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NONEXISTENCE OF A CONTINUOUS RIGHT INVERSE FOR LINEAR PARTIAL DIFFERENTIAL OPERATORS WITH CONSTANT COEFFICIENTS

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In this paper $P(D)$ will denote a linear partial differential operator of degree m with constant coefficients and $n \geqq 2$ independent variables. We let $P_m(D)$ denote the principal part of $P(D)$. We suppose that $P(D)$ acts on the Fréchet space $C^\infty(\Omega)$ of infinitely differentiable functions defined on the nonempty open subset Ω of R^n.

A theorem has evolved from the work of B. Malgrange [1] which states that $P(D)$ maps $C^\infty(\Omega)$ onto itself whenever Ω is $P(D)$-convex (Hörmander [2, Corollary 3.5.2]). Thus, it is meaningful to ask whether or not there is a continuous linear transformation R of $C^\infty(\Omega)$ into itself such that $P(D)Rf=f$ for all f in $C^\infty(\Omega)$.

If $P(D)$ is hyperbolic in the direction N, then by Theorem 5.6.4 of Hörmander [2], there does exist a continuous right inverse of $P(D)$ on the space $C^\infty(\mathsf{R}^n)$. In fact, for any f in $C^\infty(\mathsf{R}^n)$, we let $Rf=u$ denote that unique member of $C^\infty(\mathsf{R}^n)$ which satisfies $P(D)u(x)=f(x)$ for all x in R^n and $\langle N,D\rangle^k u(x)=0$ for all x in R^n satisfying $\langle x,N\rangle=0$ for $k=0,1,\ldots, m-1$, where m is the degree of $P(D)$.

On the other hand, if $P(D)$ were elliptic and nonconstant, a result of A. Grothendieck (Trèves [3, Theorem C.1]) shows that $P(D)$ has no continuous right inverse on the space $C^\infty(\Omega)$ for any nonempty open subset Ω of R^n. Also the result in [4] shows that if $P(D)$ were parabolic, it could have no continuous right inverse in $C^\infty(\Omega)$ for any nonempty open subset Ω of R^n.

It is the objective of this paper to extend the results of the previous paragraph to a wider class of partial differential operators. The author's original proof used the involved techniques of [4]. The much simpler proofs given here are due to L. Hörmander.

In what follows, we assume that there is some vector N in $\mathsf{R}^n-\{0\}$ satisfying the condition that $\langle N,\xi\rangle=0$ for all ξ in R^n for which $P_m(\xi)=0$.

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This property is clearly invariant under a change of coordinates of the form

$$x = yA + y^{(0)},$$

where A is a nonsingular n by n matrix and $y^{(0)}$ is a member of R^n. Also, assume that the partial differential operators we consider have the property that for some open subset Ω of R^n, there is a continuous linear transformation R of $C^\infty(\Omega)$ into itself such that $P(D)Rf=f$ for all f in $C^\infty(\Omega)$.

Lemma 1. *For every relatively compact open subset V of Ω, there is a compact subset K of Ω such that $Rf=0$ in V if f vanishes identically in K.*

Proof. This is an immediate consequence of the continuity of R.

Lemma 2. *Let us suppose that W is an open subset of Ω and u is a member of $C^\infty(W)$ such that $P(D)u=0$ in W. Suppose that $u=0$ in an open subset V of W. Then $u=0$ in the open subset V' of Ω defined to be the set of all x' in W such that $\xi \in \mathsf{R}^n$ and $P_m(\xi)=0$ implies $\langle x'-x,\xi\rangle=0$, and*

$$\{tx+(1-t)x' : 0 \leqq t \leqq 1\} \subset W$$

for some x in V.

Proof. This follows from Theorem 5.3.3 of Hörmander [2].

Now choose the set V in Lemma 1 as an open ball

$$V = \{x : |x-x^{(0)}| < r\} \subset\subset \Omega .$$

Let N be a vector in R^n such that $\langle N,\xi\rangle=0$ whenever $P_m(\xi)=0$. Let

$$\bar{t} = \sup\{t : 0 \leqq s < t \text{ implies } x^{(0)}+sN \in \Omega\} .$$

Now let t_1 be a positive number less than $\bar{t}$ such that if $t_1 \leqq s < \bar{t}$, then $x^{(0)}+sN \in \Omega - K$. Let t_2 be a member of $(t_1,\bar{t})$. For sufficiently small δ with $0<3\delta<r$, the 3δ-neighborhoods of the compact sets

$$I = \{x=x^{(0)}+tN : 0 \leqq t \leqq t_2\}, \quad J = \{x=x^{(0)}+tN : t_1 \leqq t \leqq t_2\}$$

belong to Ω and $\Omega - K$, respectively. Let f denote an arbitrary member of $C_c^\infty(\mathsf{R}^n)$ such that

$$f(x) = 0 \quad \text{if } \langle x-x^{(0)},N\rangle \leqq t_1 ,$$
$$f(x) = 0 \quad \text{if } |x-(x^{(0)}+tN)| \geqq \delta \text{ for all real } t ,$$

and such that the support of f is contained in a compact subset of the 3δ-neighborhood of J. Let H be the halfspace defined by

$$H = \{x \in \mathbb{R}^n : \langle x - x^{(0)}, N \rangle < t_2\} .$$

Assume that $\langle N,N\rangle = 1$. Define $u(x) = Rf(x)$ if x is in Ω, $\langle x - x^{(0)}, N\rangle \geqq 0$, and $|x - x^{(0)} - tN| \leqq 2\delta$ for some t in $(0,t_2)$. Define $u(x) = 0$ otherwise. Let W be the set,

$$W = \{x \in \Omega : \langle x - x^{(0)}, N\rangle \geqq 0,\ \delta < |x - x^{(0)} - tN| < 2\delta \text{ for some } t \in (0,t_2), \text{ and } \langle x - x^{(0)}, N\rangle < t_2\} .$$

Then Lemma 2 implies that $u(x)$ vanishes in W. Thus $u(x)$ is in $C^\infty(H)$ and $P(D)u(x) = f(x)$ for all x in H. Change coordinates so that $N = (0,\ldots,0,1)$ and so that $x_n = 0$ in the new coordinate system is equivalent to $\langle x - x^{(0)}, N\rangle = t_1$ in the old coordinate system. Then what we have shown is that for every $x^{(0)}$ with $x_n{}^{(0)} = 0$ and for every f in $C_c{}^\infty(\mathbb{R}^n)$ which vanishes for $x_n \leqq 0$ and for x whose distance from the line through $x^{(0)}$ parallel to the x_n-axis is $\geqq \delta$, there is a u in $C^\infty(H_T)$ which satisfies $Q(D)u(x) = f(x)$ for all x in $H_T = \{x : x_n < T\}$ and which also vanishes when $x_n \leqq 0$ and when the distance from x to the line through $x^{(0)}$ parallel to the x_n-axis is $\geqq \delta$, where $Q(D)$ is the representation of $P(D)$ in the new coordinate system. Now we can prove easily the following.

Lemma 3. *Let $H_T = \{x \in \mathbb{R}^n : x_n < T\}$. Then for every f in $C^\infty(\mathbb{R}^n)$ vanishing for $x_n \leqq 0$, there is a u in $C^\infty(H_T)$ such that $u(x) = 0$ for $x_n \leqq 0$ and $Q(D)u(x) = f(x)$ for every x in H_T.*

Proof. Let $\{\psi_\alpha : \alpha \in \mathfrak{A}\}$ be a partition of unity of the hyperplane $x_n = 0$ such that for each $\alpha \in \mathfrak{A}$ there is an $x^{(\alpha)}$ in $\mathbb{R}^{n-1}$ and a $\delta > 0$ such that

$$\operatorname{supp} \psi_\alpha \subset \{x' \in \mathbb{R}^{n-1} : \sum_{k=1}^{n-1} (x_k{}' - x_k{}^{(\alpha)})^2 \leqq \delta^2\} = U_\alpha .$$

Assume that $\{U_\alpha, \alpha \in \mathfrak{A}\}$ is a locally finite covering of $x_n = 0$. Let f be an arbitrary member of $C^\infty(\mathbb{R}^n)$ vanishing for $x_n \leqq 0$. Write

$$f(x) = \textstyle\sum_{\alpha \in \mathfrak{A}} f_\alpha(x), \quad \text{where} \quad f_\alpha(x) = \psi_\alpha(x') f(x'; x_n) .$$

Then there is a u_α in $C^\infty(H_T)$ such that $u_\alpha(x) = 0$ if $x_n \leqq 0$ or if $x' \notin U_\alpha$, and satisfying $Q(D)u_\alpha = f_\alpha$. Then by the local finiteness of the supports of $\{U_\alpha\}$, it follows that $u = \sum_{\alpha \in \mathfrak{A}} u_\alpha$ is a member of $C^\infty(H_T)$, $u(x) = 0$ for $x_n \leqq 0$ and $Q(D)u = f$.

We now proceed to show that $Q(D)$ is hyperbolic in the direction of the x_n-axis which will imply that $P(D)$ is hyperbolic in the direction N. To do so we use the following lemma.

LEMMA 4. *Let $Q(D)$ be a linear partial differential operator with constant coefficients in R^n with principal part $Q_m(D)$. Assume $Q_m(0,\ldots,0,1)\neq 0$. Assume that there is a $T>0$ such that for every f in $C^\infty(\mathsf{R}^n)$ vanishing for $x_n \leqq 0$ there is a u in $C^\infty(H_T)$ such that $Q(D)u(x)=f(x)$ for all x in H_T and $u(x)=0$ for $x_n \leqq 0$. Then $Q(D)$ is hyperbolic in the direction of the x_n-axis.*

PROOF. Let $t=T-\varepsilon$, where $0<T-2\varepsilon$. Let $\psi(x_n)$ be a member of $C^\infty(\mathsf{R}^1)$ such that $\psi(x_n)=1$ for $x_n \leqq t$ and $\psi(x_n)=0$ for $x_n \geqq T$. Let f be a member of $C^\infty(\mathsf{R}^n)$ which vanishes for $x_n \leqq 0$. Let $u(x)$ be a member of $C^\infty(H_T)$ such that $Q(D)u(x)=f(x)$ for all x in H_T and $u(x)=0$ for $x_n \leqq 0$. Set $v_0(x)=\psi(x_n)u(x)$. Then v_0 is a member of $C^\infty(\mathsf{R}^n)$. Now

$$f(x)-Q(D)v_0(x)=0 \quad \text{for } x_n \leqq t\,.$$

Reapplying the assumptions of the lemma after a translation, we deduce that there is a u_1 in $C^\infty(H_{t+T})$ such that

$$Q(D)u_1(x) = f(x)-Q(D)v_0(x) \quad \text{in } H_{t+T}$$

and

$$u_1(x)=0 \quad \text{for } x_n \leqq t\,.$$

Then we set

$$v_1(x) = \psi(x_n-t)\,u_1(x)$$

and conclude that

$$Q(D)\big(v_0(x)+v_1(x)\big) = f(x) \quad \text{for } x_n \leqq 2t\,,$$

$$v_0(x) = 0 \quad \text{for } x_n \leqq 0\,,$$
$$v_1(x) = 0 \quad \text{for } x_n \leqq t\,.$$

Thus, assume that we have chosen functions $v_0,v_1,\ldots,v_k$ in $C^\infty(\mathsf{R}^n)$ such that

$$Q(D)(v_0+v_1+\ldots+v_k)(x) = f(x) \quad \text{for } x_n \leqq (k+1)t\,,$$

and $v_j(x)=0$ for $x_n \leqq jt$. Let

$$w_k(x) = v_0(x)+v_1(x)+\ldots+v_k(x)\,.$$

Now we again reapply the assumptions of the lemma after a translation to deduce that there is a u_{k+1} in $C^\infty(H_{(k+1)t+T})$ such that $u_{k+1}(x)=0$ for $x_n \leqq (k+1)t$ and

$$Q(D)u_{k+1}(x) = f(x) - Q(D)w_k(x) \quad \text{for } x_n \leqq (k+1)t+T .$$

Set $v_{k+1}(x) = \psi(x_n - (k+1)t)u_{k+1}(x)$. Then $v_{k+1}(x) \in C^\infty(\mathsf{R}^n)$, $v_{k+1}(x) = 0$ for $x_n \leqq (k+1)t$ and

$$Q(D)(w_k(x) + v_{k+1}(x)) = f(x) \quad \text{for } x_n \leqq (k+2)t .$$

Set

$$u(x) = v_0(x) + v_1(x) + \dots$$

and note that $u(x) \in C^\infty(\mathsf{R}^n)$, $u(x) = 0$ for $x_n \leqq 0$, and $Q(D)u(x) = f(x)$. Thus, it follows by Lemma 5.4.1 of Hörmander [2] that $Q(D)$ is hyperbolic in the direction $(0,\dots,0,1)$.

But $Q_m(\xi) = 0$ implies $\xi_n = 0$, since $(0,\dots,0,1)$ is orthogonal to every characteristic. Now $Q_m(D)$ is hyperbolic, since $Q(D)$ is hyperbolic by Theorem 5.5.2 of Hörmander [2]. Thus, Theorem 5.5.3 of Hörmander [2]. implies that if $Q_m(D)$ were not equal to $c\xi_n{}^m$, $c \neq 0$, then there would exist a nontrivial real solution τ of the equation

$$Q_m(\xi_1,\dots,\xi_{n-1},\tau) = 0$$

for some $(\xi_1,\dots,\xi_{n-1})$ in R^{n-1}. This is impossible. Hence $Q_m(\xi) = c\xi_n{}^m$ for some $c \in \mathsf{C} - \{0\}$. Now Theorem 5.5.8 of Hörmander [2] tells us that the degree of $Q(\tau\xi + \eta)$ with respect to τ for a fixed real ξ and indeterminate η never exceeds that of $Q_m(\tau\xi + N)$. One would thus easily obtain a contradiction unless $Q(\xi)$ were a polynomial in ξ_n. Going back to our original coordinate system, we deduce the following.

THEOREM 1. *If $P(D)$ had a continuous right inverse in $C^\infty(\Omega)$, and if there were a real vector $N \neq 0$ such that $\langle N,\xi\rangle = 0$ for all ξ in R^n with $P_m(\xi) = 0$, then $P(D) = \theta(\langle N,D\rangle)$ for some suitable polynomial θ in one variable.*

Since $P(D) \neq c$ cannot satisfy the conclusions of Theorem 1 for two linearly independent vectors N_1 and N_2, we obtain the following.

COROLLARY 1. *If V is a two-dimensional subspace of R^n which is contained in every real characteristic plane of $P(D)$, then $P(D)$ has no continuous right inverse in $C^\infty(\Omega)$ for any nonempty open subset Ω of R^n unless $P(D)$ is a constant.*

REMARK 1. The above arguments can also be applied to show that if H is an open half space with a boundary whose normal N is not a characteristic of a nonconstant partial differential operator $P(D)$, then $P(D)$

has a continuous right inverse in $C^\infty(H)$ if and only if $P(D)$ is hyperbolic in the direction N.

REMARK 2. Theorem 1 shows that $\partial^2/\partial x^2 - i(\partial/\partial t)$ has no continuous right inverse on $C^\infty(\mathsf{R}_x \times \mathsf{R}_t)$, and its corollary shows that

$$\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 - i(\partial/\partial t)$$

has no continuous right inverse on $C^\infty(\mathsf{R}_x^2 \times \mathsf{R}_t)$.

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