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AN EXTREMAL PROBLEM INVOLVING FUNCTIONS AND THEIR INVERSES

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Dedicated to G.Pólya on his 85th birthday.

1. Introduction.

For $0<m<1$ let $\mathscr{F}_c(m)$ be the collection of all continuous, strictly decreasing functions $f\colon [0,1] \to [0,1]$ with $\int f = m$. (Here and throughout this article $\int g = \int_0^1 g(x)\,dx$.) We consider the problem of finding

$$I_c(m) \overset{\text{def}}{=} \sup\{\int f f^{-1} : f \in \mathscr{F}_c(m)\} .$$

We shall obtain the desired maximization by solving a more general problem involving the class

$$\mathscr{F}(m) = \{f\colon [0,1] \to [0,1] : f \text{ measurable}, \int f = m\}$$

and a generalization of the notion of inverse. These changes were made because first, there are no "maximal functions" in $\mathscr{F}_c$, while there are such in the class of step functions, and second, we shall use step functions in proving the main inequalities.

For $f \in \mathscr{F}(m)$ and λ the Lebesgue measure define

$$f^*(x) = \lambda\{t \in [0,1] : f(t) \geqq x\} .$$

and set

$$I(m) = \sup\{\int f f^* : f \in \mathscr{F}(m)\} .$$

If $f \in \mathscr{F}_c$, then $f^* = f^{-1}$, and hence the problem of finding $I(m)$ is a natural generalization of that of finding $I_c(m)$ (which will be shown to be equal to $I(m)$).

There is considerable symmetry between f and f^* in our problem: the functions occur in the same way in the integral and, as is shown in Lemma 1, if f is decreasing, $f^{**} = f$ a.e. Let $\mathscr{F}_s(m)$ be the subset of $\mathscr{F}(m)$ consisting of non increasing functions f with $f = f^*$ a.e. These functions will be called *symmetric*, a term motivated by the fact that a function in $\mathscr{F}_c \cap \mathscr{F}_s$ has a graph symmetric with respect to the line $y = x$ in the (x,y)-plane. It will be shown that $I(m)$ is realized in the class $\mathscr{F}_s(m)$.

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This work is a generalization of a discrete problem involving certain matrices with integer entries [2].

Before stating the main result, we give some more definitions. We denote by $\mathscr{F}_d(m)$ the subset of $\mathscr{F}(m)$ consisting of monotone non increasing (henceforth called "decreasing") functions. The symbols I_d and I_s will denote the corresponding maxima in $\mathscr{F}_d$ and $\mathscr{F}_s$ respectively. For $0<m<1$, we define the functions φ_m and Φ_m on $[0,1]$ by setting

$$\begin{aligned} \varphi_m(x) &= 1, && 0 \leqq x \leqq 1-(1-m)^{\frac{1}{2}}, \\ &= 1-(1-m)^{\frac{1}{2}}, && 1-(1-m)^{\frac{1}{2}} < x \leqq 1, \end{aligned} \tag{1}$$

$$\begin{aligned} \Phi_m(x) &= m^{\frac{1}{2}}, && 0 \leqq x \leqq m^{\frac{1}{2}}, \\ &= 0, && m^{\frac{1}{2}} < x \leqq 1. \end{aligned} \tag{2}$$

THEOREM 1. *For* $0<m<1$,

$$\begin{aligned} I(m) = I_d(m) = I_s(m) = I_c(m) &= \int \varphi_m{}^2 = 2m-1+(1-m)^{\frac{3}{2}}, && 0<m<\tfrac{1}{2}, \\ &= \int \varphi_{\frac{1}{2}}{}^2 = \int \Phi_{\frac{1}{2}}{}^2 = (\tfrac{1}{2})^{\frac{3}{2}}, && m=\tfrac{1}{2}, \\ &= \int \Phi_m{}^2 = m^{\frac{3}{2}}, && \tfrac{1}{2}<m<1. \end{aligned}$$

These functions are essentially (i.e. up to an equivalence in $L_1(0,1)$*) the only ones in* $\mathscr{F}(m)$ *which realize* $I(m)$.

Starting with $f \in \mathscr{F}(m)$ we show that there exists a $g \in \mathscr{F}_d(m)$ such that $\int gg^* \geqq \int ff^*$. Then we show that g may be replaced by an $h \in \mathscr{F}_s(m)$ with $\int h^2 \geqq \int gg^*$. This is proved in Section 3 by a general integral identity which is to our knowledge new. Next, by means of some geometrical lemmas we pass to symmetric step functions having at most three steps. Finally, by a standard maximization procedure, we obtain our phi functions.

2. Reduction to $\mathscr{F}_d$.

We begin by giving some properties of "star functions", each of which is obviously true for functions in $\mathscr{F}_c$.

LEMMA 1. *Let* $f \in \mathscr{F}(m)$ *and* $g \in \mathscr{F}_d(m)$. *Then*

(A) $f^* \in \mathscr{F}_d(m)$,

(B) $g^{**}=g$ a.e. *Moreover, if* g *is left continuous, then* $g^{**}=g$ *everywhere.*

PROOF. It is easy to see that f^* is decreasing. The definition of the Lebesgue integral and integration by parts give

$$m = \int f = -\int_0^{1+} t\, df^*(t) = \int f^*,$$

and thus (A) is proved.

A decreasing function may be made continuous from the left by changing its values only on a set of measure zero. We assume then that g is left continuous. For any $t \in [0,1]$ we have

$$g^*(x) = \lambda\{y: g(y) \geqq x\} \geqq t \Leftrightarrow g(t) \geqq x .$$

Applying the same reasoning to g^* and then to g, we obtain the relations

$$g^{**}(x) \geqq t \Leftrightarrow g^*(t) \geqq x \Leftrightarrow g(x) \geqq t .$$

We therefore conclude that $g^{**} = g$.

The following result is a special case of Theorem 378 in [1]. For this case the proof is very simple.

LEMMA 2. *Let $f \in \mathscr{F}(m)$ and let $g = f^{**}$. Then $g \in \mathscr{F}_d(m)$ and satisfies $\int gg^* \geqq \int ff^*$.*

PROOF. Lemma 1(A) implies f^* and g belong to $\mathscr{F}_d(m)$, and $g^* = f^*$ by Lemma 1(B). Since g is decreasing, for each a and y in $[0,1]$ we have

$$\lambda\{x\in[0,a]: g(x) \geqq y\} = \min\{a, g^*(y)\} .$$

Thus

$$\lambda\{x\in[0,a]: f(x) \geqq y\} \leqq \lambda\{x\in[0,a]: g(x) \geqq y\}$$

and consequently

$$F(a) \stackrel{\text{def}}{=} \int_0^a f(x)dx \leqq \int_0^a g(x)dx \stackrel{\text{def}}{=} G(a) .$$

Finally we have

$$\int_0^1 (g-f)f^* = \int_0^{1+} (G-F)'f^* = \int_0^{1+} (G-F)(-df^*) \geqq 0 .$$

REMARK. We would have liked to assert that $\int gg^* > \int ff^*$ if f was not (equal a.e. to) a decreasing function. This is, however, false as shown by the following function:

$$\begin{aligned} f(x) &= \tfrac{1}{2}, \quad 0 \leqq x \leqq \tfrac{1}{4} , \\ &= 1, \quad \tfrac{1}{4} < x \leqq \tfrac{1}{2} , \\ &= 0, \quad \tfrac{1}{2} < x \leqq 1 . \end{aligned}$$

We shall have to return to this point in the course of the uniqueness argument (cf. Section 5).

Let $f \in \mathscr{F}_d(m)$ for $m \in (0,1)$. There is a unique number ξ such that $f(x) > x$ for $x < \xi$ and $f(x) < x$ for $x > \xi$. We call ξ the *crossing point* of f, since, at least in the case $f \in \mathscr{F}_c$, the graph of f meets the line $y = x$ at the point (ξ, ξ). The symbol $\xi = \xi(f)$ will be reserved for the crossing point of f, and

$$\mathscr{F}_d(m, \xi) = \{f \in \mathscr{F}_d(m): f \text{ has crossing point } \xi\} .$$

The class $\mathscr{F}_s(m,\xi)$ is defined analogously. We note that a function $f \in \mathscr{F}_s$ with crossing point ξ is uniquely defined up to a set of measure zero by its values on $[0,\xi]$.

3. An integral identity and reduction to $\mathscr{F}_s$.

For every $f \in \mathscr{F}_d(m,\xi)$, define g, *the symmetrization* of f, by

$$g(x) = \tfrac{1}{2}(f(x)+f^*(x)), \quad 0 \leqq x \leqq \xi, \qquad = (\tfrac{1}{2}(f+f^*))^*(x), \quad \xi < x \leqq 1.$$

One verifies easily that $g \in \mathscr{F}_s(m,\xi)$. Since g is symmetric we have

$$\int g = m = 2\int_0^\xi g - \xi^2.$$

THEOREM 2. *Let $f \in \mathscr{F}_d(m,\xi)$ and let g be its symmetrization. Then*

(3) $$\int_0^1 g^2 = \int_0^1 ff^* + \tfrac{1}{4}\int_0^\xi |f-f^*|^2 + \tfrac{1}{2}\int_\xi^1 |f-f^*|^2.$$

REMARKS. 1. In our original draft, (3) was given as an inequality with the last term missing and the proof was effected via step functions and approximations. We are indebted to the referee for the present form of Theorem 2.

2. The theorem implies that if f is not symmetric, then its symmetrization g satisfies $\int g^2 > \int ff^*$.

3. The inequality "$g^2 \geqq ff^*$" is not valid pointwise, as is shown by taking

$$f(x) = 1, \quad 0 \leqq x \leqq \tfrac{1}{2}, \qquad = \tfrac{1}{4}, \quad \tfrac{1}{2} < x \leqq 1.$$

Then $ff^* > g^2$ in a neighborhood of $\frac{4}{5}$.

COROLLARY (for functions in $\mathscr{F}_c$). *Let f be a continuous, strictly decreasing function from* $[0,1]$ *onto* $[0,1]$ *with inverse f^{-1}. Let ξ be the point such that $f(\xi) = f^{-1}(\xi) = \xi$. Let the symmetrization g be defined by*

$$g(x) = \tfrac{1}{2}(f(x)+f^{-1}(x)), \quad 0 \leqq x \leqq \xi, \qquad = (\tfrac{1}{2}(f+f^{-1}))^{-1}(x), \quad \xi < x \leqq 1.$$

Then

$$\int_0^1 g^2 = \int_0^1 ff^{-1} + \tfrac{1}{4}\int_0^\xi |f-f^{-1}|^2 + \tfrac{1}{2}\int_\xi^1 |f-f^{-1}|^2.$$

PROOF. We have $\int g^2 = \int_0^\xi g^2 + \int_\xi^1 g^2$. The first term equals

$$\int_0^\xi (\tfrac{1}{2}(f+f^*))^2 = \int_0^\xi ff^* + \int_0^\xi (\tfrac{1}{2}(f-f^*))^2.$$

For the second term we use the following formula, which is an easy consequence of the definition of the Lebesgue integral. Let $h \in \mathscr{F}_d$, n be a positive integer and $a < b$ be points for which h is not constant in any neighborhood of a or b. Then

$$\int_a^b h^n = \int_{h(a+)+}^{h(b-)} t^n \, dh^*(t) \,. \tag{4}$$

We assume, without loss of generality, that the given function f is continuous from the left, and hence $f=f^{**}$ everywhere. We shall apply identity (4) to $\int_\xi^1 g^2$. We note first that either $g(1-)>0$ and g is constant on $[0,g(1-))$ or $g(1-)=0$. Next we note that either $g(\xi+)<\xi$ and g is constant on $(g(\xi+),\xi)$ or $g(\xi+)=\xi$. These facts and the symmetry of g imply that

$$\int_\xi^1 g^2 = \int_{g(\xi+)+}^{g(1-)} t^2 \, dg^*(t) = \int_{\xi-}^0 t^2 \, dg(t) = \tfrac{1}{2}\int_{\xi-}^0 t^2 \, df(t) + \tfrac{1}{2}\int_{\xi-}^0 t^2 \, df^*(t) \,.$$

If we apply (4) to each of the last integrals, again handling the end points with care, we find that

$$\int_\xi^1 g^2 = \tfrac{1}{2}\int_\xi^1 f^{*2} + \tfrac{1}{2}\int_\xi^1 f^2 = \int_\xi^1 ff^* + \tfrac{1}{2}\int_\xi^1 (f-f^*)^2 \,.$$

Putting the two parts together we obtain the desired identity.

4. Proof of Theorem 1 for $\mathscr{F}_s$.

The proof is effected in two stages. First, given $f \in \mathscr{F}_s(m,\xi)$ we find a step function $\tilde{f}$ in $\mathscr{F}_s(m,\xi)$ having at most two steps in $(0,\xi)$ and such that $\int \tilde{f}^2 \geqq \int f^2$. This is accomplished by use of two lemmas about centers of mass of regions. Then we vary ξ to obtain the supremum.

LEMMA 3. *Let $g,h \in \mathscr{F}_s(m)$ be essentially distinct and let $\xi(g)$, $\xi(h)$ be their respective crossing points. Let $c \in (0,\xi(g))$ be such that $g \geqq h$ on $(0,c)$ and $h \geqq g$ on $(c,\xi(g))$. Define regions R_1 and R_2 in the plane by setting*

$$R_1 = \{(x,y) : 0<x<c,\ h(x)<y<g(x)\} \,,$$
$$R_2 = \{(x,y) : c<x<\xi(h),\ \max(g(x),x)<y<h(x)\} \,.$$

Let (x_i,y_i) be the mass center of R_i, $i=1,2$, and let μ be the slope of the line joining the mass centers. Then

$$\operatorname{sgn}(\mu+1) = \operatorname{sgn}\left(\int h^2 - \int g^2\right) .$$

PROOF. R_1 and R_2 have the same two-dimensional Lebesgue measure $A>0$. Thus

$$\mu = \frac{(2A)^{-1}\left[\int_c^{\xi(g)}(h^2-g^2)\,dx + \int_{\xi(g)}^{\xi(h)}(h^2-x^2)\,dx\right] - (2A)^{-1}\int_0^c (g^2-h^2)\,dx}{(2A)^{-1}\left[\int_{\xi(h)}^{h(c)}(h^2-g^2)\,dy + \int_{\xi(g)}^{\xi(h)}(y^2-g^2)\,dy\right] - (2A)^{-1}\int_{h(c)}^{g(0)}(g^2-h^2)\,dy} \,.$$

Therefore

$$\mu+1 = D^{-1}\int(h^2-g^2), \quad \text{where } D = 2\iint_{R_2} x\,dx\,dy - \iint_{R_1} x\,dx\,dy > 0 \,.$$

Now we introduce two special functions in $\mathscr{F}_s(m,\xi)$, $s=s_\xi$, having a single step in $(0,\xi)$, and $t=t_\xi$, having two steps in $(0,\xi)$. For notational convenience we take $h=(m+\xi^2)/(2\xi)$ and $d=(m-\xi^2)/(2-2\xi)$. The functions are defined by setting

$$\begin{aligned} s_\xi(x) &= h, \quad 0\leqq x\leqq \xi\,, \\ &= \xi, \quad \xi<x\leqq h\,, \\ &= 0, \quad h<x\leqq 1\,, \\ t_\xi(x) &= 1, \quad 0\leqq x\leqq d\,, \\ &= \xi, \quad d<x\leqq \xi\,, \\ &= d, \quad \xi<x\leqq 1\,. \end{aligned}$$

We note that every $f\in\mathscr{F}_s(m,\xi)$ except s_ξ satisfies $f(0+)>h>f(\xi-)$, and every $f\in\mathscr{F}_s(m,\xi)$ except t_ξ satisfies $1>f(d-)\geqq f(d+)>\xi$.

LEMMA 4. *Let $f\in\mathscr{F}_s(m,\xi)$ and assume $f\neq s_\xi$, $f\neq t_\xi$ (in L_1). Let c be a number in $(0,\xi)$ such that $f\geqq s_\xi$ on $(0,c)$ and $s_\xi\geqq f$ on (c,ξ). Define regions $R_1,\ldots,R_4$ in the plane by setting*

$$\begin{aligned} R_1 &= \{(x,y)\colon 0<x<c,\ s(x)<y<f(x)\}\,, \\ R_2 &= \{(x,y)\colon c<x<\xi,\ f(x)<y<s(x)\}\,, \\ R_3 &= \{(x,y)\colon 0<x<d,\ f(x)<y<t(x)\}\,, \\ R_4 &= \{(x,y)\colon d<x<\xi,\ t(x)<y<f(x)\}\,. \end{aligned}$$

Let (x_i,y_i) be the mass center of R_i, $1\leqq i\leqq 4$, and let μ_{ij} be the slope of the line joining the i-th and j-th mass centers. Then $\mu_{34}<\mu_{12}$.

PROOF. We have

$$x_3 \geqq \tfrac{1}{2}d, \qquad y_3 \geqq \tfrac{1}{2}(1+f(d-)) \tag{5}$$

$$x_4 \leqq \tfrac{1}{2}(d+\xi), \quad y_4 \leqq \tfrac{1}{2}(f(d+)+\xi)\,, \tag{6}$$

and so

$$\mu_{34} \leqq \frac{\frac{1}{2}(f(d+)+\xi)-\frac{1}{2}(1+f(d-))}{\frac{1}{2}\xi} \leqq \frac{\xi-1}{\xi}.$$

Indeed, $\mu_{34}<(\xi-1)/\xi$. This is obvious if f has a discontinuity at d. In the other case, since $f\neq$ constant on $(0,\xi)$, it is not constant on one of the subintervals $(0,d)$, (d,ξ). Then we would have strict inequality in one of (5), (6).

Similar inequalities show that $\mu_{12}\geqq(\xi-1)/\xi$. (Here, however, the inequality need not be strict.) Thus $\mu_{12}>\mu_{34}$.

The geometrical lemmas will now be applied to reduce general functions in $\mathscr{F}_s(m,\xi)$ to functions with at most two steps in $(0,\xi)$.

LEMMA 5. *Let $f\in\mathscr{F}_s(m,\xi)$ be given and assume $f\neq s_\xi$, $f\neq t_\xi$. Then*

$$\max\left(\textstyle\int s_\xi^2, \int t_\xi^2\right) > \textstyle\int f^2\,.$$

PROOF. We use the notation of the preceding lemma. If $\mu_{12} > -1$, then by Lemma 3 we have $\int s_\xi^2 > \int f^2$. If $\mu_{12} \leqq -1$, then Lemma 4 implies that $\mu_{34} < -1$. Using Lemma 3 again, we see that $\int t_\xi^2 > \int f^2$. Thus one of $\int s_\xi^2$, $\int t_\xi^2$ is larger than $\int f^2$.

We now vary ξ, considered as a parameter, to maximize the L_2 norm of s_ξ. We have

$$\int s_\xi^2 = \tfrac{1}{4}m^2/\xi + m\xi - \tfrac{1}{4}\xi^3 \overset{\text{def}}{=} S(\xi) .$$

The parameter ξ is constrained to satisfy

$$(7) \qquad 1-(1-m)^{\frac{1}{2}} \leqq \xi \leqq m^{\frac{1}{2}} .$$

Indeed, since f is decreasing

$$\xi^2 \leqq \textstyle\int_0^\xi f \leqq \int_0^1 f = m ,$$

and since f is symmetric and bounded by 1,

$$\xi \geqq \textstyle\int_0^\xi f = \tfrac{1}{2}(\xi^2+m) ,$$

which implies that $\xi \geqq 1-(1-m)^{\frac{1}{2}}$.

Now $S'(\xi)$ is negative for $0<\xi<(\frac{1}{3}m)^{\frac{1}{2}}$ and positive for $(\frac{1}{3}m)^{\frac{1}{2}}<\xi<m^{\frac{1}{2}}$. Thus $S(\xi)$ achieves its maximum on $[1-(1-m)^{\frac{1}{2}}, m^{\frac{1}{2}}]$ at one of the end-points, which is to say that the maximal value of S is given by the L_2 norm of one of φ_m, Φ_m.

Next, we maximize $\int |t_\xi|^2$. We have

$$(8) \qquad \int t_\xi = m = \xi^2 + 2d(1-\xi) ,$$

from which we see that d decreases strictly from $1-(1-m)^{\frac{1}{2}}$ to 0 as ξ increases from $1-(1-m)^{\frac{1}{2}}$ to $m^{\frac{1}{2}}$. Let

$$T(\xi) \overset{\text{def}}{=} \int t_\xi^2 = d + \xi^2(\xi - d) + d^2(1-\xi) .$$

Using the constraint (8) we find that

$$T'(\xi) = (\xi-d)(2\xi-d-1) ,$$

and T' has zeros when $\xi = d$ (which occurs at $\xi = 1-(1-m)^{\frac{1}{2}}$) and when $\xi = \frac{1}{2}(1+d)$ (which occurs at $\xi = 1-(\frac{1}{3}(1-m))^{\frac{1}{2}}$). Now $\xi = 1-(\frac{1}{3}(1-m))^{\frac{1}{2}}$ might or might not lie in the open interval $(1-(1-m)^{\frac{1}{2}}, m^{\frac{1}{2}})$, but in any case $T(\xi)$ does not attain its maximum on this open interval. This is so since $T'(\xi)<0$ for $1-(1-m)^{\frac{1}{2}}<\xi<1-(\frac{1}{3}(1-m))^{\frac{1}{2}}$ and $T'(\xi)>0$ for $1-(\frac{1}{3}(1-m))^{\frac{1}{2}}<\xi<1$. Thus the maximum of T on $[1-(1-m)^{\frac{1}{2}}, m^{\frac{1}{2}}]$ is achieved by one of the quantities

$$\int \varphi_m^2 = T(1-(1-m)^{\frac{1}{2}}) \quad \text{or} \quad \int \Phi_m^2 = T(m^{\frac{1}{2}}) .$$

It remains to decide when φ_m gives a maximum and when Φ_m does. This may be done easily by use of Rolle's Theorem. We have

$$\int \varphi_m{}^2 = 2m-1+(1-m)^{\frac{3}{2}}, \quad \int \Phi_m{}^2 = m^{\frac{3}{2}}.$$

Now $\int \varphi_m{}^2$ and $\int \Phi_m{}^2$ are functions of m which are equal at $m=0$, $\frac{1}{2}$, and 1. If these functions agreed elsewhere in $[0,1]$, their second derivatives would agree at two or more points of $(0,1)$. It is easy to see, however, that the second derivatives agree only at $m=\frac{1}{2}$. By inspection of first derivatives near 0 and 1 we see that

$$\int \varphi_m{}^2 > \int \Phi_m{}^2 \quad \text{for } 0<m<\tfrac{1}{2}$$

and

$$\int \varphi_m{}^2 < \int \Phi_m{}^2 \quad \text{for } \tfrac{1}{2}<m<1.$$

Lemma 5 and the maximization of S and T show our maximal functions to be essentially unique in $\mathscr{F}_s(m)$.

5. Completion of the proof of Theorem 1.

The equality of I, I_d, I_s, and I_c is a consequence of Lemma 2, Theorem 2, and the fact that φ_m and Φ_m can be approximated by functions in $\mathscr{F}_c$.

The essential uniqueness of φ_m and Φ_m in $\mathscr{F}_d$ follows from Theorem 2 and the results of Section 4. Recall that the inequality of Lemma 2 need not be strict for functions which are not in $\mathscr{F}_d(m)$. We shall therefore use the known form of the phi functions to show them to be essentially the only maximal functions in $\mathscr{F}(m)$.

Let f be a maximal function. By Lemma 2, f^{**} is also a maximal function, and it is decreasing. Thus f^{**} is a phi function, and since phi functions are symmetric,

$$f^{***} = f^{*} = \text{a phi function}.$$

If $m>\frac{1}{2}$, $f^{**}=\Phi_m$, and if f were not (equal a.e. to) a decreasing function, f would vanish on a subset of $[0,m^{\frac{1}{2}}]$ having positive measure, and we would have $\int ff^{*}<\int \Phi_m{}^2$. Thus $f=\Phi_m$ a.e. for $m>\frac{1}{2}$.

If $m<\frac{1}{2}$, $f^{*}=\varphi_m$ and we define four sets of real numbers

$$\begin{aligned}
A &= \{t\in[0,1-(1-m)^{\frac{1}{2}}]\colon f(t)=1\},\\
B &= \{t\in[0,1-(1-m)^{\frac{1}{2}}]\colon f(t)=1-(1-m)^{\frac{1}{2}}\},\\
C &= \{t\in(1-(1-m)^{\frac{1}{2}},1]\colon f(t)=1\},\\
D &= \{t\in(1-(1-m)^{\frac{1}{2}},1]\colon f(t)=1-(1-m)^{\frac{1}{2}}\}.
\end{aligned}$$

Since the sets are disjoint and

$$\lambda(A\cup B) = 1-(1-m)^{\frac{1}{2}} = \lambda(A\cup C),$$

we have $\lambda(B) = \lambda(C)$. An easy calculation gives

$$\int ff^* = 2m - 1 + (1-m)^{\frac{3}{2}} - (1-m)\lambda(B) ,$$

which is strictly smaller than $\int \varphi_m{}^2$ unless $\lambda(B) = 0$.

Thus, if $m < \frac{1}{2}$ and $f \in \mathscr{F}(m)$ is maximal, then $f = \varphi_m$, a.e.

If $m = \frac{1}{2}$, then f^{**} is one of the phi functions, and the above reasoning shows f also to be equal a.e. to a phi function.

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