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ON A THEOREM OF DIXMIER

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A well-known theorem of Alaoglu (cf. [3, p. 84]) tells us that the closed unit ball in the Banach dual space of a normed space is compact with respect to the w^*-topology. In [1], Dixmier showed that this property is characteristic for Banach dual spaces. In this note, we shall give a short proof of a variant of Dixmier's theorem. This variant appears to be more convenient for applications [2]. Our argument is inspired by Edwards' paper [2] and is strictly elementary (in particular, we do not use the Krein–Smulian theorem).

THEOREM 1. *Let $(X,\|\cdot\|)$ be a normed space with closed unit ball Σ. Suppose there exists a (Hausdorff) locally convex topology τ for X such that Σ is τ-compact. Then X itself is a Banach dual space, that is, there exists a Banach space V such that X is isometrically isomorphic to the dual space V' of V (in particular, X is complete).*

PROOF. Let $(X,\tau)'$ and $(X,\|\cdot\|)'$ denote the dual spaces of X under τ and $\|\cdot\|$ respectively. Let V be the space of all linear functionals f on X such that f is τ-continuous on Σ. Then

$$(X,\tau)' \subseteqq V \subseteqq (X,\|\cdot\|)' . \tag{1}$$

The first inequality is obvious, and to see the second, let $f \in V$. Then $f(\Sigma)$ is the continuous image of the τ-compact set Σ, so is compact and hence bounded. Therefore f is continuous on $(X,\|\cdot\|)$, and (1) is proved. Now it is easily seen that V is a closed subspace of the Banach space $(X,\|\cdot\|)'$. Thus, V may be regarded as a Banach space in its own right.

For each x in X, define $\varphi(x)$ by the rule

$$(\varphi(x))(v) = v(x), \quad v \in V .$$

Then it is easy to see that φ is a 1-1 continuous (in fact norm-reducing) map from X into the Banach dual space V' of V. Also, since each v in V is τ-continuous on Σ, the restriction $\varphi|\Sigma$ of φ to Σ is continuous with respect to the relative τ-topology and the w^*-topology $\sigma(V',V)$. Since

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Σ is τ-compact, it follows that $\varphi(\Sigma)$ is $\sigma(V', V)$-compact. Also, this set $\varphi(\Sigma)$ is convex. By the bipolar theorem (cf. [3, 126]), it is precisely its bipolar $[\varphi(\Sigma)]^{\pi\pi}$ with respect to the duality (V', V). Note that

$$[\varphi(\Sigma)]^{\pi} = \{v \in V : (\varphi(x))(v) \leqq 1, \forall x \in \Sigma\} = \{v \in V : v(x) \leqq 1, \forall x \in \Sigma\},$$

which is just the unit ball in V, and hence $[\varphi(\Sigma)]^{\pi\pi}$ (that is, $\varphi(\Sigma)$) is the unit ball in V'. In other words, φ maps Σ onto the unit ball in V'. Therefore φ is an isometry and onto the space V'. The proof of theorem 1 is thus completed.

This theorem implies immediately the theorem of Dixmier referred to at the beginning:

THEOREM 2. *Let $(X, \|\cdot\|)$ be a Banach space with closed unit ball Σ. Suppose there exists a total subspace V of $(X, \|\cdot\|)'$ such that Σ is $\sigma(X, V)$-compact. Then X itself is a Banach dual space.*

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