MATH. SCAND. 29 (1971), 272–278

A NORM PRESERVING COMPLEX CHOQUET THEOREM

OTTE HUSTAD

Introduction.

We consider a compact Hausdorff space X and a linear subspace B of the normed space $C(X)$ consisting of all continuous, complex valued functions on X. Assume that B separates points on X and contains the constant functions. Let l be a continuous linear functional on B. Then the Bishop–de Leeuw version of the Choquet theorem (see e.g. [5]) states that there exists a complex measure m on X which is quasi-supported by the Choquet boundary of B and which represents l in the sense that $l(f)=\int f\,dm$ whenever $f \in B$. In the case where l is non-negative, the measure m is obtained from the geometric Choquet theorem by means of the evaluation map $v\colon X \to S^*$ (where S^* is the unit ball in the dual of B and where, by definition, $v(x)(f)=f(x)$ for any $f \in B$). In this case it is even true that m and l have the same norm. The general case follows from the non-negative case by decomposing l in the form $l=(l_1-l_2)+i(l_3-l_4)$; but it does not follow from this decomposition that the representing measure has the same norm as the functional l.

It is the aim of the present paper to prove that such a representing measure indeed exists. In outline, the idea behind the proof is as follows: Let T be the set of all complex numbers of absolute value one, and define the map

$$V\colon\ T \times X \to S^*\colon\ (t,x) \to tv(x)\,.$$

Applying the geometric Choquet theorem to $l \in S^*$ (we can assume that $\|l\|=1$), we get using V a measure q on $T \times X$. Then the measure m on X, defined by the formula

$$m(g) \;=\; \int t g(x)\, dq(t,x), \qquad g \in C(X)\,,$$

will have the required properties.

1. Terminology and statement of the theorem.

We retain the notation of the introduction. A measure m is always a Radon measure on some compact space Y, that is, a bounded linear

Received July 5, 1970.

functional on $C(Y)$ (or, if m is a real measure, on the space $C_R(Y)$ of all real continuous functions). The norm of m is denoted $\|m\|$. Observe that $\|m\| = |m|(Y)$, where $|m|$ denotes the *total variation* of m. We say that m is *quasi-supported* by a subset M of Y if $|m|(G) = 0$ whenever G is a compact G_δ-set in Y disjoint from M. If K is a convex set, then $\operatorname{ext} K$ is the set of the extreme points of K. We let $K(B)$ denote the set of all $l \in S^*$ such that $\|l\| = 1 = l(1)$. The *Choquet boundary* of B, $\partial_B X$, is then, by definition, the set $v^{-1}(\operatorname{ext} K(B))$.

In Section 3 we prove:

THEOREM. *Let X be a compact Hausdorff space, let $B \subset C(X)$ be a linear subspace which separates points and contains the constant functions. Let l be a continuous linear functional on B. Then there exists a complex measure m on X with the following properties:*

(i) *m is quasi-supported by the Choquet boundary of B.*
(ii) *The norm of m equals the norm of l.*
(iii) $\int f\,dm = l(f)$, $f \in B$.

2. Three lemmata.

We shall always assume that S^* is equipped with the weak*-topology. Hence S^* is a convex, compact set, and, since B separates points, the evaluation map $v\colon X \to S^*$ is a homeomorphism into S^*. It is an immediate consequence that also

$$V\colon\ T \times X \to S^*\colon\ (t,x) \to tv(x)$$

is a homeomorphism into S^*. (Here we have used the fact that B contains the constant functions). The main reason for introducing the map V is the fact, to be found for instance in [4, p. 441, proof of Lemma 6], that

$$\operatorname{ext} S^* \subset V(T \times X) . \tag{1}$$

In analogy with the definition of the Choquet boundary, we define

$$r(B) = V^{-1}(\operatorname{ext} S^*) . \tag{2}$$

The connection between $r(B)$ and the Choquet boundary of B is given by the following elementary

LEMMA 1.

$$r(B) = T \times \partial_B X . \tag{3}$$

PROOF. We first want to establish the following, probably well-known, relation

$$\operatorname{ext} K(B) = K(B) \cap \operatorname{ext} S^* . \tag{4}$$

Since the relation $\supset$ is clearly true, we have to show that $\operatorname{ext} K(B) \subset \operatorname{ext} S^*$.

Let $k \in \operatorname{ext} K(B)$, and assume

$$k = ra + (1-r)b, \quad a,b \in S^*, \quad 0<r<1 . \tag{5}$$

Then we get

$$1 = \|k\| \leqq r\|a\| + (1-r)\|b\| ,$$

and since $0<r<1$ and $\|a\|$, $\|b\| \leqq 1$, we can conclude that $\|a\| = \|b\| = 1$. Since $k \in K(B)$, we get from (5),

$$\begin{aligned} 1 = k(1) &= ra(1) + (1-r)b(1) \\ &\leqq r|a(1)| + (1-r)|b(1)| . \end{aligned}$$

It follows that $|a(1)| = |b(1)| = 1$, and since 1 is a convex combination of $a(1)$ and $b(1)$, we conclude that $a(1) = b(1) = 1$. Therefore $a,b \in K(B)$, and hence $a=b$. This shows that $k \in \operatorname{ext} S^*$, and (4) is thus proved. We next want to prove the relation

$$\{tp : t \in T,\ p \in \operatorname{ext} S^*\} \subset \operatorname{ext} S^* . \tag{6}$$

In fact, let $t \in T$ and $p \in \operatorname{ext} S^*$, and assume

$$tp = ra + (1-r)b, \quad a,b \in S^*, \quad 0<r<1 .$$

Since $|t| = 1$, we get that

$$p = r(t^{-1}a) + (1-r)(t^{-1}b) ,$$

where $t^{-1}a$, $t^{-1}b \in S^*$. Hence $t^{-1}a = t^{-1}b$, and this shows that $tp \in \operatorname{ext} S^*$.

We are now ready to prove (3). Assume first that $(t,x) \in T \times \partial_B X$. Then $v(x) \in \operatorname{ext} K(B)$, and it follows from (4) and (6) that $tv(x) \in \operatorname{ext} S^*$. This means that $(t,x) \in r(B)$. Assume conversely that $(t,x) \in r(B)$, or equivalently that $tv(x) \in \operatorname{ext} S^*$. It follows from (6) that $v(x) = t^{-1}tv(x) \in \operatorname{ext} S^*$. Since clearly $v(x) \in K(B)$, we get from (4) that $v(x) \in \operatorname{ext} K(B)$. This implies that $(t,x) \in T \times \partial_B X$.

As an immediate consequence we get the following

Corollary. *If A is a subset of $X \setminus \partial_B X$, then $T \times A$ is a subset of $T \times X \setminus r(B)$.*

Let $f \in C(X)$, and define

$$Lf\colon\ T \times X \to \mathbf{C}\colon\ (t,x) \to tf(x) .$$

Then Lf is continuous, and

$$\|Lf\| = \sup_{(t,x) \in T \times X} |tf(x)| = \|f\| . \tag{7}$$

It follows that the map

$$L\colon C(X) \to C(T\times X)\colon f \to L(f)$$

is linear and isometric. Consider the adjoint map

$$L^*\colon C^*(T\times X) \to C^*(X)\colon m \to L^*m = m\circ L .$$

Hence L^*m is a complex measure on X whenever m is a complex measure on $T\times X$. To be more explicit, L^*m is given by the formula

(8) $$L^*m(f) = \int_{T\times X} tf(x)\, dm(t,x), \quad f\in C(X) .$$

Applying (7) we get, for any measure m on $T\times X$

(9) $$\|L^*m\| \leqq \|m\| .$$

LEMMA 2. *Let m be a complex or real measure on $T\times X$, and let $G\subset X$ be a compact G_δ-set. Then*

(10) $$|L^*m|(G) \leqq |m|(T\times G) .$$

PROOF. Let $f\in C(X)$ and define

$$p(m)(f) = \int_{T\times X} f(x)\, d|m|(t,x) .$$

Then the map

$$p(m)\colon C(X) \to \mathbf{C}\colon f \to p(m)(f)$$

is a bounded positive linear functional on $C(X)$. This means that $p(m)$ is a positive measure on X. Notice that for any $f\in C(X)$

(11) $$|L^*m(f)| = \left|\int_{T\times X} tf(x)\, dm(t,x)\right| \leqq \int_{T\times X} |f(x)|\, d|m|(t,x) = p(m)(|f|) .$$

We now make appeal to a lemma in [3, p. 54 Lemme 5] to assert that

$$|L^*m|(|f|) = \sup\{|L^*m(hf)|\colon h\in C(X) \ \&\ \|h\|\leqq 1\} .$$

When we combine this equation with (11) we get

(12) $$|L^*m|(|f|) \leqq p(m)(|f|), \quad f\in C(X) .$$

It follows, in particular, that $p(m)-|L^*m|$ is a positive measure on X.

Let $\{G_n\}$ be a decreasing sequence of open sets in X such that $G=\bigcap_1^\infty G_n$. Choose continuous functions $f_n\colon X\to[0,1]$ such that $f_n=1$ on G and

$f_n = 0$ outside G_n. Applying the dominated convergence theorem to the positive measure $p = p(m)$, we get

$$p(m)(G) = \lim_{n\to\infty} \int f_n dp = \lim_{n\to\infty} \int f_n \circ pr_2 \, d|m| , \tag{13}$$

where pr_2 is the second projection

$$pr_2 \colon T \times X \to X \colon (t,x) \to x .$$

Observe that the sequence $\{f_n \circ pr_2\}$ converges boundedly pointwise to the characteristic function of $T \times G$. Hence we get from (13)

$$p(m)(G) = |m|(T \times G) .$$

From this equation, together with (12), we get

$$|L^* m|(G) \leqq p(m)(G) = |m|(T \times G) .$$

Thus we have proved (10).

LEMMA 3. *If m is a measure on $T \times X$ quasi-supported by $r(B)$, then $L^* m$ is quasi-supported by $\partial_B X$.*

PROOF. Let G be a compact G_δ-set in X disjoint from $\partial_B X$. It follows from the Corollary of Lemma 1 that $T \times G$ is disjoint from $r(B)$. Since $T \times G$ is a compact G_δ-set, we get from Lemma 2

$$0 \leqq |L^* m|(G) \leqq |m|(T \times G) = 0.$$

3. Proof of the theorem.

We can assume without loss of generality that the given l satisfies $\|l\| = 1$. Hence $l \in S^*$, and it follows from the geometric Choquet theorem (see e.g. [5, p. 30]) that there exists a probability measure p on S^* which vanishes on any G_δ-set disjoint from $\operatorname{ext} S^*$, and such that

$$l(u) = \int \hat{u}(g) \, dp(g), \quad u \in B , \tag{14}$$

where we have defined for any $u \in B$

$$\hat{u} \colon S^* \to \mathbf{C} \colon g \to g(u) .$$

We can even assert that

$$p(S^* \setminus V(T \times X)) = 0 , \tag{15}$$

because it follows from (1) that $\operatorname{ext} S^*$ is contained in the compact set $V(T \times X)$.

As a consequence of (15) we can and shall consider p as a measure on

$V(T \times X)$. Define the measure q on $T \times X$ as the *image* of p by V^{-1}. Hence, by definition,

$$q(f) = p(f \circ V^{-1}), \quad f \in C(T \times X) .$$

Then q is a probability measure on $T \times X$, and it is known (see for example [2, p. 75]) that a subset A of $T \times X$ is q-integrable if and only if $V(A)$ is p-integrable, and in that case

$$q(A) = p(V(A)) . \tag{16}$$

We now claim that q is quasi-supported by $r(B)$. In fact, let $G \subset T \times X$ be a compact G_δ-set disjoint from $r(B)$. Choose open sets G_n, $n = 1, 2, \ldots,$ in $T \times X$ such that $G = \bigcap_1^\infty G_n$. It follows that

$$V(G) = \bigcap_1^\infty V(G_n) ,$$

where $V(G_n)$ is open in $V(T \times X)$. Hence there exists open sets U_n in S^* such that $V(G_n) = V(T \times X) \cap U_n$. Put $U = \bigcap_1^\infty U_n$. Then U is a G_δ-set in S^* and

$$V(G) = V(T \times X) \cap U . \tag{17}$$

Since $V(G)$ is disjoint from $\operatorname{ext} S^*$, we get from (17) that U is disjoint from $\operatorname{ext} S^*$. Applying (16) and (17) we therefore get

$$0 \leqq q(G) = p(V(G)) \leqq p(U) = 0 .$$

This shows that q is quasi-supported by $r(B)$.

Put $m = L^* q$. It follows from Lemma 3 that m is quasi-supported by $\partial_B X$, and (9) shows that

$$\|m\| \leqq \|q\| = q(1) = 1 . \tag{18}$$

Let $u \in B$. Since $\hat{u} \circ V(t,x) = t u(x)$, we get from the definitions, and from (14) that

$$m(u) = L^* q(u) = \int \hat{u} \circ V \, dq = \int \hat{u} \circ V \circ V^{-1} \, dp = l(u) .$$

This means that m is equal to l on B. In particular, we get

$$1 = \|l\| \leqq \|m\| .$$

This shows, together with (18), that $\|m\| = \|l\|$. The measure m has thus all the required properties.

Remark. Let $F \subset \partial_B X$ be a compact set with the following property: If m is a measure on X orthogonal to B and quasi-supported by $\partial_B X$, then $|m|(F) = 0$.

It is then true that F is an *interpolation* set, which means that every continuous function on F can be extended to a function on X which belongs to B. This is a sharpening of a theorem of Bishop [1]. To prove this statement one has only to replace the Hahn–Banach theorem in Bishop's original proof with the theorem above.

REFERENCES

1. E. Bishop, *A general Rudin–Carleson theorem*, Proc. Amer. Math. Soc. 13 (1962), 140–143.
2. N. Bourbaki, *Intégration*, Chap. 6 (Act. Sci. Ind. 1281), Paris, 1959.
3. N. Bourbaki, *Intégration*, Chap. 5 (Act. Sci. Ind. 1244), Paris, 1956.
4. N. Dunford and J. T. Schwartz, *Linear operators*, Part I, Interscience Publ., New York, 1958.
5. R. R. Phelps, *Lectures on Choquet's theorem* (Van Nostrand Math. Studies 7), Van Nostrand, Princeton, 1966.

UNIVERSITY OF OSLO, NORWAY