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A LARGE BI-INVARIANT NUCLEAR FUNCTION SPACE ON A LOCALLY COMPACT GROUP

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1. Introduction.

In a fundamental work on representations of p-adic groups [1], F. Bruhat introduces a space $D(G)$ of "differentiable" functions on a general locally compact group G. This construction is based on Yamabe's approximation theorem which says that a connected locally compact group is the projective limit of Lie groups. If G is separable, Bruhat shows in [1] that $D(G)$ can be made into a nuclear complete LF-space, invariant with respect to translations and dense in the space $L(G)$ of all continuous complex functions on G with compact support.

It is natural to ask whether a space of functions on G with the properties of $D(G)$ may be constructed without the intervention of Lie-groups. There have been attempts in this direction by several authors, but it seems that not until recently has anyone obtained a space with the right properties which is also *nuclear*. This last requirement is essential for representation theory, as shown in [1]. However, in [5], T. Pytlik gives a very elegant construction of a nuclear space Φ of functions on a locally compact group G which is assumed to be metrizable and σ-compact. Now this space Φ, while being left-invariant, need not be right invariant, and it may also be very small. In this paper we will show that it is possible to use Pytlik's construction as a basis for a construction-process that eventually leads to a bi-invariant nuclear LF-space Y, which is dense in $L(G)$. We also show that the left (and right) regular representation of G on Y is continuous.

We shall assume that the group G is second countable. Integration on G will always be with respect to a fixed left Haar measure, which we denote by μ. By Δ we denote the modular function on G, and e is the identity element in G. If f is a function on G, then $\tilde{f}(x)=\overline{f(x^{-1})}$, $x \in G$. If C is a subset of G, and if $F(G)$ is a class of functions on G, then $F_C(G)$ denotes the class of functions in $F(G)$ which vanish outside C. If f is a function on G, and $x,y \in G$, then $f_x(y)=f(xy)$, $f^x(y)=f(yx)$.

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Standard results from abstract harmonic analysis will be used without explicit quotation. The general reference is [3].

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2. Construction of the spaces E^j, F^j.

Let G be a locally compact group, satisfying the second axiom of countability. Once and for all we choose a basis of relatively compact symmetric neighbourhoods $\{U_n\}_{n=0}^{\infty}$ of the identity element e in G, such that

$$\mu(U_0) \leqq 1 \quad \text{and} \quad U_{n+1}^2 \subseteqq U_n ,$$

$n=0,1,\ldots$. A sequence of functions $\{\psi_k\} \subseteqq L(G)$ is an approximative identity on G if

$$\psi_k \geqq 0, \quad \int \psi_k = 1 \text{ for all } k, \quad \text{and} \quad \operatorname{supp}(\psi_k) \downarrow \{e\} .$$

An approximate identity $\{\psi_k\}$ is *subordinate* the basis $\{U_n\}$ if $\operatorname{supp}(\psi_k) \subseteqq U_k$; $k=1,2,\ldots$. In [5] an approximative identity $\{\psi_n\}$, subordinate to a basis $\{U_n\}$, with the following additional properties, is constructed: $\psi_n = \tilde{\psi}_n$ for all n, and if $f \in L^2(G)$ satisfies $f * \psi_n = 0$ for some n, then $f=0$. Let C be a relatively compact subset of G.

Lemma 2.1. *If $f \in L_C{}^2(G)$ and $\psi_n * f = 0$ for some n, then $f=0$.*

Proof. Suppose $f \in L_C{}^2(G)$ and $\psi_n * f = 0$. Then $0 = (\psi_n * f)^{\sim} = \tilde{f} * \psi_n$. Since f has compact support, $\tilde{f} \in L^2(G)$. Hence $\tilde{f} = 0$, so $f=0$.

For any function f on G, let $\check{f}(x) = f(x^{-1})$. The map $f \to \check{f}$ is linear and $(f*g)^{\vee} = \check{g} * \check{f}$ whenever convolution is defined. Let C be a relatively compact subset of G. We identify $L^2(C)$ and $L_C{}^2(G)$. If $f \in L^2(C)$, then

$$\|\check{f}\|_2 \leqq \sup_{u \in C} \Delta(u^{-1}) \, \|f\|_2 ,$$

so $^{\vee}$ is a bicontinuous map of $L^2(C)$ onto $L^2(C^{-1})$. In [5] the linear maps

$$T_n : L^2(G) \to L^2(G), \quad n=1,2,\ldots ,$$

are defined by $T_n f = f * \psi_n$. Now define

$$S_n : L^2(G) \to L^2(G), \quad n=1,2,\ldots ,$$

by $S_n f = \psi_n * f$.

Lemma 2.2. *Let V be a relatively compact subset of G. Then*

(a) $T_n : L^2(VU_n) \to L^2(VU_{n-1})$,

(b) $S_n : L^2(U_n V) \to L^2(U_{n-1} V)$

are Hilbert–Schmidt.

PROOF. (a) is proved in [5], lemma 2. That S_n maps $L^2(U_n V)$ into $L^2(U_{n-1}V)$ is clear since ψ_n has support in U_n, and $U_n^2 \subseteqq U_{n-1}$. If $f \in L^2(U_n V)$, then

$$S_n f = \psi_n * f = (T_n f)^\vee .$$

Since $^\vee$ is continuous, (b) follows from (a).

We observe that because of lemma 2.1 and the preceding remark, the linear operators T_n and S_n are injective for all n. Clearly

$$T_{j+1} \ldots T_n(L^2(VU_n)) \subseteqq L^2(VU_j)$$

and

$$S_{j+1} \ldots S_n(L^2(U_n V)) \subseteqq L^2(U_j V) .$$

For $j = 0, 1, \ldots$ we define for $n > j$

$$E^j_{V,n} = T_{j+1} \ldots T_n(L^2(VU_n)) ,$$
$$F^j_{V,n} = S_{j+1} \ldots S_n(L^2(U_n V)) .$$

The elements of $E^j_{V,n}$ (resp. $F^j_{V,n}$) belong to $L(G)$ and have their support in VU_j (resp. $U_j V$). We define, for $j = 0, 1, \ldots$,

$$E^j_V = \bigcap_{n>j} E^j_{V,n}, \quad F^j_V = \bigcap_{n>j} F^j_{V,n} .$$

Let L and R denote the left and right regular representations of G on $L^2(G)$, respectively. I.e., for $f \in L^2(G)$

$$(L_x f)(y) = f(x^{-1}y), \quad x, y \in G ,$$
$$(R_x f)(y) = f(yx), \quad x, y \in G .$$

We observe that L_x and T_n (resp. R_x and S_n) commute for all x and n. Indeed:

$$R_x S_n f = (S_n f)^x = (\psi_n * f)^x = \psi_n * f^x = \psi_n * (R_x f) = S_n R_x f ,$$

where $f \in L^2(G)$. A similar argument works for L_x and T_n.

LEMMA 2.3. *If $V_1 \subseteqq V_2$ are relatively compact subsets of G, then*

(a) *$E^j_{V_1} \subseteqq E^j_{V_2}$ and $F^j_{V_1} \subseteqq F^j_{V_2}$ for all $j \geqq 0$.*

If $x \in G$, then

(b) *$E^j_{xV} = L_x(E^j_V)$ and $F^j_{Vx^{-1}} = R_x(F^j_V)$ for all $j \geqq 0$.*

PROOF. This is a straight forward verification, and the proof is omitted.

LEMMA 2.4. *For any relatively compact set $V \subseteqq G$ and $j \geqq 0$ we have $f \in E^j_V$ if and only if $\tilde{f} \in F^j_{V^{-1}}$.*

PROOF. Suppose $f \in E^j_V$ and let $n > j$ be arbitrary. There is $h \in L^2(VU_n)$ such that

$$f = T_{j+1} \ldots T_n h = h * \psi_n * \ldots * \psi_{j+1} .$$

Hence $\tilde{f} = \psi_{j+1} * \ldots * \psi_n * \tilde{h}$, and $\tilde{h} \in L^2(U_n V^{-1})$ since U_n is symmetric. Consequently $\tilde{f} = S_{j+1} \ldots S_n \tilde{h}$ belongs to $F^j_{V^{-1},n}$. Since this is true for all $n > j$ it follows that $\tilde{f} \in F^j_{V^{-1}}$. By symmetry and the fact that $f^{\sim\sim} = f$ the lemma follows.

Next we want to show that if $V \neq \emptyset$, then all the spaces E^j_V, F^j_V, $j > 0$, are non-zero. In [5] this is done for the space E^0_V. Since we are dealing with several spaces at the same time, and also want to obtain some extra information, we give a complete proof, based on the argument of [5].

PROPOSITION 2.5. *There is a sequence* $\{\varphi_j\}_{j \geqq 0}$ *such that* $\varphi_j \in E^j_{\{e\}}$, $\varphi_j \geqq 0$, $\operatorname{supp}(\varphi_j) \subseteqq U_j$ *and* $\int \varphi_j = 1$. *Moreover, if* $i > j \geqq 0$, *then* $\varphi_j = T_{j+1} \ldots T_i \varphi_i$.

PROOF. We define the double sequence

$$\varphi_{j,k} = T_{j+1} \ldots T_k \psi_{k+1} ,$$

$j = 0, 1, \ldots,\ k = j+1, j+2, \ldots$. We obtain

$$\begin{aligned} \|\varphi_{j,k}\|_\infty &= \sup_{x \in G} |T_{j+1} \ldots T_k \psi_{k+1}(x)| \\ &= \sup_{x \in G} \left| \int T_{j+2} \ldots T_k \psi_{k+1}(y) \psi_{j+1}(y^{-1}x)\, dy \right| \\ &\leqq \|\psi_{j+1}\|_\infty \int T_{j+2} \ldots T_k \psi_{k+1}(y)\, dy = \|\psi_{j+1}\|_\infty , \end{aligned}$$

where we have used the properties of the functions ψ_i and the fact that the Haar-integral is a multiplicative linear functional on $L(G)$. Now $\operatorname{supp}(\varphi_{j,k}) \subseteqq U_j$ for all $k > j$, so

$$\|\varphi_{j,k}\|_2 \leqq \|\psi_{j+1}\|_\infty\, \mu(U_j)^{\frac{1}{2}} .$$

Hence, for each $j \geqq 0$, the sequence $\{\varphi_{j,k}\}_{k>j}$ is bounded in $L^2(U_j)$. We observe that

$$(*) \qquad \varphi_{j,k} = T_{j+1} \varphi_{j+1,k}$$

for all $j \geqq 0$ and $k \geqq j+2$. Each of the operators T_j is Hilbert–Schmidt, hence compact. So in particular the sequence $\varphi_{0,k} = T_1 \varphi_{1,k}$, $k = 2, 3, \ldots$, contains a norm-convergent sub-sequence $\{\varphi_{0,k'}\}$ in $L^2(U_0)$. Let $\varphi_0 = \lim_{k'} \varphi_{0,k'}$. For any fixed $j \geqq 1$ let $\{\varphi_{j,k''}\}_{k''}$ be a convergent subsequence of the sequence $\varphi^j_{k'} = T_{j+1} \varphi_{j+1,k'}$, $k' \geqq j+2$. We put $\varphi_j = \lim_{k''} \varphi_{j,k''}$. We now assert that $\varphi_j \in E^j_{\{e\}}$ and that $\varphi_j = T_{j+1} \varphi_{j+1}$ for all $j \geqq 0$. Indeed, using (*) one obtains

$$(**)\qquad \begin{aligned} T_1 \dots T_j \varphi_j &= \lim_{k''} T_1 \dots T_j \varphi_{j,k''} \\ &= \lim_{k''} \varphi_{0,k''} = \varphi_0 \end{aligned}$$

Since $\varphi_j \in L^2(U_j)$ it follows that $\varphi_0 \in E^0_{\{e\},j}$ for all $j \geqq 1$, hence $\varphi_0 \in E^0_{\{e\}}$. Now all T_j are invertible so (**) implies that $T_{j+1} \dots T_i \varphi_i = \varphi_j$ for all $i > j$. Hence $\varphi_j \in E^j_{\{e\}}$ for all $j \geqq 0$.

By construction $\varphi_{j,k} \geqq 0$ for all j,k, and since $\varphi_{j,k''} \to \varphi_j$ in $L^2(U_j)$, a subsequence will converge pointwise almost everywhere. Hence, by continuity $\varphi_j \geqq 0$. Next, $\mu(U_j) < \infty$ so convergence in $L^2(U_j)$ implies convergence in $L^1(U_j)$, so that

$$\int \varphi_j = \lim_{k''} \int \varphi_{j,k''} .$$

But

$$\int \varphi_{j,k} = \int T_{j+1} \dots T_k \psi_{k+1} = \int \psi_{k+1} = 1 .$$

It follows that $\int \varphi_j = 1$ for all $j \geqq 0$. The proof is complete.

By lemma 2.4 and the result above it immediately follows that $F^j_{\{e\}}$ is non-zero for each $j \geqq 0$. In fact $\tilde{\varphi}_j \in F^j_{\{e\}}$, and the sequence $\{\tilde{\varphi}_j\}$ has exactly the same properties as the sequence $\{\varphi_j\}$. The only thing which perhaps isn't obvious is that $\int \tilde{\varphi}_j = 1$. But clearly

$$\int \tilde{\varphi}_j = \lim_{k''} \int \tilde{\varphi}_{j,k''}, \quad \int \tilde{\varphi}_{j,k} = \int \psi_{j+1} * \dots * \psi_{k+1} = 1 ,$$

so this is also true.

Corollary 2.6. *If $V \neq \emptyset$, then E^j_V and F^j_V are non-zero for all $j \geqq 0$.*

Proof. Immediate from lemma 2.3, prop. 2.5 and the preceding remark.

To topologize the spaces E^j_V and F^j_V we introduce the maps

$$\tau_{j,n} = (T_{j+1} \dots T_n)^{-1}, \quad \sigma_{j,n} = (S_{j+1} \dots S_n)^{-1}$$

for $0 \leqq j < n$. Then $\tau_{j,n}$ (resp. $\sigma_{j,n}$) is a bijection of $E^j_{V,n}$ (resp. $F^j_{V,n}$) onto $L^2(VU_n)$ (resp. $L^2(U_n V)$). We define families of norms on $E^j_{V,n}$ and $F^j_{V,n}$ by:

$$\begin{aligned} \pi^j_{V,n}(f) &= \|\tau_{j,n}(f)\|_2, \quad f \in E^j_V , \\ \varkappa^j_{V,n}(f) &= \|\sigma_{j,n}(f)\|_2, \quad f \in F^j_V , \end{aligned}$$

for $n > j$. We give E^j_V (resp. F^j_V) the locally convex topology determined by the family of norms $\{\pi^j_{V,n}\}_{n>j}$ (resp. $\{\varkappa_{V,n}\}_{n>j}$).

Lemma 2.7. *The map $f \to \tilde{f}$ is an anti-linear homeomorphism of E^j_V onto $F^j_{V^{-1}}$ ($j \geqq 0$).*

PROOF. The algebraic property is clear from lemma 2.4. Let $f \in E_V^j$ and let $n > j$ be given. Put $h = \tau_{j,n}(f)$ so $h \in L^2(VU_n)$. Then $\tilde{h} \in L^2(U_n V^{-1})$ and $\sigma_{j,n}(f) = \tilde{h}$. Hence

$$\varkappa_{V^{-1},n}^j(\tilde{f})^2 = \|\tilde{h}\|_2^2 = \int_G |h(u^{-1})|^2 \, du = \int_{VU_n} |h(u)|^2 \Delta(u^{-1}) \, du .$$

Let $m = \inf_{u \in VU_n} \Delta(u^{-1})$, $M = \sup_{u \in VU_n} \Delta(u^{-1})$. Then

$$m \|h\|_2^2 \leqq \|\tilde{h}\|_2^2 \leqq M \|h\|_2^2 ,$$

so

$$m^{\frac{1}{2}} \pi_{V,n}^j(f) \leqq \varkappa_{V^{-1},n}^j(\tilde{f}) \leqq M^{\frac{1}{2}} \pi_{V,n}^j(f) .$$

The proof is complete.

PROPOSITION 2.8. *For each non-void relatively compact set $V \subseteqq G$, and each $j \geqq 0$, the space E_V^j (resp. F_V^j) is a locally convex nuclear Frechet space. The linear map $\tau_{j,n}$ (resp. $\sigma_{j,n}$) is a homeomorphism of E_V^j onto E_V^n (resp. F_V^j onto F_V^n), when $n > j$.*

PROOF. In [5], lemma 4 it is proved that E_V^0 $(= \Phi_V)$ is nuclear. E_V^0 is a countably normed space, hence metrizable. To see that it is complete, let $\{f_k\}$ be a Cauchy-sequence in E_V^0. Then $\{f_k\}$ is Cauchy with respect to each of the norms $\pi_{V,n}^0$ for each $n \geqq 1$, and $E_{V,n}^0$ is a Hilbert-space, in particular complete, in the norm $\pi_{V,n}^0$. Hence, for $n \geqq 1$ there is a function $h^n \in E_{V,n}^0$ such that $\pi_{V,n}^0(h^n - f_k) \to 0$ as $k \to \infty$. Suppose $n > m$ and let $i_{m,n}$ be the inclusion map of $E_{V,n}^0$ into $E_{V,m}^0$. Then

$$i_{m,n} = \tau_{0,m}^{-1} \circ (T_{m+1} \ldots T_n) \circ \tau_{0,n} ;$$

$\tau_{0,m}$ and $\tau_{0,n}$ are isometries and

$$T_{m+1} \ldots T_n : L^2(VU_n) \to L^2(VU_m)$$

is continuous by lemma 2.2. Hence $i_{m,n} : E_{V,n}^0 \to E_{V,m}^0$ is continuous. Since $f_k \to h^n$ in $E_{V,n}^0$ it follows that $f_k \to h^n$ also in $E_{V,m}^0$. But then $h^n = h^m$. Since n and m were arbitrary, we obtain

$$h^1 = h^2 = \ldots h^n = \ldots = h$$

with $h \in E_{V,n}^0$ for all $n \geqq 1$. Hence $h \in E_V^0$ and $f_k \to h$ in E_V^0. This shows that E_V^0 is complete, and therefore a nuclear Frechet space.

Next, we show that $\tau_{j,n}$ is a linear homeomorphism of E_V^j onto E_V^n. (The argument for $\sigma_{j,n}$ is similar and is omitted.) It is routine to check that $\tau_{j,n}(E_V^j) = E_V^n$. Now let $f \in E_V^j$; and let $0 \leqq j < n < k$. Then

$$\pi_{V,k}^n(\tau_{j,n}(f)) = \|\tau_{n,k}(\tau_{j,n}(f))\|_2 = \|\tau_{j,k}(f)\|_2 = \pi_{V,k}^j(f) .$$

Hence $\tau_{j,n}$ is continuous as this equality holds for all $k > n$. It also shows that $\tau_{j,n}^{-1}$ is an isometry of $E_{V,k}^{n}$ onto $E_{V,k}^{j}$ for $k > n$. If $n \geqq k > j$ we may regard $\tau_{j,n}^{-1}$ as the isometry $T_{j+1} \dots T_n$ of $E_{V,n+1}^{n}$ onto $E_{V,n+1}^{j}$ followed by the injection of $E_{V,n+1}^{j}$ into $E_{V,k}^{j}$. The latter is continuous by an argument similar to the one used to show that $i_{n,m}$ is continuous. It follows that $\tau_{j,n}$ is a homeomorphism. This fact and lemma 2.7 now yields the proposition.

Let $\mathscr{C}$ be the family of relatively compact subsets of G. We define

$$E^j = \bigcup_{V \in \mathscr{C}} E_V^j, \quad F^j = \bigcup_{V \in \mathscr{C}} F_V^j$$

for all $j \geqq 0$. If $V_1 \subseteqq V_2$ with $V_1, V_2 \in \mathscr{C}$, the injection of $E_{V_1}^j$ into $E_{V_2}^j$ is easily seen to be a homeomorphism. Hence, with the inductive topology, E^j is the inductive limit of the spaces E_V^j; $V \in \mathscr{C}$. Similarly F^j is the inductive limit of the F_V^j's.

Since G is second countable, there is a sequence $\{V_n\}$ of open relatively compact sets, $V_n \subseteqq V_{n+1}$, such that

$$G = \bigcup_{n=1}^{\infty} V_n .$$

LEMMA 2.9. *For each* $j \geqq 0$, E^j *(resp.* F^j*) is the strict inductive limit of the sequence* $\{E_{V_n}^j\}$ *(resp.* $\{F_{V_n}^j\}$*).*

PROOF. Let $G^j = \lim\operatorname{ind}\{E_{V_n}^j\}$. Clearly $G^j = E^j$ as linear spaces. G^j is a strict inductive limit since each $E_{V_n}^j$ is complete and the injection $E_{V_n}^j \to E_{V_{n+1}}^j$ is a homeomorphism. It is clear that the topology of E^j is weaker than the topology of G^j. Conversely, let W be any convex, circled neighborhood of 0 in G^j. Let V be any relatively compact subset of G. There is $V_n \supseteqq V$, hence $E_V \subseteqq E_{V_n}$ and the injection is continuous. So $W \cap E_V$ is a neighborhood of 0 in E_V. Hence W is a neighborhood of 0 in E^j. A similar argument works for F^j. The proof is complete.

We say that a linear space G is an LF-space (resp. strict LF-space) if G is the inductive limit (resp. strict inductive limit) of a sequence

$$G_1 \subseteqq \dots \subseteqq G_n \subseteqq G_{n+1} \subseteqq \dots$$

of Frechet-spaces.

COROLLARY 2.10. *For each* $j \geqq 0$, E^j *and* F^j *are nuclear strict* LF-*spaces.*

In particular it is clear that E_V^j (resp. F_V^j) for each $V \in \mathscr{C}$ carries the relative topology of E^j (resp. F^j).

We may also note that all E^j, F^j are contained in $L(G)$, in fact, if $f \in E_V^j$ say, then $\operatorname{supp}(f) \subseteqq \overline{VU_j}$. By prop. 2, the spaces E^j, and spaces

F^j, are all linearly homeomorphic. By lemma 2.7 $f \to \tilde{f}$ is an anti-linear homeomorphism of E^j onto F^j.

It will be convenient to express the topology on E^j and F^j somewhat differently. For $V \in \mathscr{C}$; $j \geqq 0$ and $n \geqq j$, let

$$p^j_{V,n}(f) = \|\tau_{jn}(f)\|_\infty ; \quad f \in E^j_{V,n} ,$$

where $\tau_{jj}(f) = f$, and let

$$q^j_{V,n}(f) = \|\sigma_{jn}(f)\|_\infty ; \quad f \in F^j_{V,n} ,$$

where $\sigma_{jj}(f) = f$. I claim that the system of norms $\{p^j_{V,n}\}_n$ (resp. $\{q^j_{V,n}\}_n$) determine the topology of E^j_V (resp. F^j_V). Indeed; we have $\tau_{jn}(f) \in L(G)$ and $\operatorname{supp}(\tau_{jn}(f)) \subseteqq \overline{VU_0}$ for all $n > j$, $f \in E^j_V$. Hence

$$\|\tau_{jn}(f)\|_2^2 = \int_{VU_0} |\tau_{jn}(f)(x)|^2 \, dx \leqq \|\tau_{jn}(f)\|_\infty^2 \mu(VU_0) ,$$

so

$$\pi^j_{V,n}(f) \leqq \mu(VU_0)^{\frac{1}{2}} \, p^j_{V,n}(f) .$$

On the other hand, for $n \geqq j$;

$$\begin{aligned} p^j_{V,n}(f) = \|\tau_{jn}(f)\|_\infty &= \|\tau_{j,n+1}(f) * \psi_{n+1}\|_\infty \\ &\leqq \|\tau_{j,n+1}(f)\|_2 \, \|\psi_{n+1}\|_\infty \, \mu(U_{n+1})^{\frac{1}{2}} \\ &\leqq \pi^j_{V,n+1}(f) \, \|\psi_{n+1}\|_\infty . \end{aligned}$$

So the two norm-systems $\{p^j_{V,n}\}_n$ and $\{\pi^j_{V,n}\}_n$ are equivalent. Analogously, the family $\{q^j_{V,n}\}_n$ will determine the topology of F^j_V. On account of this fact, we will refer to the topology of E^j (resp. F^j) as the inductive topology of uniform convergence with respect to the operators τ_{jn} (resp. σ_{jn}); $n \geqq j$.

LEMMA 2.11. *Let $h \in L_C^1(G)$ with* $\operatorname{supp}(h) \subseteqq K$. *In this case:*

(a) *if $f \in E^j_V$, then $h * f \in E^j_{KV}$ and*

$$p^j_{KV,n}(h * f) \leqq p^j_{V,n}(f) \, \|h\|_1, \quad n \geqq j .$$

(b) *if $g \in F^j_V$, then $g * h \in F^j_{VK}$ and*

$$q^j_{VK,n}(g * h) \leqq q^j_{V,n}(g) \, \|\Delta^{-1} h\|_1, \quad n \geqq j .$$

PROOF. We prove (a), the proof of (b) is similar. First, let $n > j$, and observe that $\tau_{jn}(f)$ is continuous with support in VU_n. Hence $h * \tau_{jn}(f)$ belongs to $L^2(KVU_n)$, and

$$T_{j+1} \cdots T_n(h * \tau_{jn}(f)) = T_{j+1} \cdots T_n(\tau_{jn}(h * f)) = h * f ,$$

so $h*f \in E^j_{KV,n}$ for all $n>j$. Hence $h*f \in E^j_{KV}$. Next for $n \geqq j$,

$$\|\tau_{jn}(h*f)\|_\infty = \|h*\tau_{jn}(f)\|_\infty \leqq \|\tau_{jn}(f)\|_\infty \|h\|_1 ,$$

from which (a) follows.

LEMMA 2.12. *For* $f \in E^j$ *(resp.* $g \in F^j$*);* $j \geqq 0$*, the map* $x \to f_x$ *(resp.* $x \to g^x$*) is continuous of* G *into* E^j *(resp.* F^j*).*

PROOF. In [5], theorem 2, it is proved that $x \to f_x$ is continuous for $f \in E^0$ $(=\Phi)$. (That $f_x \in E^0$ for all $x \in G$ follows from lemma 2.3.) The same argument works for all E^j and F^j.

LEMMA 2.13. *Let* $f \in E^j_V$, $g \in F^j_V$. *Then*

(a) $p^j_{xV,n}(L_x f) = p^j_{V,n}(f), \quad n \geqq j$

(b) $q^j_{Vx^{-1},n}(R_x g) = q^j_{V,n}(g), \quad n \geqq j$

PROOF. We prove (a), the proof of (b) is similar. By lemma 2.3, $L_x f \in E^j_{xV}$ and hence

$$p^j_{xV,n}(L_x f) = \|\tau_{jn}(L_x f)\|_\infty = \|L_x \tau_{jn}(f)\|_\infty = \|\tau_{jn}(f)\|_\infty = p^j_{V,n}(f) .$$

3. Construction of a large biinvariant nuclear space.

Take E^j and F^j as defined in section 2. Let Y^j be the linear subspace of $L(G)$ consisting of elements with representations

$$h = \sum_{i=1}^{\infty} \lambda_i f_i * g_i , \tag{S}$$

where $\{\lambda_i\}$, $\{f_i\}$ and $\{g_i\}$ are sequences in $\mathbf{C}$, E^j and F^j respectively, such that $\sum_{i=1}^\infty |\lambda_i| < 1$, $f_i \to 0$ in E^j, and $g_i \to 0$ in F^j, and where the series (S) converges in $L(G)$. We give $L(G)$ the inductive topology of uniform convergence on compacta.

We are going to show that the linear span Y of $\bigcup_{j=1}^\infty Y^j$ is dense in $L(G)$, is invariant with respect to left and right translations, and can be provided with a nuclear topology finer than the topology of $L(G)$ in such way that it becomes an LF-space.

We shall use the following notation: If E and F are locally convex linear spaces, then $E \otimes_\varepsilon F$, $E \otimes_\pi F$ and $E \otimes_\iota F$ will denote their tensor product equipped with the topology of bi-equicontinuous convergence: ε, the projective topology π, or the inductive topology ι, respectively. The completion of $E \otimes F$ in the ε, π and ι-topology, will be denoted by $E \tilde{\otimes} F$, $E \hat{\otimes} F$ and $E \overline{\otimes} F$, respectively. (For general facts about tensor-products and nuclear spaces, we refer to [2] and [6].)

We shall need the following general result:

PROPOSITION 3.1. *Let E and F be strict inductive limits of increasing sequences $\{E_i\}$, $\{F_i\}$ respectively, such that for each i, E_i and F_i are nuclear Frechet-spaces. Then $E\overline{\otimes}F$ is strict inductive limit of the sequence $\{E_i\hat{\otimes}F_i\}$. In particular $E\overline{\otimes}F$ is a nuclear, strict* LF-*space.*

PROOF. Since $E_i \subseteq E$, $F_i \subseteq F$ for all i, we may regard $E_i \otimes F_i$ as a linear subspace of $E \otimes F$ by the canonical injection. Clearly $E_i \otimes F_i \subseteq E_{i+1} \otimes F_{i+1}$, $i=1,2,\ldots$. By prop. 14 in [2] it follows that $E\otimes_\iota F$ is the inductive limit of the sequence $\{E_i \otimes_{\iota_i} F_i\}$, where ι_i denotes the inductive tensor product topology on $E_i \otimes F_i$. Hence ι restricted to $E_i \otimes F_i$ is weaker than ι_i. On the other hand ι is stronger than ε, so the restriction of ι to $E_i \otimes F_i$ is stronger than the restriction of ε to $E_i \otimes F_i$. But the latter topology on $E_i \otimes F_i$ concides with the ε-topology of $E_i \otimes F_i$ itself [6, Prop. 43.7]. Now E_i is nuclear, hence $E_i \otimes_\varepsilon F_i$ is isomorphic to $E_i \otimes_\pi F_i$ [6, thm. 50.1]. Since E_i and F_i are Frechet-spaces, we also obtain that the projective and inductive tensor product topologies coincide. So $E_i \otimes_\varepsilon F_i$ is isomorphic to $E_i \otimes_{\iota_i} F_i$. Hence we also obtain that the restriction of ι to $E_i \otimes F_i$ is stronger than ι_i. Consequently $E_i \otimes_{\iota_i} F_i$ carries the relativized ι-topology. Hence $E_i \overline{\otimes} F_i$ is simply the closure of $E_i \otimes F_i$ in $E\overline{\otimes}F$. By what's been said above:

$$(*) \qquad E_i \tilde{\otimes} F_i = E_i \hat{\otimes} F_i = E_i \overline{\otimes} F_i \,,$$

that is, these spaces are topologically and linearly isomorphic. By the first equality, the canonical injection

$$E_i \hat{\otimes} F_i \to E_{i+1} \hat{\otimes} F_{i+1}$$

is a topological isomorphism into [6, prop. 43.7]. So

$$G = \lim \operatorname{ind} \{E_i \hat{\otimes} F_i\}$$

is a strict inductive limit of Frechet-spaces, hence a strict LF-space, hence complete. By the second equality in (*)

$$G = \lim \operatorname{ind} \{E_i \overline{\otimes} F_i\} \,.$$

By the last part of prop. 14 in [2] we therefore know that the inductive limit topology of G coincides with the relativized ι-topology of $E\overline{\otimes}F$. But G is dense in $E\overline{\otimes}F$, and closed since it is complete. Hence $G = E\overline{\otimes}F$. Now $E_i \hat{\otimes} F_i$ is nuclear for each i [6, prop. 50.1], and countable inductive limits of nuclear spaces are nuclear [6, prop. 50.1], so $E\overline{\otimes}F$ is nuclear. The proof is complete.

Now let E^j, F^j, $j \geqq 0$, be as in section 2. For each $j \geqq 0$ we shall equip $E^j \otimes F^j$ with the inductive tensor product topology. Let us choose, once and for all, a sequence $\{V_n\}$ of open relatively compact subsets of G such that $V_n \subseteq V_{n+1}$ for all n and $\bigcup_{n=1}^{\infty} V_n = G$. Let $K_n = \overline{V}_n$ for all n. Then $L(G)$ is the strict inductive limit of Banach-spaces $L_{K_n}(G)$, and is therefore complete.

For each $j \geqq 0$, $(f,g) \to f*g$ is a bilinear map of $E^j \times F^j$ into $L(G)$. It therefore determines a unique linear map

$$A^j : E^j \otimes F^j \to L(G)$$

such that

$$A^j(\textstyle\sum_{i=1}^{k} f_i \otimes g_i) = \sum_{i=1}^{k} f_i * g_i ,$$

where $f_i \in E^j$, $g_i \in F^j$, $i = 1,\ldots,k$.

LEMMA 3.2. *For each* $j \geqq 0$,

$$A^j : E^j \otimes_\iota F^j \to L(G)$$

is continuous.

PROOF. To simplify notation, let us write $E_n^j = E_{V_n}^j$, $F_n^j = F_{V_n}^j$. By lemma 2.9 we know that E^j (resp. F^j) is the strict inductive limit of the sequence $\{E_n^j\}$ (resp. $\{F_n^j\}$). Since all E_n^j, F_n^j are nuclear Frechet-spaces (prop. 2.8) the assumptions of prop. 3.1 are satisfied. As observed in the proof, we then have

$$E^j \otimes_\iota F^j = \lim \operatorname{ind}\{E_n^j \otimes_\pi F_n^j\} .$$

Hence, to prove continuity of A^j we need only show that the restriction A_n^j of A^j to $E_n^j \otimes_\pi F_n^j$ is continuous into $L(G)$ for each n. Let z be an element of $E_n^j \otimes F_n^j$. Then we have an expression $z = \sum_{i=1}^{k} f_i \otimes g_i$, with $f_i \in E_n^j$, $g_i \in F_n^j$, $i = 1,\ldots,k$. Then

$$h = A_n^j z = \textstyle\sum_{i=1}^{k} f_i * g_i$$

has support in the closure of $V_n U_0^2 V_n$ for any $j \geqq 0$, $n \geqq 1$, and

$$\begin{aligned} \|h\|_\infty = \|\textstyle\sum_{i=1}^{k} f_i * g_i\|_\infty &\leqq \textstyle\sum_{i=1}^{k} \|f_i * g_i\|_\infty \\ &\leqq \mu(V_n U_0) \textstyle\sum_{i=1}^{k} \|f_i\|_\infty \|g_i\|_\infty \\ &= \mu(V_n U_0) \textstyle\sum_{i=1}^{k} p_{V_n,j}^j(f_i)\, q_{V_n,j}^j(g_i) . \end{aligned}$$

Hence

$$\|h\|_\infty \leqq \mu(V_n U_0)(p_{V_n,j}^j \otimes q_{V_n,j}^j)(z)$$

which proves the assertion.

Since A^j is continuous and $L(G)$ is complete, we may extend A^j to a continuous linear operator of $E^j \overline{\otimes} F^j$ into $L(G)$. We denote this extension also by A^j, and its restriction to $E_n^j \hat{\otimes} F_n^j$ by A_n^j.

It will be convenient to characterize the elements of Y^j in an other way. We first make some observations. Let E, F be two Frechet-spaces. Then every element z of $E\hat{\otimes}F$ is the sum of an absolutely convergent series

$$z = \sum_{i=1}^{\infty} \lambda_i x_i \otimes y_i ,$$

where $\{\lambda_i\}$ is a sequence of complex numbers such that $\sum_{i=1}^{\infty} |\lambda_i| < 1$, and $\{x_i\}$ (resp. $\{y_i\}$) is a sequence converging to zero in E (resp. F). [6, thm. 45.1]. Conversely, let $\{\lambda_i\}$, $\{x_i\}$ and $\{y_i\}$ be sequences in C, E and F respectively, such that $\sum_{i=1}^{\infty} |\lambda_i| < 1$, $x_i \to 0$ in E and $y_i \to 0$ in F. We claim that the series $\sum_{i=1}^{\infty} \lambda_i x_i \otimes y_i$ then will converge absolutely in $E\hat{\otimes}F$ to an element z. Let p,q be arbitrary continuous semi-norms on E, F respectively, and put $r = p\otimes q$. Since $E\hat{\otimes}F$ is complete, it suffices to show that

$$\sum_{i=1}^{\infty} r(\lambda_i x_i \otimes y_i) < \infty .$$

But

$$\sum_{i=1}^{\infty} r(\lambda_i x_i \otimes y_i) = \sum_{i=1}^{\infty} |\lambda_i| p(x_i) q(x_i) .$$

Since $p(x_i) \to 0$ and $q(x_i) \to 0$ the latter series converges.

PROPOSITION 3.3. *For every j we have*

$$Y^j = A^j(E^j \overline{\otimes} F^j)$$

and if we provide Y^j with the quotient topology $\mathscr{T}$, then the series (S) *converges in Y^j.*

PROOF. Suppose $h \in A^j(E^j \overline{\otimes} F^j)$. There is an $n \geqq 1$ and $z \in F_n^j \hat{\otimes} F_n^j$ such that $h = A^j z$. Then z may be written

$$z = \sum_{i=1}^{\infty} \lambda_i f_i \otimes g_i ,$$

$\sum_{i=1}^{\infty} |\lambda_i| < 1$, $f_i \to 0$ in E^j and $g_i \to 0$ in F^j. The series converges in $E_n^j \hat{\otimes} F_n^j$ and A^j is continuous, so $h = \sum_{i=1}^{\infty} \lambda_i f_i * g_i$. That is, h belongs to Y^j.

Conversely, suppose that h belongs to Y^j, with a series representation $h = \sum_{i=1}^{\infty} \lambda_i f_i * g_i$. Since $\{f_i\}$ and $\{g_i\}$ converges to zero in E^j and F^j respectively, they are in particular bounded. E^j and F^j are strict LF-spaces so there is an n such that $\{f_i\}$ and $\{g_i\}$ are contained in E_n^j and F_n^j respectively, and converges to zero there. By remarks above it now follows that the series $\sum_{i=1}^{\infty} \lambda_i f_i \otimes g_i$ converges to an element z in $E_n^j \hat{\otimes} F_n^j$. By continuity of A^j we get

$$A^j z = \sum_{i=1}^{\infty} \lambda_i f_i * g_i = h,$$

which proves that $A^j(E^j \overline{\otimes} F^j) = Y^j$.

Since $E^j \hat{\otimes} F^j$ is a strict LF-space by prop. 3.1, Y^j becomes an LF-space with defining sequence

$$Y_n^j = A_n^j(E_n^j \hat{\otimes} F_n^j), \quad n = 1, 2, \ldots ,$$

when Y_n^j is given the quotient topology $\mathscr{T}_n$ from $E_n^j \otimes F_n^j$. Indeed, Y_n^j is a Frechet-space for each n, $Y_n^j \subseteq Y_{n+1}^j$, and the injection $Y_n^j \to Y_{n+1}^j$ is continuous by definition of quotient topologies. Also, the injection Γ_n of Y_n^j into Y^j is continuous, and $\bigcup_{n=1}^{\infty} Y_n^j = Y^j$. To see that $\mathscr{T}$ is the inductive topology with respect to $\{\mathscr{T}_n\}$, let B be a linear operator of Y^j into a locally convex space X, such that $B \circ \Gamma_n$ is continuous for each n. Then $B \circ A_n^j = (B \circ \Gamma_n) \circ A_n^j$ is continuous, hence $B \circ A^j$ is continuous. But then B is continuous. So $Y^j = \lim \operatorname{ind} \{Y_n^j\}$. Moreover Y^j is nuclear by prop. 50.1 in [6]. We have proved

Lemma 3.4. *Y^j is nuclear* LF-*space*, $j \geqq 0$.

Now let $Y = [\bigcup_{j=0}^{\infty} Y^j] =$ the linear span of $\bigcup_{j=0}^{\infty} Y^j$ in $L(G)$. We give Y the inductive topology.

Proposition 3.5. *Y is a nuclear* LF-*space.*

Proof. For $k = 1, 2, \ldots$, let $H_k = [\bigcup_{j=0}^{k} Y_k^j]$. We claim that

1) $H_k \subseteq H_{k+1}$, $\quad k = 1, 2, \ldots$,
2) $Y = \bigcup_{k=1}^{\infty} H_k$.

Indeed,

$$H_k \subseteq [\bigcup_{j=0}^{k+1} Y_k^j] \subseteq [\bigcup_{j=0}^{k+1} Y_{k+1}^j] = H_{k+1} ,$$

which proves 1).

All H_k are linear spaces, and the sequence $\{H_k\}$ is increasing, so to verify 2) it suffices to show that if $h \in Y^j$ for any $j \geqq 0$, then $h \in H_k$ for some k. So let $h \in Y^j$. There is n such that $h \in Y_n^j$. Let $k = \max(j, n)$. Then

$$h \in Y_k^j \subseteq [\bigcup_{i=0}^{k} Y_k^i] = H_k ,$$

and 2) is valid. We now give each H_k the (finite) inductive system topology. This makes H_k into a nuclear Frechet space. Let Y_H denote Y with the inductive limit topology defined by the sequence $\{H_k\}$. We show that Y_H and Y are topologically isomorphic.

To see that the identity map of Y_H into Y is continuous, it suffices to show that the injection $H_k \to Y$ is continuous for all k. Since in turn each H_k has an inductive topology, it suffices to show that the injection $Y_k^j \to Y, j \leqq k$, is continuous. But the injections $Y_k^j \to Y^j \to Y$ are both continuous, so $Y_H \to Y$ is continuous. Conversely, to see that the identity map $Y \to Y_H$ is continuous, it suffices to show that $Y^j \to Y_H$ is continuous for all $j \geqq 0$. Again, this means that it suffices to show that $Y^j_k \to Y_H$ is continuous for all j,k. We have two cases:

a) $j \leqq k$. The maps $Y^j_k \to H_k \to Y_H$ are both continuous.

b) $j > k$. The maps $Y^j_k \to Y^j_k \to H_j \to Y_H$ are all continuous.

Hence the identity map $Y \to Y_H$ is a homeomorphism. Since Y_H is an inductive limit of an increasing sequence of nuclear Frechet-spaces, the proposition follows.

PROPOSITION 3.6. *Y is dense in $L(G)$.*

PROOF. Let the sequence $\{\varphi_j\}_{j \geqq 0}$ be as in prop. 2.5. Let $\gamma_j = \varphi_j * \tilde{\varphi}_j$, so $\gamma_j \in Y^j$ for all $j \geqq 0$. Hence $\{\gamma_j\}_{j \geqq 0} \subseteqq Y$. Clearly,

$$\gamma_j \geqq 0, \quad \operatorname{supp}(\gamma_j) \subseteqq U_{j-1}, \quad \int \gamma_j = 1 ,$$

$j \geqq 1$, by prop. 2.5 and the subsequent remarks. So $\{\gamma_j\}$ is an approximative identity for G. Let $f \in L(G)$. Then

$$\gamma_j * f = \varphi_j * (\tilde{\varphi}_j * f)$$

belongs to Y^j since $\tilde{\varphi}_j * f \in F^j$ by lemma 2.11 (b). Hence $\gamma_j * f \in Y$ and $\gamma_j * f$ converges uniformly to f on a compact set containing the support of $\gamma_j * f$ and f for all j. The proof is complete.

PROPOSITION 3.7. *Let $h \in Y$. We have:*

(i) *if $\varphi \in L^1_C(G)$, then $\varphi * h$ and $h * \varphi$ belong to Y,*

(ii) *$\tilde{h} \in Y$,*

(iii) *if $x \in G$, then h_x and h^x belong to Y.*

Each of the statements above is true with Y^j in place of Y, $j \geqq 0$, when $h \in Y^j$.

PROOF. It clearly suffices to prove the proposition for arbitrary Y^j. So let $h \in Y^j$ be given along with $\varphi \in L^1_C(G)$. By definition h has a series representation

(S) $\quad h = \sum_{i=1}^{\infty} \lambda_i f_i * g_i, \quad \sum_{i=1}^{\infty} |\lambda_i| < 1; \; f_i \to 0$ in E^j, $g_i \to 0$ in F^j.

Now convolution to the left or right on $L(G)$ by elements φ in $L^1_C(G)$ is a continuous linear operator on $L(G)$. Hence we have

$$\varphi * h = \sum_{i=1}^{\infty} \lambda_i (\varphi * f_i) * g_i , \tag{1}$$

$$h * \varphi = \sum_{i=1}^{\infty} \lambda_i f_i * (g_i * \varphi) , \tag{2}$$

where the series in (1), (2) converges in $L(G)$. To see that $\varphi * h$ (resp. $h * \varphi$) belongs to Y^j it therefore suffices to show that $\varphi * f_i \to 0$ in E^j (resp. $g_i * \varphi \to 0$ in F^j). But this follows directly from lemma 2.11. So (i) is proved.

Next, recall that $f \to \tilde{f}$ is an anti-linear homeomorphism of E^j onto F^j (lemma 2.7). Hence, with $f = \sum_{i=1}^{\infty} \lambda_i f_i * g_i$ as above, we must have $\tilde{f}_i \to 0$ in F^j, $\tilde{g}_i \to 0$ in E^j. So the function $k = \sum \bar{\lambda}_i \tilde{g}_i * \tilde{f}_i$ belongs to Y^j. By evaluation we verify that $k = \tilde{f}$. This proves (ii).

Finally, let $x \in G$, and $h = \sum_{i=1}^{\infty} \lambda_i f_i * g_i$ as above. The sequence $\{f_i\}$ is bounded in E^j and therefore belongs to E_n^j for some n, and converges to zero in E_n^j. By lemma 2.13, $(f_i)_x \to 0$ in E^j. The series $\sum_{i=1}^{\infty} \lambda_i (f_i)_x * g_i$ clearly converges to h_x in $L(G)$, so $h_x \in Y^j$. The proof for h^x is similar. This completes the proof of the proposition.

Because of the last result we may define the left and right regular representations λ and ϱ respectively, of G on Y.

PROPOSITION 3.8. *The maps*

(i) $(x,h) \to \lambda_x h$,

(ii) $(x,h) \to \varrho_x h$,

with $x \in G$, $h \in Y$, *are continuous of* $G \times Y$ *into* Y.

PROOF. We prove (i); (ii) is proved similarly. Prop. 3.5 shows that Y is an LF-space, hence barrelled. By [4, lemma 3, p. 24] it therefore suffices to show that

a) λ_x is a continuous linear map on Y,

b) $x \to \lambda_x h$ is continuous of G into Y for all $h \in Y$.

By the definition of Y and the fact that λ leaves each Y^j invariant (prop. 3.7) it suffices to prove (a) and (b) with arbitrary Y^j $(j \geqq 0)$ instead of Y. As before, let L be the left regular representation of G on E^j. For each $x \in G$, L_x is continuous on E^j (lemma 2.13), so $L_x \otimes I$, with I the identity map on F^j, is continuous on $E^j \overline{\otimes} F^j$ [2, p. 75]. We claim that

$$(*) \qquad A^j (L_x \otimes I) = \lambda_x A^j .$$

Let z be an element in $E^j \overline{\otimes} F^j$. There is n such that $z \in E_n^j \hat{\otimes} F_n^j$, hence $z = \sum_{i=1}^{\infty} \lambda_i f_i \otimes g_i$, with $\{\lambda_i\}$, $\{f_i\}$ and $\{g_i\}$ as before. $L_x \otimes I$ is continuous, so we get:

$$A^j (L_x \otimes I) z = A^j \textstyle\sum_{i=1}^{\infty} \lambda_i (L_x f_i) \otimes g_i = \sum_{i=1}^{\infty} \lambda_i (L_x f_i) * g_i$$

which by the proof of (iii) in prop. 3.7 is equal to

$$\lambda_x \left(\textstyle\sum_{i=1}^{\infty} \lambda_i f_i * g_i \right) = \lambda_x A^j z$$

which proves the claim. Since A^j is open, this implies that λ_x is continuous on Y^j.

For the proof of (b) we first observe that by (*) it suffices to show that $x \to (L_x \otimes I)z$, $x \in G$, is continuous for all $z \in E^j \overline{\otimes} F^j$. Let $0 \neq z \in E^j \overline{\otimes} F^j$ be given. There is n such that $z = \sum_{i=1}^{\infty} \lambda_i f_i \otimes g_i$ with $\{\lambda_i\} \subseteq l^1$, $\{f_i\}$ converges to zero in E_n^j and $\{g_i\}$ converges to zero in F_n^j. We have

$$(L_x \otimes I)z = \sum_{i=1}^{\infty} \lambda_i (L_x f_i) \otimes g_i ,$$

so if $x \to y$ in G, we may choose E_n^j large enough to include all $L_x f_i$ and $L_y f_i$, $i = 1, 2, \ldots$, as $x \to y$. Let $\varepsilon > 0$ and norms $p = p^j_{V_n, m}$, $q = q^j_{V_n', m'}$ on E_n^j, F_n^j, respectively, be given. Let $\varepsilon' = \varepsilon \left(\sum_{i=1}^{\infty} |\lambda_i| q(g_i)\right)^{-1}$, and choose an integer k such that $p(f_i) < \frac{1}{2}\varepsilon'$ for $i > k$. Then, by lemma 2.13 we obtain for $i > k$

$$(**) \qquad p(L_x f_i - L_y f_i) \leqq p(L_x f_i) + p(L_y f_i) = 2p(f_i) < \varepsilon' .$$

By lemma 2.12 there is a neighborhood U of y in G such that

$$(***) \qquad p(L_x f_i - L_y f_i) < \varepsilon' \quad \text{for } i = 1, \ldots, k \text{ if } x \in U .$$

Hence, if $x \in U$ we get by (**) and (***):

$$\begin{aligned} (p \otimes q)[(L_x \otimes I)z - (L_y \otimes I)z] &= (p \otimes q) \sum_{i=1}^{\infty} \lambda_i (L_x f_i - L_y f_i) \otimes g_i \\ &\leqq \sum_{i=1}^{\infty} |\lambda_i| p(L_x f_i - L_y f_i) q(g_i) \\ &< \sum_{i=1}^{\infty} |\lambda_i| \varepsilon' q(g_i) = \varepsilon . \end{aligned}$$

Since norms of the type $p \otimes q$ determines the topology on $E_n^j \hat{\otimes} F_n^j$ and the injection of $E_n^j \hat{\otimes} F_n^j$ into $E^j \overline{\otimes} F^j$ is continuous, (b) follows. The proof is complete.

Summarizing, we have proved the following

THEOREM 3.9. *The space $Y \subseteq L(G)$ is a nuclear* LF-*space, and is dense in $L(G)$. Y is a two-sided ideal in $L^1_C(G)$ with respect to convolution, and is closed with respect to the operation* ˜. *The left and right regular representations of G on Y are well defined and jointly continuous.*

ADDED IN PROOF. We have been informed by S. M. Newberger and C. A. Akemann that there is a gap in Lemma 9 in the author's paper: "Physical states on a C^*-algebra", Acta Math. 122 (1969), 161–172. Theorem 1 of this paper must therefore be considered unproved. However, in a forthcoming paper, Akemann, Elliott and Newberger have been able to fill the gap for a large class of C^*-algebras.

REFERENCES

1. F. Bruhat, *Distributions sur un groupe localement compact et applications à l'étude des représentations des groupes p-adiques*, Bull. Soc. Math. France 89 (1961), 43–75.
2. A. Grothendieck, *Produit tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. 16, Providence, 1955.
3. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis* I (Grundlehren Math. Wissensch. 115), Springer-Verlag, Berlin · Göttingen · Heidelberg, 1963.
4. R. T. Moore, *Measurable, continuous and smooth vectors for semigroups and group representations* Mem. Amer. Math. Soc. 78, Providence, 1968.
5. T. Pytlik, *A nuclear space of functions on a locally compact group*, Bull. Acad. Pol. Sci. Sér. Sci. Math. Astronom. Phys. 17 (1969), 161–166.
6. F. Treves, *Topological vector spaces, distributions and kernels*, Academic Press, New York · London, 1967.

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