand, by the assumption and the continuity of multiplication, we have

$$(\operatorname{supp}\nu)S(\mu) = \operatorname{supp}\nu\ .$$

With similar argument as before, we have

$$\nu L(\mu) = L(\mu)\ .$$

Therefore,

$$\operatorname{supp} L(\mu) = (\operatorname{supp}\nu)K(S(\mu)) \subset (\operatorname{supp}\nu)S(\mu) = \operatorname{supp}\nu\ .$$

We conclude that $\operatorname{supp} L(\mu) = \operatorname{supp}\nu$.

(e) implies (a). Suppose $\operatorname{supp} L(\mu) = \operatorname{supp}\nu$. Thus $L(\mu)$ and ν are idempotent measures supported on the compact simple semigroup $\operatorname{supp}\nu$. Suppose first that $\operatorname{supp}\nu$ is a group. Then we see $L(\mu) = \nu$. Suppose next that $\operatorname{supp}\nu$ is not a group. The results of Pym [5, C6.3] then give that $L(\mu)$, ν are primitive idempotent measures on $\operatorname{supp}\nu$. But

$$L(\mu) = L(\mu)\nu = \nu L(\mu)\ ,$$

so that $L(\mu) = \nu$ and (e) implies (a).

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