MATH. SCAND. 29 (1971), 307–316

A PROPERTY OF BIHARMONIC FUNCTIONS WITH DIRICHLET FINITE LAPLACIANS

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Consider a noncompact orientable C^∞ manifold R of dimension $m \geqq 2$ with C^∞ Riemannian metric $ds^2 = \sum_{i,j=1}^m g_{ij}(x)\, dx^i dx^j$. A *biharmonic function* u on R is a C^4 solution of the fourth order elliptic equation $\Delta^2 u = 0$. Here

$$(1)\qquad \Delta_x \cdot \;=\; -g(x)^{-\frac{1}{2}} \sum_{i=1}^m \frac{\partial}{\partial x^i}\left(\sum_{j=1}^m g(x)^{\frac{1}{2}} g^{ij}(x) \frac{\partial \cdot}{\partial x^j}\right)$$

is the Laplace–Beltrami operator associated with the metric tensor g_{ij}, with $g = \det(g_{ij})$ and $(g^{ij}) = (g_{ij})^{-1}$. We are interested in the space $W(R) = \{u \in C^4(R) \mid \Delta^2 u = 0\}$ and its various subspaces.

If R is the interior of a compact bordered Riemannian manifold $\bar{R}$, then every $u \in W(\bar{R}) = \{u \in W(R) \cap C(\bar{R}) \mid \Delta u \in C(\bar{R})\}$ admits the *Riesz-type representation*

$$(2)\qquad u \;=\; H_u^R + \int_R G(\cdot, x)\, \Delta_x u(x)\, dx\;,$$

where H_u^R is the solution of the harmonic Dirichlet problem on R with boundary values $u \mid \partial R$, $G(x,y)$ is the harmonic Green's function on R, and $dx = g^{\frac{1}{2}} dx^1 \ldots dx^m$ is the Riemannian volume element of R. The representation (2) reduces the study of $W(\bar{R})$ to that of

$$H(\bar{R}) \;=\; \{u \in C^2(R) \cap C(\bar{R}) \mid \Delta u = 0 \text{ on } R\}$$

which is more accessible to explicit treatment than $W(\bar{R})$.

For this reason it is of compelling importance to distinguish subclasses of $W(R)$ and of Riemannian manifolds R for which the representation (2) is valid. Some straightforward observations on this problem were made in [2], [3], and [4]. In this paper we study the metric growth of Δu for $u \in W(R)$ such that Δu has a finite Dirichlet integral and u possesses a certain boundedness property. The result to be proven is, roughly

Received September 28, 1970.

The work was sponsored by the U.S. Army Research Office – Durham, Grant DA-ARO-D-31-124-70-G7, University of California, Los Angeles.

speaking, that $\Delta u \in L^2(R, G\cdot dx)$. From this it follows that such a u admits the representation (2) for all R with $1 \in L^2(R, G\cdot dx)$. That a condition on R cannot be altogether dispensed with is established by a counterexample.

Subclasses of $W(R)$.

1. We denote by P, B, and D the classes of nonnegative, bounded, and Dirichlet finite functions, and we let BD stand for $B \cap D$. If no confusion with the class of continuous functions is to be feared, we also use C for BD. Given a class Y of functions we denote by Y_Δ the class $\{u \in C^2 \mid \Delta u \in Y\}$, and consider the subclasses $WXY_\Delta(R)$ of $W(R)$ with $X, Y = P, B, D$, and C. Thus $u \in WXY_\Delta$ means that $u \in W$, $u \in X$, and $\Delta u \in Y$.

A regular subregion Ω of R is a relatively compact subregion of R whose boundary $\partial\Omega$ is a C^∞ hypersurface. We denote by $B(x,\varepsilon)$ a parametric ball about x with radius ε, so small that $B(x,\varepsilon)$ is a regular region of R.

2. Let $u \in W(R)$ and take a regular region Ω of R. For an arbitrary point $x \in \Omega$ choose a ball $B = B(x,\varepsilon)$ with $\bar{B} \subset \Omega$. Denote by $G_\Omega(\cdot,\cdot)$ the harmonic Green's function on Ω with flux -1 about its pole, and by H_u^Ω the harmonic function on Ω continuous on $\bar{\Omega}$ with $H_u^\Omega \mid \partial\Omega = u$. By Green's formula,

$$\int_{\Omega-\bar{B}} \left[(u(y)-H_u^\Omega(y))\Delta_y G_\Omega(x,y) - G_\Omega(x,y)\Delta_y(u(y)-H_u^\Omega(y))\right] dy$$
$$= -\int_{\partial\Omega-\partial B} \left[(u(y)-H_u^\Omega(y)) {*d_y} G_\Omega(x,y) - G_\Omega(x,y) {*d_y}(u(y)-H_u^\Omega(y))\right].$$

On letting $\varepsilon \to 0$ we obtain

$$u(x) = H_u^\Omega(x) + \int_\Omega G_\Omega(x,y)\Delta_y u(y)\, dy \tag{3}$$

for every $x \in \Omega$ (cf. e.g. [1]).

The transition to the limit $\Omega \to R$ so as to obtain

$$u(x) = H_u^R(x) + \int_R G_R(x,y)\Delta u(y)\, dy\,, \tag{4}$$

with the conventional notation $H_u^R = \lim_{\Omega\to R} H_u^\Omega$, is possible if and only if

$$\int_R G_R(x,y)\,|\Delta u(y)|\, dy < \infty \tag{5}$$

for some and hence for all x in R. This is the reason why we are interested in what could be called the metric growth $\int_R G_\Omega(x,y)\,|\Delta u(y)|^p\,dy$ of Δu as $\Omega \to R$. We shall write G for G_R.

The class WDC_Δ.

3. We first consider the class $WDC_\Delta(R)$ consisting of all $u \in W(R)$ with finite $D(u)$ and $\sup_R|\Delta u| + D(\Delta u)$, where

$$D(f) = D_R(f) = \int_R df\wedge * df$$

is the Dirichlet integral. We shall prove:

THEOREM 1. *The metric growth of the Laplacian of u in $WDC_\Delta(R)$ is so slow that*

$$\int_R G(x,y)\,|\Delta u(y)|^2\,dy < \infty\,. \tag{6}$$

4. Fix an $x \in R$ and a ball $B = B(x,\varepsilon)$. Let $\psi \in C^\infty(R)$ such that $0 \leqq \psi \leqq 1$, $\psi\,|\,R-\bar{B} = 1$, and $\psi\,|\,B(x,\frac{1}{2}\varepsilon) = 0$. Since $\Delta u \in BD$, the functions

$$\varphi(y) = \psi(y)G(x,y)\Delta u(y) \quad \text{and} \quad \varphi_\Omega(y) = \psi(y)G_\Omega(x,y)\Delta u(y)$$

are in the class $BD(R)$, with Ω an arbitrary regular region containing $\bar{B}$, and $G_\Omega(x,y)$ extended to R by $G_\Omega\,|\,R-\Omega = 0$. Since

$$\lim_{\Omega\to R} D_R(G(x,\cdot) - G_\Omega(x,\cdot)) = 0$$

(cf. e.g. [5]), we conclude that

$$\lim_{\Omega\to R} D_R(\varphi - \varphi_\Omega) = 0\,. \tag{7}$$

From $d(\varphi_\Omega * du) = d\varphi_\Omega\wedge * du - \varphi_\Omega \Delta u * 1$ and $\varphi_\Omega\,|\,\partial\Omega = 0$, it follows by Stokes' formula that

$$\int_R \psi(y)\,G_\Omega(x,y)\,|\Delta u(y)|^2\,dy = \int_R d\varphi_\Omega \wedge * du\,.$$

By Schwarz's inequality,

$$\int_{R-B} G_\Omega(x,y)\,|\Delta u(y)|^2\,dy \leqq (D_R(\varphi_\Omega)\,D_R(u))^{\frac{1}{2}}\,.$$

Since $G_\Omega(x,\cdot)$ converges increasingly on R to $G(x,\cdot)$, we conclude by (7) that

$$\int_{R-B} G(x,y)\,|\Delta u(y)|^2\,dy \leqq (D_R(\varphi)\,D_R(u))^{\frac{1}{2}} < \infty\,. \tag{8}$$

On B, we set $K=\sup_B(\Delta u)^2 g^{\frac{1}{2}}$, and obtain

$$G(x,y)\,|\Delta u(y)|^2\,dy \leqq K\,l_m(y)\,dy^1\ldots dy^m\,,$$

where $l_m(y)=c_1|y|^{2-m}$ for $m\geqq 3$, and $l_m(y)=c_2\log(2\varepsilon/|y|)$ for $m=2$, with the c_i constants. Therefore

$$\int_B G(x,y)\,|\Delta u(y)|^2\,dy < \infty\,. \tag{9}$$

We remark in passing that (9) is valid for every $u\in C^2(\bar{B})$. From (8) and (9), assertion (6) follows.

5. In the statement of Theorem 1 and in the above proof we implicitly assumed the existence of the Green's function $G(\cdot,\cdot)$ on R. This is justified by the following observation. Suppose R is in the class O_G of Riemannian manifolds R that do not carry Green's functions. Since O_G is contained in the class O_{HD} of Riemannian manifolds R such that

$$HD(R) = \{u\in C^2(R)\mid \Delta u=0,\ D_R(u)<\infty\}$$

reduces to constants (cf. e.g. [5]), every $u\in WDD_\Delta$ satisfies $\Delta u=c$, a constant. If $c\neq 0$, then $R\notin O_{QD}$, where $Q=\{v\in C^2(R)\mid \Delta v=1\}$. However $R\notin O_{QD}$ is characterized by $\int_R G(x,y)\,dx\,dy<\infty$ (see [4]). This is clearly a contradiction and we conclude that $\Delta u=0$ for every $u\in WDD_\Delta$ and a fortiori for every $u\in WDC_\Delta$. If we set $G(\cdot,\cdot)\equiv\infty$ for $R\in O_G$ and make the convention that $0\cdot\infty=0$, then (6) is still valid.

Since $u\in WDD_\Delta$ on $R\in O_G$ belongs to HD, we deduce again by $O_G\subset O_{HD}$ that $u=\text{const}$. We therefore can express the above relations by the following inclusion scheme:

$$O_G \subset O_{WDD\Delta}\begin{cases}\subset O_{WDC_\Delta}\,,\\ \subset O_{WCD_\Delta}\,,\end{cases} \tag{10}$$

where O_{WXY_Δ} is the class of those R for which $WXY_\Delta(R)$ reduces to constants.

The class WCD_Δ.

6. From the class WDC_Δ we turn to the class $WCD_\Delta(R)$ consisting of all $u\in W(R)$ with finite $\sup_R|u|+D_R(u)$ and $D_R(\Delta u)$. This class is technically more difficult to treat than the former. As a counterpart of Theorem 1 we shall prove:

THEOREM 2. *The metric growth of the Laplacian of u in $WCD_\Delta(R)$ is so slow that*

$$\int_R G(x,y)\,|\Delta u(y)|^2\,dy < \infty . \tag{11}$$

If $R \in O_G$, then as in the case of Theorem 1, we have $\Delta u = 0$ for every $u \in WCD_\Delta$ (cf. (10)), and (11) can be considered trivial. Thus we may suppose $R \notin O_G$ in the proof given in Nos. 7–10.

7. Fix an $x \in R$ and a ball $B = B(x,\varepsilon)$. Let Ω be a regular region of R with $\Omega \supset \bar{B}$. We set

$$g_\Omega = \int_\Omega G_\Omega(\cdot,y)\,\Delta u(y)\,dy$$

on Ω and recall (3):

$$u = H_u^\Omega + g_\Omega \tag{12}$$

on Ω. Since we are assuming $2\sup_R|u| = K_1 < \infty$, we have, by the maximum principle, $\sup_\Omega |H_u^\Omega| \leqq \frac{1}{2}K_1$ and consequently

$$\sup_\Omega |g_\Omega| \leqq K_1 . \tag{13}$$

Further immediate consequences of (12) are

$$\Delta g_\Omega = \Delta u \tag{14}$$

on Ω,

$$g_\Omega \,|\, \partial\Omega = 0 , \tag{15}$$

and, by Stokes' formula,

$$D_\Omega(H_u^\Omega) + D_\Omega(g_\Omega) = D_\Omega(u) \leqq D_R(u) < \infty . \tag{16}$$

8. Since $G_\Omega(\cdot,\cdot)$ converges increasingly on R to $G(\cdot,\cdot)$, and $G(\cdot,x)$ is bounded on $R - B$,

$$K_2 = \sup_{\Omega \supset \bar{B}} \sup_{y \in \Omega - B} G_\Omega(y,x) < \infty .$$

Stokes' formula applied to $\Delta_y u(y) * (G_\Omega(x,y)\,d_y g_\Omega(y))$ on $\Omega - \bar{B}$ reads

$$\int_{\partial\Omega - \partial B} \Delta u * (G_\Omega dg_\Omega) = \int_{\Omega - \bar{B}} d(\Delta u * (G_\Omega dg_\Omega)) . \tag{17}$$

By (12) and (16) we see that the $\partial g_\Omega(y)/\partial y_i$, $i = 1,\ldots,m$, are uniformly convergent in $B(x,2\varepsilon)$. Therefore there exists a constant K_3 such that

$$\left| \int_{\partial B} \Delta u * (G_\Omega dg_\Omega) \right| \leqq K_3 \tag{18}$$

for every Ω. Since the right-hand side of (17) is

$$\int_{\Omega-\bar{B}} G_\Omega d\Delta u\wedge *dg_\Omega + \int_{\Omega-\bar{B}} \Delta u\, dG_\Omega\wedge *dg_\Omega - \int_{\Omega-\bar{B}} G_\Omega \Delta u\, \Delta g_\Omega dy ,$$

relations (14), (17), and (18) imply

$$(19)\qquad \int_{\Omega-\bar{B}} G_\Omega (\Delta u)^2\, dy \leqq K_3 + \left| \int_{\Omega-\bar{B}} G_\Omega d\Delta u\wedge *dg_\Omega \right| + \left| \int_{\Omega-\bar{B}} \Delta u\, dG_\Omega\wedge *dg_\Omega \right| .$$

9. We are going to evaluate the last two terms of the right-hand side of (19). We start with the first. By Schwarz's inequality and (16)

$$\left(\int_{\Omega-\bar{B}} G_\Omega d\Delta u\wedge *dg_\Omega \right)^2 = \left(\int_{\Omega-\bar{B}} (G_\Omega^{\frac{1}{2}}\, d\Delta u)\wedge *(G_\Omega^{\frac{1}{2}}\, dg_\Omega) \right)^2$$
$$\leqq \int_{\Omega-\bar{B}} (G_\Omega^{\frac{1}{2}}\, d\Delta u)\wedge *(G_\Omega^{\frac{1}{2}}\, d\Delta u) \cdot \int_{\Omega-\bar{B}} (G_\Omega^{\frac{1}{2}}\, dg_\Omega)\wedge *(G_\Omega^{\frac{1}{2}}\, dg_\Omega)$$
$$\leqq K_2^2\, D_{\Omega-\bar{B}}(\Delta u)\, D_{\Omega-\bar{B}}(g_\Omega) \leqq K_2^2\, D_R(\Delta u)\, D_R(u) .$$

Hence on setting $K_4 = K_2 (D_R(\Delta u) D_R(u))^{\frac{1}{2}}$ we obtain

$$(20)\qquad \left| \int_{\Omega-\bar{B}} G_\Omega\, d\Delta u\wedge *dg_\Omega \right| \leqq K_4 .$$

10. To evaluate the last term in (19), observe that $\Delta u dG_\Omega \wedge *dg_\Omega = \Delta u dg_\Omega \wedge *dG_\Omega$. Again by Stokes' formula

$$(21)\qquad \int_{\partial\Omega-\partial B} \Delta u g_\Omega *dG_\Omega = \int_{\Omega-\bar{B}} g_\Omega d\Delta u\wedge *dG_\Omega + \int_{\Omega-\bar{B}} \Delta u dg_\Omega\wedge *dG_\Omega .$$

Since g_Ω and the $\partial G_\Omega(z,y)/\partial y_i$, $i=1,\ldots,m$, are uniformly convergent on ∂B, there exists a constant K_5 such that

$$(22)\qquad \left| \int_{\partial B} \Delta u\, g_\Omega *dG_\Omega \right| \leqq K_5$$

for every Ω. From this and (21) we obtain

$$(23)\qquad \left| \int_{\Omega-\bar{B}} \Delta u\, dG_\Omega\wedge *dg_\Omega \right| = \left| \int_{\Omega-\bar{B}} \Delta u\, dg_\Omega\wedge *dG_\Omega \right|$$
$$\leqq K_5 + \left| \int_{\Omega-\bar{B}} g_\Omega d\Delta u\wedge *dG_\Omega \right| .$$

In the same fashion as in No. 9, Schwarz's inequality, (13), and (16) yield

$$\left| \int_{\Omega-\bar{B}} g_\Omega d\Delta u \wedge * dG_\Omega \right|^2 \leqq K_1^2 D_{\Omega-\bar{B}}(\Delta u)\, D_{\Omega-\bar{B}}(G_\Omega) \leqq K_1^2 D_R(\Delta u)\, D_{\Omega-\bar{B}}(G_\Omega)\,.$$

Since $D_{R-\bar{B}}(G(x,\cdot)) = \lim_{\Omega\to R} D_{\Omega-\bar{B}}(G_\Omega(x,\cdot))$, there exists a constant K_6 such that $K_5 + K_1(D_R(u) D_{\Omega-\bar{B}}(G_\Omega))^{\frac{1}{2}} \leqq K_6$ for every Ω. From (23) it now follows that

$$\left| \int_{\Omega-\bar{B}} \Delta u \; dG_\Omega \wedge * dg_\Omega \right| \leqq K_6\,. \tag{24}$$

By (19), (20), and (24), we have

$$\int_{\Omega-\bar{B}} G_\Omega(x,y)(\Delta u(y))^2\, dy \leqq K_7$$

with $K_7 = K_3 + K_4 + K_6$ for every regular region Ω. On letting $\Omega \to R$ we obtain

$$\int_{R-\bar{B}} G(x,y)(\Delta u(y))^2\, dy \leqq K_7\,.$$

By (9) we already know that $\int_{\bar{B}} G(x,y)(\Delta u(y))^2 dy < \infty$. The proof of (11) is thus complete.

11. We consider a measure $d_G x = G(z,x)\,dx$ on R, and the corresponding L^2-space $L^2(R, d_G x)$. In view of Harnack's inequality, the location of z is immaterial provided it is fixed. Theorems 1 and 2 may be reformulated as follows:

$$\Delta(WDC_\Delta(R)) \cup \Delta(WCD_\Delta(R)) \subset L^2(R, d_G x)\,. \tag{25}$$

The Riesz-type representation.

12. Let $Q(R) = \{u \in C^2(R) \mid \Delta u = 1\}$ and consider the class O_{QP} of Riemannian manifolds R for which $QP(R) = \emptyset$. It is known that $R \notin O_{QP}$ is equivalent to $\int_R G(x,y)\,dy < \infty$ (see [4]), that is, $1 \in L^2(R, d_G x)$. We have seen that if $R \notin O_{QP}$, then every $u \in WXY_D$ $(X, Y = B, C)$ admits the Riesz-type representation (2) (cf. [3]). We shall prove that the same is true for WDC_Δ and WCD_Δ. The latter case only is nontrivial.

THEOREM 3. *If $R \notin O_{QP}$, then every u in $WCD_\Delta(R)$ (resp. $WDC_\Delta(R)$) has the Riesz-type representation*

(26) $$u = h+\int_R G(\cdot,y)\Delta u(y)dy$$

on R, with $h \in HC(R)$ *(resp.* $HD(R)$*).*

For the proof, let $\langle\cdot,\cdot\rangle$ be the inner product on $L^2(R,d_Gy)$. Since 1 and $|\Delta u|$ are in $L^2(R,d_Gy)$, Schwarz's inequality yields

$$\int_R G(x,y)\,|\Delta u(y)|\,dy = \langle 1,|\Delta u|\rangle \leqq (\langle 1,1\rangle\langle|\Delta u|,|\Delta u|\rangle)^{\frac{1}{2}} < \infty .$$

In view of (16), (26) follows on letting $\Omega \to R$ in (3).

13. One might suspect that the Riesz-type representation be valid at least for every $u \in WBB_\Delta(R)$ without any condition on R. That this is not the case can be seen by the following very simple example.

EXAMPLE. *Let R be the Riemannian manifold whose base manifold is the punctured disk* $\{z \mid 1<|z|<\infty\}$ *about* ∞ *and whose metric is given by* $ds^2=\lambda(z)\,|dz|^2$ *with* $\lambda(z)=|z|^{-1}$. *Then the function*

(27) $$u(z) = \tfrac{1}{2}(z+\bar z)/(z\bar z)^{\frac{1}{2}}$$

belongs to $WBB_\Delta(R)$ *(and actually to* $WBC_\Delta(R)$*) but does not admit a Riesz-type representation.*

To see this, denote by $\Delta_e= -4\partial^2/\partial z\partial\bar z$ the Euclidean Laplacian. The Laplace–Beltrami operator on R is $\Delta=\lambda(z)^{-1}\Delta_e$ and the volume element $dV(z)=\frac{1}{2}i\lambda(z)\,dz\wedge d\bar z$. Thus $H(R)=\{u \mid \Delta_e u=0\}$, and the Euclidean Dirichlet integral is identical with $D_R(\cdot)$.

By a simple computation we see that $\Delta_e u(z)=\frac{1}{2}(z+\bar z)/(z\bar z)^{3/2}$, which in turn gives

(28) $$\Delta u(z) = \tfrac{1}{2}(1/z+1/\bar z) .$$

Therefore $\Delta u \in HB(R)$. Moreover

$$d\Delta u(z)\wedge * d\Delta u(z)/dV(z) = (z\bar z)^{-2}$$

and we conclude that $D_R(\Delta u)<\infty$. Since $|u|<1$, we have $u \in WBC_\Delta(R)\subset WBB_\Delta(R)$.

To see that u has no Riesz-type representation we only have to show that

(29) $$a = \int_{|z|>1} G(w,z)\,|\Delta u(z)|\,dV(z) = \infty .$$

Here $w \in R$ is fixed with $|w|>2$, and $G(w,\cdot)$ is the harmonic Green's function on R with pole at w. Clearly there exists a positive number ε such that $G(w,z) \geqq \varepsilon$ for $|z| \geqq 2$. Therefore

$$a \geqq \varepsilon \int_{|z|>2} \frac{|z+\bar{z}|}{2z\bar{z}} \frac{1}{|z|} (\tfrac{1}{2} i dz \wedge d\bar{z})$$

$$= \varepsilon \int_{|z|>2} \frac{d|z|}{|z|} \int_0^{2\pi} |\sin(\arg z)| \, d \arg z = \infty .$$

Observe that $\bar{R} = \{z \mid 1 \leqq |z| < \infty\}$ can be considered as a bordered Riemannian manifold with compact border $|z| = 1$ and the ideal boundary ∞. The function u is in $W(\bar{R})$ and $\Delta u \in H(\bar{R})$. Clearly Δu is also harmonic at ∞ and $\Delta u(\infty) = 0$ but $\Delta u(z) = -\Delta u(-\bar{z})$. This accounts for the intricate behavior of u near ∞ and leads to (29). Harmonically the ideal boundary ∞ of R is of a quite simple nature but biharmonically it is very involved. This shows that biharmonic classification (see [3], [4]) heavily depends on the metric structure of the manifold in addition to its harmonic structure.

For a general R with $R \notin O_{QP}$, every $u \in WB_{\Delta}(R)$ admits (26) since

$$\int_R G_R(x,y) \, |\Delta_y u(y)| \, dy \leqq c \int_R G_R(x,y) \, dy < \infty$$

($c = \sup_R |\Delta u|$) and the transition from (3) to (4) is legitimate.

Without the condition $R \notin O_{QP}$ this conclusion is no longer true, as is also shown by our example. In fact,

$$\int_{|z|>1} G(w,z) \, dV(z) \geqq \varepsilon \int_{|z|>2} |z|^{-1} (\tfrac{1}{2} i dz \wedge d\bar{z}) = 2\pi\varepsilon \int_{|z|>2} d|z| = \infty ,$$

and our R belongs to O_{QP}. In this sense $R \notin O_{QP}$ is inevitable to assure that every $u \in WB_{\Delta}(R)$ possess a Riesz-type representation. The significance of Theorem 3 lies in the fact that the same condition is sufficient to admit a Riesz-type representation of every $u \in WCD_{\Delta}(R)$, with Δu not necessarily bounded.

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