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ON VARIATIONS OF METRICS

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Introduction.

In 1960 Yamabe [7] published the following theorem: "Any compact C^∞ Riemannian manifold of dimension $n \geqq 3$ can be deformed conformally to a C^∞ Riemannian manifold of constant scalar curvature". In 1966 Aubin [1] used Yamabe's result to prove the following theorem: "Any compact C^∞ Riemannian manifold of dimension $n \geqq 3$ carries a metric with constant negative scalar curvature", thus demonstrating a topological insignificance of the scalar curvature. In 1968 Trudinger [6] showed an error in Yamabe's proof, but proved his theorem under a condition on the original Riemannian manifold, which is satisfied e.g. if the integral of the scalar curvature is negative. In his paper [1] Aubin proved that a metric with a negative integral of the corresponding scalar curvature can be constructed on a Riemannian manifold with constant nonnegative scalar curvature. Thus Trudinger was able to obtain Aubin's theorem in the case of a Riemannian manifold with constant nonnegative curvature. In 1968 Aubin [2] published a proof of Yamabe's theorem, using a different method. Unfortunately however, his proof is also incomplete, as already noticed by others. So, Yamabe's theorem is still in doubt and as we shall have an opportunity to observe, probably is false.

In this paper we give a complete proof of Aubin's theorem. Moreover, we think it worthwhile to discuss the variation of metrics in a general setting. In the first section we introduce the variation integrals involved and investigate their critical points. In the second section we compute the variation integrals for special variations and prove the existence of a metric with negative total scalar curvature. This result, together with Trudingers result, which we prove by a different method and with a more precisely stated condition in the third section, proves Aubin's theorem.

1. Variational problems.

Let M denote a compact C^∞ manifold. The space of Riemannian

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metrics on M is an open convex subset of the linear space $C^\infty(L_s{}^2(TM, \mathsf{R}))$ of symmetric bilinear forms on M. Given a function F on the space of metrics, we denote the variational derivative of F at a point g and in the direction of a symmetric bilinear form h, by

$$D_h F(g) = (d/dt)F(\alpha(t))|_{t=0} ,$$

where $\alpha(t)$ is a curve in the space of metrics with $\alpha(0)=g$ and $\alpha'(0)=h$, for example $\alpha(t)=g+th$. The problem is to construct functions F such that the critical points g of F, that is where $D_hF(g)=0$ for all h, are interesting metrics and then if possible, to prove the existence of such metrics. We consider first the volume and the total scalar curvature:

$$V(g) = \int_M \omega(g), \quad J(g) = \int_M K(g)\,\omega(g) ,$$

where $\omega(g)$ denotes the volume element of M and $K(g)$ the scalar curvature with respect to the metric g. Theorems 1 and 2 are more or less well-known. The references [3], [5] were pointed out by the referee.

THEOREM 1. *Let M be of dimension $n \geqq 3$ and suppose J has zero variational derivative for all conformal variations of the metric g, keeping the volume $V(g)$ constant. Then the mean curvature $K(g)$ is constant on M. If J has zero variational derivative for all variations of g, keeping $V(g)$ constant, then $K(g)$ is constant and g is an Einstein metric, i.e. Ricci curvature $S(g)=n^{-1}K(g)g$.*

REMARK. For $n=2$ we have $S(g)=n^{-1}K(g)g$ for all metrics g, $J(g)$ is independent of g, and in fact by the Gauss–Bonnet Theorem we have $J(g)=2\pi\chi(M)$, where $\chi(M)$ is the Euler characteristic of M.

PROOF OF THEOREM 1. Let $g+th$ be a variation of g, such that $D_hV(g)=0$. Let h_{ik} and g_{ik} be the components of h and g in some coordinate system. We have the following formulas for the components of the Christoffel Symbols, the curvature tensor and the Ricci tensor:

$$\begin{aligned} \Gamma^r_{ik} &= \tfrac{1}{2}g^{r\nu}(D_i g_{k\nu} + D_k g_{i\nu} - D_\nu g_{ik}) , \\ R^r_{jik} &= D_j\Gamma^r_{ik} - D_i\Gamma^r_{jk} + \Gamma^r_{j\nu}\Gamma^\nu_{ik} - \Gamma^r_{i\nu}\Gamma^\nu_{jk} , \\ S_{ik} &= R^r_{rik}, \quad K = g^{ik}S_{ik} . \end{aligned}$$

The volume element is $\omega = (\det g)^{\frac{1}{2}}dx$, where $\det g$ denotes the determinant of g_{ik}. Then we compute the following formulas for variational derivatives:

$$D_h g^{ik} = -g^{i\nu} h_{\nu\mu} g^{\mu k}, \quad D_h (\det g)^{\frac{1}{2}} = \tfrac{1}{2} (\det g)^{\frac{1}{2}} g^{ik} h_{ik} ,$$
$$D_h R^r_{jik} = \nabla_j (D_h \Gamma^r_{ik}) - \nabla_i (D_h \Gamma^r_{jk}) ,$$
$$D_h \Gamma^r_{ik} = \tfrac{1}{2} g^{r\nu} (\nabla_i h_{k\nu} + \nabla_k h_{i\nu} - \nabla_\nu h_{ik}) .$$

Now we obtain for $J(g) = \int_M (\det g)^{\frac{1}{2}} K(g)\, dx$:

$$\begin{aligned} D_h J(g) &= \int_M (\det g)^{\frac{1}{2}} [\tfrac{1}{2} g^{ik} h_{ik} K - g^{i\nu} h_{\nu\mu} g^{\mu k} S_{ik} + \\ &\qquad + g^{ik} (\nabla_r (D_h \Gamma^r_{ik}) - \nabla_i (D_h \Gamma^r_{rk}))]\, dx \\ &= \int_M (\det g)^{\frac{1}{2}} (\tfrac{1}{2} g^{\nu\mu} K - g^{i\nu} g^{\mu k} S_{ik}) h_{\nu\mu}\, dx \end{aligned}$$

and for $V(g) = \int_M (\det g)^{\frac{1}{2}}$:

$$D_h V(g) = \int_M \tfrac{1}{2} (\det g)^{\frac{1}{2}} g^{ik} h_{ik}\, dx .$$

Now let $D_h J(g) = 0$ for all conformal deformations of g such that $D_h V(g) = 0$. Then $h = n^{-1} g^{ik} h_{ik} g$, so

$$D_h J(g) = \int_M (\det g)^{\frac{1}{2}} (\tfrac{1}{2} - n^{-1}) K g^{ik} h_{ik}\, dx ,$$

which implies that K is constant for $n > 2$. If the variational derivative of J vanishes for all variations of g, keeping $V(g)$ constant, then we obtain

$$\tfrac{1}{2} g^{\nu\mu} K - g^{\nu i} g^{\mu k} S_{ik} = \lambda g^{\nu\mu} ,$$

with some constant λ. Contraction with $g_{\nu\mu}$ gives $\lambda = (\frac{1}{2} - n^{-1}) K$, so K is constant for $n > 2$ and $S_{ik} = n^{-1} K g_{ik}$. If $n = 2$, a direct computation shows that this equation for the Ricci tensor holds for all g (with K variable in general) and then we obtain $D_h J(g) = 0$ for all g, or J is independent of g.

The question, whether a given manifold carries an Einstein metric is by Theorem 1 reduced to a variational problem, although certainly a difficult one. In fact we shall see that J takes all real values, so that a minimum or a maximum does not exist, but of course, critical values of J may still exist. The product of spheres $S^1 \times S^2$ does not admit an Einstein metric, but it is not known, whether there is a simply connected compact manifold, which does not carry an Einstein metric (see problem 4 (Eells–Sampson) in [4]). For the existence of an Einstein metric we need at least some topological condition, but are such conditions sufficient?

We mention here another variational integral taking nonnegative values only, but having critical points of the same kind as J:

$$I(g) = \int_M |K(g)|^{\frac{1}{2}n}\, \omega(g) .$$

THEOREM 2. *Theorem* 1 *holds with* J *replaced by* I, *where the volume may either be kept constant or not.*

PROOF. Using our previous results, we have

$$\begin{aligned} D_hI(g) &= \textstyle\int_M (\det g)^{\frac12}|K|^{\frac12 n-2}\big((\tfrac12 g^{\nu\mu}K-\tfrac12 n g^{\nu i}g^{\mu k}S_{ik})Kh_{\nu\mu}+ \\ &\qquad\qquad + \tfrac12 n g^{ik}(\nabla_\nu D_h\Gamma^\nu_{ik}-\nabla_i D_h\Gamma^\nu_{\nu k})K\big)\,dx \\ &= \tfrac12\textstyle\int_M (\det g)^{\frac12}\big((g^{\nu\mu}K-ng^{i\nu}g^{\mu k}S_{ik})K|K|^{\frac12 n-2}+ \\ &\qquad\qquad + n(g^{i\nu}g^{k\mu}-g^{\nu\mu}g^{ik})\nabla_i d_k(K|K|^{\frac12 n-2})\big)h_{\nu\mu}\,dx\,. \end{aligned}$$

For conformal deformations we obtain

$$\begin{aligned} D_hI(g) &= \tfrac12(1-n)\textstyle\int_M(\det g)^{\frac12}g^{\nu\mu}\nabla_\nu d_\mu(K|K|^{\frac12 n-2})g^{ik}h_{ik}\,dx \\ &= \tfrac12(1-n)\textstyle\int_M \Delta(K|K|^{\frac12 n-2})\,\mathrm{tr}\,h\;\omega(g)\,, \end{aligned}$$

where Δ denotes the Laplacian with respect to the metric g. Thus $D_hI(g)=0$, for conformal deformations, implies

$$\Delta(K|K|^{\frac12 n-2}) = \lambda$$

with some constant λ and $\lambda=0$, if we do not keep the volume fixed. However, in any case we obtain $\lambda=0$ and $K=$constant. Now, if $D_hI(g)=0$ for all variations, possibly keeping the volume fixed, then we obtain:

$$g^{\nu\mu}K = ng^{i\nu}g^{\mu k}S_{ik}$$

and g is an Einstein metric.

Aubin uses the variational integral I in his attempt to prove Yamabe's theorem. However, he uses strong constraints on the variations, which affect the Euler–Lagrange equation. It is Aubin's treatment of the Euler–Lagrange equation and subsequently his proof of the constancy of K, which is incomplete.

2. Special variations.

We wish to investigate the values of the variation integral J, which are taken under special variations of the metric, starting from a given metric $\langle\cdot,\cdot\rangle$ on M. Thus M is now a Riemannian manifold and we denote the covariant differentiation by ∇, the curvature tensor by R, the Ricci tensor by S and the volume element by ω, all corresponding to the fixed metric $\langle\cdot,\cdot\rangle$. Any metric g can be written as

$$g_A(v,\mu) = \langle Av,u\rangle, \quad A\in C^\infty(L(TM,TM))$$

with $A(x) : T_xM \to T_xM$ positive and self-adjoint for all x in M. Then we have for the covariant derivative of a vector field X in the direction of another vector field Y with respect to the new metric

$$\nabla_A X\, Y = \nabla X\, Y + \Gamma_A(X,Y)$$

with

$$\Gamma_A(u,v) = \tfrac{1}{2}A^{-1}\big(\nabla A(u)v + \nabla A(v)u - \nabla^* A(u,v)\big)\,,$$
$$\langle \nabla^* A(u,v), w\rangle = \langle u, \nabla A(w)v\rangle = \langle v, \nabla A(w)u\rangle\,.$$

For the volume element and the curvature tensor corresponding to the new metric, we get

$$\omega_A = (\det A)^{\frac{1}{2}}\,\omega\,,$$
$$R_A(v,u)w = R(v,u)w + \nabla\Gamma_A(v,u,w) - \nabla\Gamma_A(u,v,w) + {} + \Gamma_A(v,\Gamma_A(u,w)) - \Gamma_A(u,\Gamma_A(v,w))\,.$$

Using an orthonormal basis $e_1,\ldots,e_n$ of TM at some point to contract R, we obtain the following formulas for the Ricci tensor and the scalar curvature:

$$\begin{aligned} S_A(u,v) &= \textstyle\sum_j \langle R_A(e_j,u)v, e_j\rangle\,,\\ K_A &= \textstyle\sum_i S_A(e_i, A^{-1}e_i)\\ &= \textstyle\sum_i S(e_i,A^{-1}e_i) + \sum_{ij}\big(\langle \nabla dA(e_i,e_j,A^{-1}e_i), A^{-1}e_j\rangle + {}\\ &\qquad + \langle \Gamma_A(e_i,e_j), A\Gamma_A(A^{-1}e_i, A^{-1}e_j)\rangle - {}\\ &\qquad - \langle \Gamma_A(e_j,A^{-1}e_j), A\Gamma_A(A^{-1}e_i,e_i)\rangle\big)\,, \end{aligned}$$

with $dA(u,v) = \nabla A(u,v) - \nabla A(v,u)$.

We now consider the special case $A = \psi + d\varphi \otimes d^*\varphi$, that is

$$Av = \psi v + d\varphi(v)\, d^*\varphi, \qquad d^*\varphi = \operatorname{grad}\varphi\,,$$

where ψ and φ are C^∞ functions on M and $\psi > 0$. This includes in particular the conformal deformations of the original Riemannian manifold ($\varphi = 0$). We have

$$\begin{aligned} A^{-1} &= \psi^{-1}(1 - \alpha\, d\varphi\otimes d^*\varphi), \qquad \alpha = (\psi + \|d\varphi\|^2)^{-1}\,,\\ A(d^*\varphi) &= \alpha^{-1} d^*\varphi, \qquad \det A = \alpha^{-1}\psi^{n-1}, \qquad \omega_A = \alpha^{-\frac{1}{2}}\psi^{\frac{1}{2}(n-1)}\omega\,,\\ \nabla A(u)v &= d\psi(u)v + \nabla d\varphi(u,v)\, d^*\varphi + d\varphi(v)\, \nabla d^*\varphi(u)\,,\\ \nabla^* A(u,v) &= \langle u,v\rangle d^*\psi + d\varphi(u)\, \nabla d^*\varphi(v) + d\varphi(v)\, \nabla d^*\varphi(u)\,,\\ A\Gamma_A(u,v) &= \tfrac{1}{2}\big(d\psi(u)v + d\psi(v)u - \langle u,v\rangle d^*\psi\big) + \nabla d\varphi(u,v)\, d^*\varphi\,,\\ \nabla dA(w,u)v &= \nabla d\psi(w,u)v - \nabla d\psi(w,v)u + \nabla d\varphi(w,v)\, \nabla d^*\varphi(u) + {}\\ &\quad + d\varphi(v)\, \nabla^2 d^*\varphi(w,u) - \nabla d\varphi(w,u)\, \nabla d^*\varphi(v) - d\varphi(u)\, \nabla^2 d^*\varphi(w,v)\,. \end{aligned}$$

Using these formulas we can easily compute the scalar curvature K_A and then the variation integral $J(A) = \int_M K_A \omega_A$. For conformal deformations ($\varphi = 0$), we get

LEMMA 1. *For conformal deformations of the Riemannian manifold M, $\langle\cdot,\cdot\rangle$ by a C^∞ positive function ψ, we have*

$$K_\psi = u^{-p}\left(u^2K - 4\frac{n-1}{n-2}u\Delta u\right)$$

with $p = 2n/(n-2)$, $u = \psi^{\frac{1}{4}(n-2)}$ and we have

$$J(u) = \int_M K_\psi \omega_\psi = \int_M \left(u^2K + 4\frac{n-1}{n-2}\|du\|^2\right)\omega ,$$
$$V(u) = \int_M \omega_\psi = \int_M u^p \omega .$$

Moreover J is bounded below for conformal deformations with $V =$ constant.

The formulas in the Lemma follow easily from the more general formula for K_A and previous formulas for the special variation A, putting $\varphi = 0$ there. J is easily seen to be bounded below for any scalar curvature K, as we shall have an opportunity to demonstrate in Section 3. We shall now see that for more general deformations J can take arbitrary large negative values.

LEMMA 2. *We have for variations of the type $A = \psi + d\varphi\otimes d^*\varphi$, $\psi > 0$:*

$$\begin{aligned}K_A = {} & \psi^{-1}K - \alpha\psi^{-1}S(d^*\varphi, d^*\varphi) - \alpha^2\psi^{-1}\Delta\varphi\langle d\beta, d\varphi\rangle + \\ & + \tfrac{1}{4}\psi^{-3}(2\alpha^2\psi^2 - 2(n-3)\alpha\psi - (n-2)(n-7))\|d\psi\|^2 + \\ & + \tfrac{1}{4}\alpha\psi^{-3}(2(2-n)\alpha\psi + (n-2)(n-7))\gamma^2 + \\ & + \tfrac{1}{2}\alpha\psi^{-2}(2\alpha\psi - n + 2)\langle d\beta, d\psi\rangle + \tfrac{1}{2}\alpha^2\psi^{-1}\|d\beta\|^2 - \\ & - \tfrac{1}{2}\alpha^2\psi^{-2}(n-2)\gamma\langle d\beta, d\varphi\rangle - \alpha^2\psi^{-1}\gamma\Delta\varphi - (n-2)\psi^{-2}\Delta\psi \\ & + \alpha\psi^{-2}(n-2)\,\mathrm{Div}\,(\gamma d\varphi) + \alpha\psi^{-1}\,\mathrm{Div}\,(\Delta\varphi d\varphi - \tfrac{1}{2}d\beta - d\psi) ,\end{aligned}$$

with $\alpha = (\psi + \|d\varphi\|^2)^{-1}$, $\beta = \|d\varphi\|^2$, $\gamma = \langle d\psi, d\varphi\rangle$.

For variations of the type $B = \theta^{4/(n-2)}(\psi + d\varphi\otimes d^*\varphi)$ with $\theta = \alpha^{\nu-\frac{1}{4}}\psi^{\mu-\frac{1}{4}(n-3)}$ for some ν and μ, we have

$$\begin{aligned}J(B) = \int_M K_B\omega_B &= \int_M \left(\theta^2K_A + 4\frac{n-1}{n-2}\|d\theta\|^2\right)\omega_A \\ &= \int_M \alpha^{2\nu-1}\psi^{2\mu-2}\big[\psi^2(K - \alpha S(d^*\varphi, d^*\varphi)) +\end{aligned}$$

$$+\tfrac{1}{4}\big((n-1)/(n-2)\big)\psi\|(4\nu-1)\alpha\psi d\beta-\big(4\mu-n+3-(4\nu-1)\alpha\psi\big)d\psi\|^2+$$
$$+\tfrac{1}{4}\big(2(1-4\nu)\alpha^2\psi^2-\big(2n-6-8\mu+4(n-2)(2\nu-1)\big)\alpha\psi-$$
$$-(n-2)(n-8\mu-3)\big)\|d\psi\|^2+$$
$$+\tfrac{1}{4}\alpha(n-2)\big(2(4\nu-1)\alpha\psi+n-3-8\mu\big)\langle d\psi,d\varphi\rangle^2+$$
$$+\tfrac{1}{2}\alpha\psi\big(2(1-3\nu)\alpha\psi+2\mu-(n-2)(4\nu-1)\big)\langle d\beta,d\psi\rangle+$$
$$+\tfrac{1}{2}\alpha^2\psi^2(1-2\nu)\|d\beta\|^2+\tfrac{1}{2}(n-2)(4\nu-1)\alpha^2\psi\langle d\beta,d\varphi\rangle\langle d\psi,d\varphi\rangle+$$
$$+\alpha\psi\Delta\varphi\langle(2\nu-1)\alpha\psi d\beta-\big(2\mu-(2\nu-1)\alpha\psi\big)d\psi,d\varphi\rangle\big]\omega\,.$$

The formula for K_A follows by direct computation from previous results and $J(B)$ is computed by first using Lemma 1 and then the formula for K_A and a rather lengthy but elementary computation involving partial integration. We now want to choose ν,μ and the functions ψ and φ, such that $J(B)<0$.

For a given φ and a positive constant c, we choose

$$q = 2\mu/(2\nu-1) > 1$$

and define ψ by the equation

$$\beta = c\psi^q-\psi, \qquad \psi \geqq \exp\left(-\frac{\ln c}{q-1}\right).$$

Then

$$\alpha = c^{-1}\psi^{-q} \quad \text{and} \quad d\beta = \left(\frac{q}{\alpha\psi}-1\right)d\psi\,.$$

The last term under the integral for $J(B)$ in Lemma 2 is then zero and we obtain:

$$J(B) = c^{1-2\nu}\int_M\left(K-\alpha\, S(d^*\varphi,d^*\varphi)+\frac{\alpha}{4\psi^2}\,(n-2)(2q+n-3)\langle d\psi,d\varphi\rangle^2+\right.$$
$$\left.+\|d\psi\|^2\frac{1}{4\psi^2}\left(\frac{n-1}{n-2}\,\psi(n-3+q)^2+2(1-q)\alpha\psi-(n-2)(2q+n-3)\right)\right)\omega$$
$$\leqq c^{1-2\nu}\int_M\left((|K|+\|S\|)-\right.$$
$$\left.-\alpha\varphi\left(2\psi\left(\frac{q}{\alpha\psi}-1\right)\right)^{-2}\left(2(n-1)q-2-\frac{n-1}{n-2}\alpha^{-1}(n-3+q)^2\right)\|d\beta\|^2\right)\omega.$$

Now we choose $q=2$ and φ such that $\beta\leqq n^{-1}$. Then we have for c sufficiently large:

$$J(B) \leqq c^{1-2\nu}(\text{const} - f_n(c)\|d\beta\|_0^2) \quad \text{with } f_n(c) > 0 .$$

Thus $J(B) < 0$ for $\|d\beta\|_0^2 = \int_M \|d\beta\|^2 \omega$ sufficiently large.

THEOREM 3. *Let M be a compact C^∞ manifold of dimension $n > 2$. Then there exists a Riemannian metric on M with negative J. Moreover J can be made to take arbitrary values, keeping the volume of M constant.*

PROOF. From the last estimate on $J(B)$ it is clear that J can be made to take arbitrary negative values by a suitable choice of ν and $\|d\beta\|_0$, even if we keep the volume fixed, since it involves only norm conditions on β. By Lemma 1, we can obtain arbitrary positive values using conformal deformations, if we start with a metric with negative J.

3. Critical points.

We want to investigate the existence of critical points of the variation integral J under conformal deformations, leaving the volume fixed. Thus M is now a Riemannian manifold and we keep the notations of Section 2.

We denote by $H^k(M)$, $k = 0, 1, 2, \ldots$, the Sobolev spaces of functions on M. $H^k(M)$ is a Hilbert space with $C^\infty(M)$ as a dense subset and with the inner product

$$\langle u, v\rangle_k = \sum_{\nu=0}^k \int_M \langle \nabla^\nu u, \nabla^\nu v\rangle \omega .$$

We denote the corresponding norm by $\|\cdot\|_k$, and $L^q(M)$ denotes the Lebesgue space of functions on M with the norm

$$|u|_q = (\int_M |u|^q \omega)^{1/q} .$$

We have $J(u) = \int_M (u^2 K + \varkappa \|du\|^2)\omega$ by Lemma 1, with $\varkappa = 4(n-1)/(n-2)$, so J is a quadratic form on $H^1(M)$. With $p = 2n/(n-2)$ as in Lemma 1, we have $2 < p \leqq 6$ and

$$1 - \tfrac{1}{2}n = -(n/p), \quad \text{which implies} \quad H^1(M) \subset L^p(M) ,$$

by the Sobolev embedding Theorem. The volume function $V(u) = \int_M |u|^p \omega = |u|_p{}^p$ is therefore a C^2 function on $H^1(M)$ and the set $\Omega \subset H^1(M)$ with $|u|_p = 1$ is a C^2 submanifold. We have

$$\|\xi\|_0 \leqq |\xi|_p \leqq c_p \|\xi\|_1, \quad \xi \in H^1(M) ,$$

where c_p denotes the smallest constant with this property. Thus $\|u\|_0$ is bounded by 1 for $u \in \Omega$ (for convenience we choose M to have volume $= 1$). Moreover we have

$$J(u) \geqq \varkappa \|u\|_1{}^2 + (\min K - \varkappa)\|u\|_0{}^2 ,$$

so $J(u)$ is bounded below, as stated in Lemma 1. Moreover $\|u\|_1$ is bounded on sets where J is bounded. We have

$$\begin{aligned} dJ(u)\,\xi &= \textstyle\int_M (2Ku\xi + 2\varkappa\langle du, d\xi\rangle)\,\omega = 2\langle Ku - k\Delta u, \xi\rangle_0 ,\\ dV(u)\,\xi &= p\textstyle\int_M u|u|^{p-2}\omega . \end{aligned}$$

Now $\langle \xi,\eta\rangle_1 = \langle \xi,(1-\Delta)\eta\rangle_0$, so the gradients of V and J, as functions on $H^1(M)$, are

$$\begin{aligned} \operatorname{grad} V(u) &= pT(u|u|^{p-2}) ,\\ \operatorname{grad} J(u) &= 2T(Ku - \varkappa\Delta u) = 2\varkappa u - 2T\big((\varkappa - K)u\big) , \end{aligned}$$

with $T = (1-\Delta)^{-1}$. Let $-G(u)$ be the gradient of $J_\Omega = J_{|\Omega}$, the restriction of J to the submanifold Ω of $H^1(M)$. Then

$$G(u) = -\operatorname{grad} J(u) + \lambda(u) \operatorname{grad} V(u) ,$$

with

$$\lambda(u) = \langle \operatorname{grad} J(u), \operatorname{grad} V(u)\rangle_1 \|\operatorname{grad} V(u)\|_1{}^{-2} .$$

We obtain the following estimates for $u, v \in \Omega$:

$$\begin{aligned} &\langle G(u) - G(v), u - v\rangle_1 \\ &\quad = -2\varkappa\|u-v\|_1{}^2 + 2\varkappa\langle \varkappa - K, (u-v)^2\rangle_0 + p\langle \lambda(u)u|u|^{p-2} - \lambda(v)v|v|^{p-2}, u - v\rangle_0 \\ &\quad \leqq -2\varkappa\|u-v\|_1{}^2 + 2\varkappa|k-K|_\infty\|u-v\|_0{}^2 + p|\lambda(u)-\lambda(v)|\;|u-v|_p + \\ &\qquad + p\lambda(v)\langle u|u|^{p-2} - v|v|^{p-2}, u-v\rangle_0 ,\\ &|\langle u|u|^{p-2} - v|v|^{p-2}, u-v\rangle_0| = (p-1)\langle \textstyle\int_0^1 |v + s(u-v)|^{p-2} ds,\ (u-v)^2\rangle_0 \\ &\qquad \leqq (p-1)|u-v|_p{}^2 \textstyle\int_0^1 \big((1-s)|v|_p + s|u|_p\big)^{p-2} ds \\ &\qquad \leqq (p-1)c_p{}^2\|u-v\|_1{}^2 . \end{aligned}$$

Let $u(t)$ be some gradient curve in Ω of the vector field $G(u)$ such that $J(u(t))$ converges to the infimum J_0 of J_Ω for $t \to \infty$. Then $\|G(u(t))\|_1$ converges to 0 and $\|u(t)\|_1$ is bounded, since $J(u(t))$ is bounded. Moreover, we conclude from the equality

$$\langle u, G(u)\rangle_1 = -J(u) + \tfrac{1}{2}p\lambda(u)$$

that $\lambda(u(t))$ converges to $2J_0/p$ for $t \to \infty$. Then, choosing a sequence $t_i \to \infty$ such that $u(t_i) = u_i$ converges in $H^0(M)$, we obtain convergence of u_i in $H^1(M)$ from the estimates for $\|u-v\|_1$ just given, provided

$$(p-1)c_p{}^2 J_0 < \varkappa .$$

The limit u of u_i is then a minimum point of J_Ω and satisfies the equation $G(u)=0$, that is, u is a weak (H^1) solution of the equation

$$Ku - \varkappa \Delta u = J_0 u|u|^{p-2} .$$

Since $|u|$ is also a H^1 function with the same J and V values as u, $|u|$ is also a minimum point for J_Ω and satisfies the same differential equation (although $G(u)=0$ does not imply $G(|u|)=0$ at a nonminimal critical point u).

THEOREM 4. *Suppose the infimum J_0 of J_Ω satisfies the inequality*

$$J_0 < 4\frac{n-1}{n-2}c_p^{-2} ,$$

where $1/c_p$ is the infimum of $\|u\|_1$ with $|u|_p=1$. Then J_Ω takes its minimal value at a C^∞ positive function u, satisfying

$$Ku - 4\frac{n-1}{n-2}\Delta u = J_0 u^{p-1} .$$

Moreover, a multiplication of the metric on M by $u^{4/(n-1)}$ is a conformal deformation of M to a Riemannian manifold with constant scalar curvature equal to J_0 (and volume equal to 1).

PROOF. We have already proved that the minimal value is taken at a function $u \geqq 0$ in $H^1(M)$ satisfying $G(u)=0$. Then u is of class C^∞ by well-known regularity results [4, Theorem 3] and u is positive by the maximum principle for elliptic equations ($\Delta u \leqq cu$, in a domain with $u \leqq 1$).

COROLLARY. *Let M be a compact C^∞ Riemannian manifold of dimension $n \geqq 3$ and with negative total scalar curvature, $\int_M K\omega < 0$. Then M can be conformally deformed to a Riemannian manifold with constant negative scalar curvature.*

PROOF. We have $J_0 \leqq J(1) = \int_M K\omega < 0$, so Theorem 4 can be applied.

THEOREM 5. *Let M be a compact C^∞ manifold of dimension $n \geqq 3$. Then there exists a C^∞ metric on M with constant negative scalar curvature.*

PROOF. By Theorem 3, there is a metric with negative total scalar curvature and then we can apply the Corollary of Theorem 4.

The question is still open, whether it is in general possible to deform a compact Riemannian manifold conformally to a manifold with constant scalar curvature. For example if $K \geqq \varkappa$, then for $u \in \Omega$:

$$J(u) \geqq \varkappa \|u\|_1^2 \geqq \varkappa c_p^{-2} |u|_p^2 = \varkappa c_p^{-2},$$

so the condition in Theorem 4 is not satisfied. Moreover it follows that $J(u)$ does not take its minimum if $\|u\|_1$ does not take its minimum c_p^{-1} and if there is a minimizing sequence u_i for the variational problem of minimizing $\|u\|_1$, keeping $|u|_p$ constant, such that $\|u_i\|_0$ converges to zero. I think this holds, due to the fact that the inclusion $L^p(M) \subset H^1(M)$ is not compact, but I have not made a serious attempt of proving it. In the absence of a minimum point, one could try to look for another critical point. However, a nonminimal critical point is possibly a function taking both signs and moreover probably so, because of the similarity to the problem of finding critical points of $\|du\|_0^2$ on $\|u\|_0^2 = 1$, which are the eigenfunctions of the Laplacian: $\Delta u = -\lambda u$.

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