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MEASURE SPACES CONNECTED BY CORRESPONDENCES

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Contents.

0. Introduction.

We consider Theorem 5.3 on the construction of projective limits of probability spaces as the main result of the present paper. Such a result is well known when the probability spaces are connected by continuous functions (some recent references are Théorème 4.1 of [2], Theorem 3.2 of [5] and Théorème p. 206 of [6]). The main feature of our theorem is that the probability spaces are connected by (topologically nice) correspondences. We have not introduced correspondences just in order to generalize for the sake of generalization in itself, but because we are convinced that for many applications, especially to stochastic processes, one is forced to leave the classical setup. As an illustration of what we have in mind, consider the Skorohod space $D[0,1]$ which is a well suited "target space" for the realization of many stochastic processes. Here, the projections are not continuous, and it is more realistic to replace them by the induced correspondences (cf. section 2). Thus one should replace the projection

$$x \to x(t) = x(t+)$$

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by the correspondence

$$x \to \{x(t-), x(t+)\}\,.$$

The notion of a measure preserving correspondence plays a fundamental role in most parts of the paper. This notion is introduced and studied in Section 3. As was pointed out to me by A. Hede Madsen and M. Niss, V. Strassen has been working with somewhat the same ideas, cf. [7].

We shall employ without much further comment or explanation the conventions and definitions of [9].

1. Preliminaries on correspondences.

Let X and Y be sets (non empty). φ is called a *correspondence* from X to Y if φ, to each element x of X assigns a non empty subset $\varphi(x)$ of Y. We shall reserve the letters φ and ψ for correspondences, whereas ordinary functions will be denoted by the letter π. Thus a symbol like $\pi\colon X \to Y$ always refers to a function, and $\varphi\colon X \to Y$ always refers to a correspondenee from X to Y. In case each set $\varphi(x)$ is a one-point set, we shall allow ourselves to say that φ is a function.

Let $\varphi\colon X \to Y$ be a correspondence. For a subset A of X, the *image* of A under φ is defined by

$$\varphi(A) = \bigcup_{x \in A} \varphi(x)\,.$$

For a subset B of Y we define two kinds of inverse images, the *strong inverse* and the *weak inverse*; these are given by

$$\varphi^s(B) = \{x \,:\, \varphi(x) \subseteq B\}, \quad \varphi^w(B) = \{x \,:\, \varphi(x) \cap B \neq \emptyset\}\,.$$

The simple biimplication

$$A \subseteq \varphi^s(B) \Leftrightarrow \varphi(A) \subseteq B$$

will often be useful. By the "duality relation"

$$\varphi^w(\complement B) = \complement \varphi^s(B)\,,$$

properties expressed in terms of strong inverses can be translated into properties expressed in terms of weak inverses and vice versa.

The *graph* of φ is the subset of $X \times Y$ consisting of the pairs (x,y) with $y \in \varphi(x)$.

If $\varphi\colon X \to Y$ is a correspondence with $\varphi X = Y$, then φ^w is a correspondence $Y \to X$ whose graph is the set of (y,x) with $(x,y) \in$ graph of φ.

Assume now that X and Y are topological spaces. We assume that all topological spaces to be considered are Hausdorff spaces. $\varphi\colon X \to Y$

is said to be *upper semicontinuous* (u.s.c.) if the strong inverse of any open subset of Y is open in X. Below we list four conditions all equivalent to the upper semicontinuity of φ:

$$\varphi^w F \in \mathscr{F}(X), \ \forall F \in \mathscr{F}(Y); \tag{1.1}$$

$$\forall N(\varphi x) \, \exists N(x): \ \ \varphi(N(x)) \subseteqq N(\varphi x); \tag{1.2}$$

$$\overline{\varphi^w B} \subseteqq \varphi^w(\bar{B}), \ \forall B \subseteqq Y; \tag{1.3}$$

$$(\varphi^s B)^\circ \supseteqq \varphi^s(\mathring{B}), \ \forall B \subseteqq Y. \tag{1.4}$$

We call φ *open* if one of the following three equivalent conditions holds:

$$\varphi G \in \mathscr{G}(Y), \ \forall G \in \mathscr{G}(X); \tag{1.5}$$

$$\overline{\varphi^w B} \supseteqq \varphi^w(\bar{B}), \ \forall B \subseteqq Y \tag{1.6}$$

$$(\varphi^s B)^\circ \subseteqq \varphi^s(\mathring{B}), \ \forall B \subseteqq Y. \tag{1.7}$$

φ is *compact-valued* if $\varphi(x)$ is compact for all x in X.

1.8 LEMMA. *A correspondence $\varphi: X \to Y$ is compact-valued if and only if it has the following smoothness property: For every class $\mathscr{F}$ of closed subsets of Y which filters downward, say $\mathscr{F} \downarrow F_0$, it is true that the class of sets $\varphi^w F$ with F in $\mathscr{F}$ filters downward towards the set $\varphi^w F_0$.*

We shall only have occasion to employ the "only if" part, and since this part of the lemma is quite easy to establish, we shall not give the details of the proof.

φ is said to *preserve compact nets* if, for any net (x_α, y_α) on the graph of φ for which the net (x_α) is compact, it is true that the net (y_α) is compact too. As for the notion of compact nets, see P7 of [9].

The upper semicontinuous compact-valued correspondences will play an important role in the sequel. We find the following, probably well-known lemma convenient:

1.9 LEMMA. *Let X and Y be topological spaces and assume that to each element x of X we have assigned a subset $\varphi(x)$ of Y (it is not known in advance that $\varphi(x)$ is non empty). Then φ is an upper semicontinuous compact-valued correspondence if and only if φ possesses the following properties:*

(i) $\{x : \varphi(x)$ *is non empty*$\}$ *is dense in* X.
(ii) *The graph of* φ *is closed.*
(iii) φ *preserves compact nets.*

PROOF. Firstly assume that φ is an u.s.c. compact-valued correspondence. It is immediate that φ has property (i). Let (x_α, y_α) be a net on the graph of φ with (x_α) compact. We may and do assume that (x_α, y_α) is a universal net. Then, for some x, we have $x_\alpha \to x$. We shall prove that $y_\alpha \to y$ for some $y \in \varphi(x)$. If we assume the contrary, then a simple compactness argument ($\varphi(x)$ is compact!) together with the fact that (y_α) is universal tell us that there exists a neighbourhood $N(\varphi x)$ of $\varphi(x)$ such that

$$y_\alpha \in \complement N(\varphi x), \quad \text{eventually}.$$

Employing the upper semicontinuity of φ and the convergence of x_α to x, we are soon led to a contradiction. Thus, for some $y \in \varphi(x)$, the net (y_α) converges to y. Actually, this argument proves the validity of (ii) as well as of (iii).

Then assume that (i), (ii), and (iii) hold. It is easy to see that $\varphi(x)$ is non empty for all x, and thus φ is a correspondence. By (ii) it follows that $\varphi(x)$ is closed for all x, and since, by (iii), $\varphi(x)$ is net-compact, $\varphi(x)$ must be compact for all x. To prove that φ is u.s.c., let $F \in \mathscr{F}(Y)$ be given and consider a convergent net (x_α) on $\varphi^w F$, say $x_\alpha \to x$. To each α we choose $y_\alpha \in \varphi(x_\alpha)$ such that $y_\alpha \in F$. By (iii) we can find y such that some subnet of (y_α) converges to y. From (ii) it follows that $y \in \varphi(x)$. We also have $y \in F$, hence $x \in \varphi^w F$. We have seen that $\varphi^w F$ is closed.

1.10 LEMMA. *Let* $\varphi: X \to Y$ *be* u.s.c. *and compact-valued. Then the following properties hold*:
(i) φK *is compact in* Y *for all* K *compact in* X.
(ii) $\{\overline{\varphi G} : G \supseteq K\} \downarrow \varphi K$ *for all* $K \in \mathscr{K}(X)$.

(i) follows from the lemma just proved, and (ii) follows in a straight forward manner from (i).

Let $\varphi: X \to Y$ and $\psi: Y \to Z$ be correspondences. The composite correspondence $\psi \circ \varphi = \psi\varphi$ from X to Z is then defined by

$$\psi\varphi(x) = \psi(\varphi(x)).$$

Since we always have

$$(\psi\varphi)^s A = \varphi^s(\psi^s A),$$

it is seen that if φ and ψ are u.s.c. then so is $\psi\varphi$. Also, if φ and ψ are u.s.c. and compact-valued, then so is $\psi\varphi$.

In the same way as one can determine a topology by demanding that certain functions are continuous, one can determine a topology by demanding that certain correspondences are u.s.c. Let there be given an abstract set X, an index set I, a family (X_i) of topological spaces indexed by I, and, lastly, for each $i \in I$ a correspondence $\varphi_i \colon X \to X_i$. By the *weak topology* on X determined by the correspondences φ_i we understand the weakest topology on X rendering all the correspondences φ_i u.s.c. Of course, it is not automatic that the weak topology is Hausdorff, however, in the application we have in mind this will be so. By $\mathscr{G}_0$ we denote the class of sets $\varphi_i{}^s(G)$ with $i \in I$ and $G \in \mathscr{G}(X_i)$, and by $\mathscr{G}_1$ we denote the class of finite intersections of sets in $\mathscr{G}_0$. The topology in question on X then has $\mathscr{G}_1$ as a base. By $[N]$ we denote the condition that whenever $K \in \mathscr{K}(X)$ and $N(K)$ is a neighbourhood of K, we can find $G \in \mathscr{G}_1$ with $K \subseteq G \subseteq N(K)$.

1.11 Lemma. *Let the topology of X be the weak topology determined by the correspondences $\varphi_i \colon X \to X_i$; $i \in I$. Let Y be a topological space and φ a correspondence $Y \to X$. Assume that $\varphi_i \circ \varphi$ is* u.s.c. *for each $i \in I$.*

(i): *If φ is a function then φ is* u.s.c. (*that is, φ considered as a function is continuous*).

(ii): *If condition $[N]$ holds and if φ is compact-valued, then φ is* u.s.c.

The proof will be omitted.

It is not true in general that the correspondence φ of the above lemma is u.s.c., and it does not help very much if one assumes that the correspondences are compact-valued. Let us show this by giving a concrete example:

1.12 Example. The index set is the set consisting of the two elements 1 and 2, the set X consists of three elements a, b and c, the topological space X_1 and also the space X_2 is the discrete space with the two points 0 and 1, the topological space Y is the one-point compactification of the natural numbers, and the correspondences φ, φ_1 and φ_2 are given by

$$\begin{aligned} \varphi(n) &= \{a\} && \text{for } n \neq \infty\,, \\ &= \{b,c\} && \text{for } n = \infty\,, \end{aligned}$$

$$\begin{aligned} \varphi_1(a) &= \varphi_1(b) = \{0\}, & \varphi_1(c) &= \{1\}\,, \\ \varphi_2(a) &= \varphi_2(c) = \{1\}, & \varphi_2(b) &= \{0\}\,. \end{aligned}$$

It is seen that the weak topology on X determined by φ_1 and φ_2 is the discrete topology, that $\varphi_1 \circ \varphi$ and $\varphi_2 \circ \varphi$ are u.s.c., and that φ fails to be u.s.c. In fact, φ does not have a closed graph. An example of somewhat the same nature can be constructed such that φ will neither preserve compact nets nor have a closed graph (we can even arrange it so that φ is compact-valued, and yet the image of a compact set under φ need not be compact).

2. Correspondences induced by functions.

Let X and Y be topological spaces and $\pi: X \to Y$ a function from X to Y. If $y = \pi(x)$, one often has an interpretation as "if the system is in state x then the action π leeds to the value y". This interpretation may, however, be meaningless since small changes in x may possibly give rise to other y-values. We therefore define φ, the *correspondence induced by* π, by taking as $\varphi(x)$ the set of those $y \in Y$ for which there exists some net (x_α, y_α) with $x_\alpha \to x$, $y_\alpha \to y$ and $y_\alpha = \pi(x_\alpha)$ for all α. In other words, φ is the correspondence whose graph is the closure in $X \times Y$ of the graph of π. We shall therefore also employ the notation $\varphi = \bar{\pi}$. As is easily seen,

$$\varphi(x) = \bigcap_{N(x)} \overline{\pi(N(x))}\,, \tag{2.1}$$

the intersection being taken over all neighbourhoods $N(x)$ of x. Intuitively, one should think of $\varphi(x)$ as the realistic image of x (in contrast to the unrealistic image $\pi(x)$). Of course, if the graph of π is already closed, for instance if π is continuous, then we get nothing new since then $\varphi(x) = \{\pi(x)\}$ holds for all x.

Note, that the definition of the induced correspondence also makes sense if π is a correspondence. Actually, many of the considerations below can equally well be carried out in this more general setting.

The situation we shall be particularly interested in is that in which X is a function space on which we aim at realizing a stochastic process. If T is the time interval, then X is a subset of R^T. If $i = \{t_1, \ldots, t_n\}$ is a finite subset of T, then we denote by π_i the projection $X \to \mathsf{R}^i$. Thus $\pi_i(x)$ is the function $i \to \mathsf{R}$ whose value at $t \in i$ is $x(t)$. The induced correspondence we denote φ_i. Let us demonstrate by the example $X = D[0,1]$ that φ_i can be strictly larger than π_i: Put

$$\pi = \pi_{\frac{1}{2}}, \quad \varphi = \varphi_{\frac{1}{2}}\,.$$

Let t_n be a sequence of time points strictly decreasing to $\frac{1}{2}$ and put $x_n = 1_{[t_n, 1]}$, $x = 1_{[\frac{1}{2}, 1]}$. Then x_n converges to x in $D[0,1]$ and $\pi(x_n)$ converges to 0. Thus $\varphi(x)$ contains both 1 $(= x(\frac{1}{2}))$ and 0 $(= x(\frac{1}{2}-))$.

2.2 LEMMA. *Let $\pi: X \to Y$ be given and denote by φ the induced correspondence. Consider the three properties:*

$$\forall x \, \exists N(x) : \overline{\pi(N(x))} \text{ is compact}; \tag{2.3}$$

$$\forall x \, \forall N(x) \, \exists N'(x) : \overline{\pi(N'(x))} \subseteq \pi(N(x)); \tag{2.4}$$

$$\pi \text{ is open}. \tag{2.5}$$

(i). *If* (2.3) *holds, then φ is* u.s.c. *and compact-valued, and φ can be characterized as the smallest* u.s.c. *compact-valued correspondence containing π.*

(ii). *If* (2.3) *and* (2.4) *hold, then we have*

$$\varphi^w F = \overline{\pi^{-1} F}, \ \forall F \in \mathscr{F}(Y). \tag{2.6}$$

(iii). *If* (2.4) *and* (2.5) *hold, then φ is open.*

PROOF. The assertion (i) follows without much difficulty from Lemma 1.9.

To prove (ii), assume that (2.3) and (2.4) hold and consider a closed subset F of Y. We indicate by the notation $N_0(\cdot)$ that the neighbourhood in question is one for which the set $\pi(N_0(\cdot))$ is relatively compact. We have

$$\begin{aligned} \overline{\pi^{-1}F} &= \{x : N(x) \cap \pi^{-1}F \neq \emptyset, \forall N(x)\} \\ &= \{x : \pi(N(x)) \cap F \neq \emptyset, \forall N(x)\} \\ &= \{x : \overline{\pi(N_0(x))} \cap F \neq \emptyset, \forall N_0(x)\} \end{aligned}$$

and, by a standard compactness argument, this set is identical with the set

$$\{x : \textstyle\bigcap_{N_0(x)} \overline{\pi(N_0(x))} \cap F \neq \emptyset\} = \varphi^w F .$$

This proves (ii).

If (2.4) holds, then, for any subset A of X, we have

$$\varphi(A) = \textstyle\bigcap_{N(A)} \pi(N(A)) .$$

Thus $\varphi G = \pi G$ holds for any open subset G of X.

From this fact follows (iii).

The lemma tells us which topological properties one can expect of induced correspondences. Note, that if $X \subseteq \mathsf{R}^T$ is a function space, then (2.3) for all π_i's is equivalent to the requirement that for each $t \in T$ the mapping $x \to x(t)$ be locally bounded.

The following lemma is needed later on.

2.7 LEMMA. *Let* $\pi_1\colon X \to Y$ *and* $\pi_2\colon Y \to Z$ *be given functions and assume that* $\bar{\pi}_1$ *is* u.s.c. *compact-valued and that* π_2 *is continuous. Then the formula*

$$\overline{\pi_2\circ\pi_1} = \bar{\pi}_2\circ\bar{\pi}_1 \ (=\pi_2\circ\bar{\pi}_1)$$

holds.

PROOF. In the proof we shall not distinguish notationally between a correspondence and its graph.

In order to prove the inclusion $\overline{\pi_2\circ\pi_1}\subseteq\bar{\pi}_2\circ\bar{\pi}_1$, it is enough to prove that $\bar{\pi}_2\circ\bar{\pi}_1$ is closed. Let (x_α,z_α) be a convergent net on $\bar{\pi}_2\circ\bar{\pi}_1$, say $(x_\alpha,z_\alpha) \to (x,z)$. We can find a net (y_α) such that, for each α,

$$(x_\alpha,y_\alpha) \in \bar{\pi}_1 \quad \text{and} \quad (y_\alpha,z_\alpha) \in \bar{\pi}_2\,.$$

Since $\bar{\pi}_1$ preserves compact nets, (y_α) has a convergent subnet. Assume for simplicity that (y_α) itself converges, say $y_\alpha \to y$. Since $(x_\alpha,y_\alpha) \to (x,y)$, we see that $(x,y) \in \bar{\pi}_1$, and since $(y_\alpha,z_\alpha) \to (y,z)$, we see that $(y,z) \in \bar{\pi}_2$. Thus $(x,z) \in \bar{\pi}_2\circ\bar{\pi}_1$ as desired.

To prove the remaining inclusion $\bar{\pi}_2\circ\bar{\pi}_1\subseteq\overline{\pi_2\circ\pi_1}$, assume that $(x,z) \in \bar{\pi}_2\circ\bar{\pi}_1$. Then we can find y with $(x,y) \in \bar{\pi}_1$ and $(y,z) \in \bar{\pi}_2$. Since $(x,y) \in \bar{\pi}_1$, there is a net (x_α,y_α) on π_1 converging to (x,y). By the continuity of π_2, $(x_\alpha,\pi_2(y_\alpha))$ is a net on $\pi_2\circ\pi_1$ converging to (x,z), hence (x,z) belongs to $\overline{\pi_2\circ\pi_1}$.

By simple examples one may show that none of the inclusions

$$\overline{\pi_2\circ\pi_1} \subseteq \bar{\pi}_2\circ\bar{\pi}_1 \quad \text{or} \quad \bar{\pi}_2\circ\bar{\pi}_1 \subseteq \overline{\pi_2\circ\pi_1}$$

need hold if one drops the assumptions on π_1 and π_2.

3. Image measures.

Let (X,μ) be a probability space and Y some set with a measurable structure. For a function $\pi\colon X \to Y$ we are often interested in knowing, for subsets B of Y, the probability that the action π leeds to a value in the set B. This leeds to the usual definition of the image measure η: $\eta B=\mu(\pi^{-1}B)$. What happens if we no longer deal with a function but with a correspondence $\varphi\colon X \to Y$? Recall one of the possible interpretations of φ: φ represents the value of x after some action or measurement, but due to some special circumstances, a particular measurement of x may leed to any value in the set $\varphi(x)$. We are interested in knowing, for subsets B of Y the probability ηB that a measurement leeds to a value in the set B. For x in the set $\varphi^s B$ we are sure that the value will be in B, and if, for some x, the value does lie in B, then x must be an element of $\varphi^w B$; thus ηB must satisfy the inequalities

$$\mu(\varphi^s B) \leqq \eta B \leqq \mu(\varphi^w B) .$$

With the above background the reader is, so we hope, willing to accept the following definition:

3.1 Definition. Let X and Y be topological spaces and $\varphi: X \to Y$ a correspondence. For $\mu \in \mathfrak{M}_+(X; r)$ we define a set $\varphi(\mu)$, called the set of *image measures* of μ under φ, as the set of those $\eta \in \mathfrak{M}_+(Y; r)$ for which the two inequalities

$$\mu^*(\varphi^s B) \leqq \eta^* B \tag{3.2}$$

and

$$\eta_* B \leqq \mu_*(\varphi^w B) \tag{3.3}$$

hold for every subset B of Y.

An upper star in (3.2) indicates outer measure, and a lower star in (3.3) indicates inner measure.

Clearly, it would be possible to extend the definition to more general situations, for instance one need not limit oneself to a topological situation nor to totally finite measures.

Note, that we have imposed no measurability conditions on φ. Therefore, we can not be sure that $\varphi(\mu)$ is non empty (perhaps, one could call φ μ-measurable if $\varphi(\mu)$ is non empty).

The set $\varphi(\mu)$ is convex.

If (X,μ) and (Y,η) are given, we shall say that φ is *measure preserving* if $\eta \in \varphi(\mu)$.

3.4 Lemma. *Let $\mu \in \mathfrak{M}_+(X; r)$, $\eta \in \mathfrak{M}_+(Y; r)$ and $\varphi: X \to Y$ be given. Then the following conditions are all equivalent:*

$$\varphi \text{ is measure preserving}; \tag{3.5}$$

$$\mu X = \eta Y, \quad \mu^*(\varphi^s B) \leqq \eta^* B, \forall B \subseteqq Y; \tag{3.6}$$

$$\mu X = \eta Y, \quad \mu^*(\varphi^s G) \leqq \eta G, \quad \forall G \in \mathscr{G}(Y); \tag{3.7}$$

$$\mu X = \eta Y, \quad \mu^* A \leqq \eta^*(\varphi A), \forall A \subseteqq X . \tag{3.8}$$

Proof. (3.5) $\Leftrightarrow$ (3.6) $\Rightarrow$ (3.7) is clear. (3.7) $\Rightarrow$ (3.8) follows from

$$\mu^* A \leqq \inf_{\varphi^s G \supseteqq A} \mu^*(\varphi^s G) \leqq \inf_{G \supseteqq \varphi A} \eta G = \eta^*(\varphi A) ,$$

and (3.8) $\Rightarrow$ (3.6) follows from

$$\mu^*(\varphi^s B) \leqq \eta^*(\varphi(\varphi^s B)) \leqq \eta^* B .$$

The lemma is proved.

It follows from (3.8) that if μ is a unit mass, say $\mu = \varepsilon_x$ then $\varphi(\mu)$ consists of all regular probability measures η with $\eta^*(\varphi x) = 1$.

We shall mostly deal with u.s.c. correspondences; (3.7) then takes the form $\mu(\varphi^s G) \leqq \eta G$. In this case one can also relax (3.8) by only paying attention to closed subsets F of X, since

$$\mu(\varphi^s G) = \sup\{\mu F : F \subseteq \varphi^s G\} \leqq \sup\{\eta^*(\varphi F) : \varphi F \subseteq G\} \leqq \eta G .$$

In the same way one proves the following result:

3.9 Lemma. *Let μ be tight, η regular and φ* u.s.c. *Assume also that $\mu X = \eta Y$. Then φ is measure preserving if and only if $\mu K \leqq \eta^*(\varphi K)$ holds for all compact subsets K of X.*

In this lemma we may in fact relax the condition on φ only assuming that $\varphi^s G$ is measurable for all $G \in \mathscr{G}(Y)$.

3.10 Lemma. *Assume that $\mu \in \mathfrak{M}_+^1(X; \tau, r)$, that $\eta \in \mathfrak{M}_+^1(Y; r)$, that φ is* u.s.c. *compact-valued and that Y is regular. If*

$$\mu(\varphi^w F) \geqq \eta(\mathring{F})$$

holds for all $F \in \mathscr{F}(Y)$, then φ is measure preserving.

Proof. This follows from the inequalities

$$\eta F \leqq \inf\nolimits_{G \supseteq F} \eta G \leqq \inf\nolimits_{G \supseteq F} \mu(\varphi^w \bar{G}) = \mu(\varphi^w F)$$

where we have utilized Lemma 1.8.

3.11 Lemma. *Let φ be* u.s.c. *and compact-valued.*

(i): *If μ is tight, then every measure in $\varphi(\mu)$ is tight.*

(ii): *If μ is τ-smooth, then every measure in $\varphi(\mu)$ is τ-smooth.*

Proof. (i): Let $\eta \in \varphi(\mu)$. To $\varepsilon > 0$ we can find $K \in \mathscr{K}(X)$ with $\mu K > \mu X - \varepsilon$. For the compact set φK we then have

$$\eta(\varphi K) \geqq \mu K > \mu X - \varepsilon .$$

Since η is regular, we can conclude that η is tight.

(ii): Let $\eta \in \varphi(\mu)$. Consider a family $\mathscr{F} \subseteq \mathscr{F}(X)$ with $\mathscr{F} \downarrow \emptyset$. By Lemma 1.8 and the τ-smoothness of μ we have

$$\inf\{\mu(\varphi^w F) : F \in \mathscr{F}\} = 0 .$$

It follows that $\inf\{\eta F : F \in \mathscr{F}\} = 0$. By the regularity of η we now conclude that η is τ-smooth.

The lemma just proved would not hold had we not demanded in the definition of image measures that these be regular.

3.12 Lemma. *If μ and η are τ-smooth and φ measure preserving, then the supports satisfy*

$$\operatorname{supp}(\eta) \subseteq \overline{\varphi(\operatorname{supp}(\mu))} .$$

Proof. Let $F \in \mathscr{F}(X)$ satisfy $\mu F = \mu X$. Then, from (3.8) we obtain $\eta(\overline{\varphi F}) = \eta Y$, hence $\operatorname{supp}(\eta) \subseteq \overline{\varphi F}$. Apply this with $F = \operatorname{supp}(\mu)$.

A correspondence $x \to \varphi(x)$ induces a correspondence $\mu \to \varphi(\mu)$, or does it? We have not proved that $\varphi(\mu)$ is non empty for all μ. The two results below show that under certain circumstances $\mu \to \varphi(\mu)$ is indeed a correspondence.

3.13 Theorem. *Let $\varphi\colon X \to Y$ be* u.s.c. *and compact-valued. Then the forming of image measures $\mu \to \varphi(\mu)$ defines a correspondence $\mathfrak{M}_+(X; t) \to \mathfrak{M}_+(Y; t)$ which is* u.s.c. *and compact-valued too, when we provide $\mathfrak{M}_+(X; t)$ and $\mathfrak{M}_+(Y; t)$ with the topologies of weak convergence* (*cf.* [9]).

Proof. We shall appeal to Lemma 1.9.

If μ has finite support, then it is easy to check that $\varphi(\mu)$ is non empty. Thus (i), Lemma 1.9 holds (apply (iii), Theorem 11.1 of [9]).

To prove that $\mu \to \varphi(\mu)$ has a closed graph, let (μ_α) be a convergent net on

$$\mathfrak{M}_+(X; t)\colon\ \mu_\alpha \to_w \mu ,$$

let (η_α) be a convergent net on

$$\mathfrak{M}_+(Y; t)\colon\ \eta_\alpha \to_w \eta ,$$

and let, for each α, $\eta_\alpha \in \varphi(\mu_\alpha)$. We are to prove that $\eta \in \varphi(\mu)$. To do this we shall apply Lemma 3.9. We have

$$\eta Y = \lim \eta_\alpha Y = \lim \mu_\alpha X = \mu X .$$

For $K \in \mathscr{K}(X)$ we find

$$\begin{aligned}\mu K = \inf_{G \supseteqq K} \mu G &\leqq \inf_{G \supseteqq K} \liminf_\alpha \mu_\alpha G \\ &\leqq \inf_{G \supseteqq K} \limsup_\alpha \eta_\alpha(\overline{\varphi G}) \\ &\leqq \inf_{G \supseteqq K} \eta(\overline{\varphi G}) = \eta(\varphi K)\,.\end{aligned}$$

We now conclude that $\eta \in \varphi(\mu)$.

To finish the proof, we shall show that $\mu \to \varphi(\mu)$ preserves compact nets. For this, we appeal to the characterization of compact nets given in Theorem 9.1 of [9]. Let (μ_α) be a compact net on $\mathfrak{M}_+(X\,;t)$ and let, for each α, η_α be a measure in $\varphi(\mu_\alpha)$. Let $\mathscr{G} \subseteqq \mathscr{G}(Y)$ dominate $\mathscr{K}(Y)$ and let ε be positive. The class of sets $\varphi^s G$ with G in $\mathscr{G}$ is a subclass of $\mathscr{G}(X)$ dominating $\mathscr{K}(X)$. Since (μ_α) is compact, we can find finitely many sets from $\mathscr{G}$, say $G_1,\ldots,G_n$, such that

$$\min_{\nu=1,\ldots,n} \mu_\alpha(\complement\, \varphi^s(G_\nu)) < \varepsilon$$

holds eventually. Then

$$\min_{\nu=1,\ldots,n} \eta_\alpha(\complement G_\nu) < \varepsilon$$

also holds eventually. This argument together with the observation that $\limsup \eta_\alpha Y < \infty$ holds, tells us that (η_α) is compact.

3.14 Corollary. (cf. Lemme 2 of [1]). *Let $\varphi\colon X \to Y$ be a correspondence with $\varphi X = Y$ such that the image of every closed set (in X) is closed (in Y) and such that the weak inverse image of every one-point set is compact (in X). Then, to any measure $\eta \in \mathfrak{M}_+(Y\,;t)$ there exists at least one measure $\mu \in \mathfrak{M}_+(X\,;t)$ such that $\eta \in \varphi(\mu)$.*

Proof. Apply Theorem 3.13 to the correspondence $\varphi^w\colon Y \to X$.

3.15 Corollary. *Let $\varphi\colon X \to X$ be an* u.s.c. *compact-valued correspondence from the compact space X to itself. Then there exists a $\mu \in \mathfrak{M}_+{}^1(X\,;t)$ which is invariant under φ, that is, $\mu \in \varphi(\mu)$.*

Proof. Apply the Kakutani fixed-point theorem to the correspondence $\varphi\colon \mathfrak{M}_+{}^1(X\,;t) \to \mathfrak{M}_+{}^1(X\,;t)$.

3.16 Theorem. *Let X and Y be regular and $\varphi\colon X \to Y$* u.s.c. *and compact-valued. Then $\mu \to \varphi(\mu)$ defines an* u.s.c. *and compact-valued correspondence from $\mathfrak{M}_+(X;\,\tau)$ to $\mathfrak{M}_+(Y;\,\tau)$ when we provide these spaces of measures with the topology of weak convergence.*

PROOF. The pattern of the proof is the same as before. Again, there are only difficulties in proving that the graph is closed and that compact nets are preserved.

With the same notations as before, we can prove that the graph is closed by employing the inequalities

$$\begin{aligned}\eta F = \inf_{G \supseteqq F} \eta G &\leqq \inf_{G \supseteqq F} \liminf_\alpha \eta_\alpha G \\ &\leqq \inf_{G \supseteqq F} \limsup_\alpha \eta_\alpha \bar{G} \leqq \inf_{G \supseteqq F} \limsup_\alpha \mu_\alpha(\varphi^w \bar{G}) \\ &\leqq \inf_{G \supseteqq F} \mu(\varphi^w \bar{G}) = \mu(\varphi^w F)\,.\end{aligned}$$

To prove that compact nets are preserved we shall appeal to Theorem 9.2 of [9]. Let (μ_α) be a compact net on $\mathfrak{M}_+(X;\tau)$ and let, for each α, $\eta_\alpha \in \varphi(\mu_\alpha)$. Let $\mathscr{F} \subseteq \mathscr{F}(Y)$ satisfy $\mathscr{F} \downarrow \emptyset$.

Then the class of sets $\varphi^w F$ with $F \in \mathscr{F}$ is a subclass of $\mathscr{F}(X)$ filtering downwards to the empty set. Since (μ_α) is compact, we conclude that

$$\inf_{F \in \mathscr{F}} \limsup_\alpha \mu_\alpha(\varphi^w F) = 0$$

holds, and it follows that

$$\inf_{F \in \mathscr{F}} \limsup_\alpha \eta_\alpha F = 0$$

holds. This together with $\limsup \eta_\alpha Y < \infty$ shows that (η_α) is compact.

Clearly, Theorem 3.16 admits a corollary analogous to Corollary 3.14.

4. Identification of measures.

For the next sections we need a refinement of the results of P 19 of [9].

4.1 DEFINITIONS. Let X be a topological space, D an abstract set whose elements we denote by the letter α and let there be given a mapping $\alpha \to (G_\alpha, F_\alpha)$ of D into $\mathscr{G}(X) \times \mathscr{F}(X)$ such that $G_\alpha \subseteqq F_\alpha$ holds for all $\alpha \in D$.

The mapping is *latticelike* if (4.2) and (4.3) below hold:

$$\forall \alpha_1,\alpha_2 \exists \alpha:\ G_{\alpha_1} \cup G_{\alpha_2} \subseteqq G_\alpha \subseteqq F_\alpha \subseteqq F_{\alpha_1} \cup F_{\alpha_2}; \tag{4.2}$$

$$\forall \alpha_1,\alpha_2 \exists \alpha:\ G_{\alpha_1} \cap G_{\alpha_2} \subseteqq G_\alpha \subseteqq F_\alpha \subseteqq F_{\alpha_1} \cap F_{\alpha_2}\,. \tag{4.3}$$

The mapping *separates points* if (4.4) holds:

$$\forall x_1 \neq x_2 \exists \alpha:\ x_1 \in G_\alpha,\ x_2 \notin F_\alpha\,. \tag{4.4}$$

The mapping *separates compact sets* if (4.5) holds (K_1,K_2 denoting compact sets):

$$(4.5) \qquad \forall K_1 \cap K_2 = \emptyset \, \exists \alpha : \; K_1 \subseteq G_\alpha \subseteq F_\alpha \subseteq \complement K_2 \, .$$

The mapping *separates points and closed sets* if (4.6) holds (F's denoting closed sets):

$$(4.6) \qquad \forall x \notin F \, \exists \alpha : \; x \in G_\alpha, \; F_\alpha \subseteq \complement F \, .$$

The mapping *almost separates points and closed sets* if (4.7) holds:

$$(4.7) \qquad \forall x \notin F \, \exists N(x) \, \forall y \in F \, \exists \alpha : \; N(x) \subseteq G_\alpha, \; y \notin F_\alpha \, .$$

4.8 LEMMA (identification of tight measures). *Let $D \to \mathscr{G}(X) \times \mathscr{F}(X)$ be a latticelike mapping separating points. Then the mapping also separates compact sets.*

As a corollary we have that, if μ_1 and μ_2 are tight measures on X for which $\mu_1 G_\alpha \leqq \mu_2 F_\alpha$ holds for all $\alpha \in D$, then $\mu_1 \leqq \mu_2$ holds; if, furthermore $\mu_1 X = \mu_2 X$ holds, then μ_1 and μ_2 are identical.

PROOF. Let K_1 and K_2 be disjoint compact sets. Fix, for some time, $y \in K_2$. Choose to each $x \in K_1$ an index $\alpha(x) \in D$ such that

$$x \in G_{\alpha(x)}, \quad y \notin F_{\alpha(x)} \, .$$

We can find finitely many points in K_1, say $x_1, \ldots, x_n$, such that

$$K_1 \subseteq G_{\alpha(x_1)} \cup \ldots \cup G_{\alpha(x_n)}$$

holds. Choose $\alpha \in D$ such that

$$G_{\alpha(x_1)} \cup \ldots \cup G_{\alpha(x_n)} \subseteq G_\alpha \subseteq F_\alpha \subseteq F_{\alpha(x_1)} \cup \ldots \cup F_{\alpha(x_n)}$$

holds. Then $K_1 \subseteq G_\alpha$, $y \notin F_\alpha$. What we have seen is this: To any $y \in K_2$ there exists $\alpha(y) \in D$ with

$$K_1 \subseteq G_{\alpha(y)} \quad \text{and} \quad y \notin F_{\alpha(y)} \, .$$

We can find finitely many points in K_2, say $y_1, \ldots, y_m$, such that

$$F_{\alpha(y_1)} \cap \ldots \cap F_{\alpha(y_m)} \subseteq \complement K_2$$

holds. Now choose $\alpha \in D$ such that

$$G_{\alpha(y_1)} \cap \ldots \cap G_{\alpha(y_m)} \subseteq G_\alpha \subseteq F_\alpha \subseteq F_{\alpha(y_1)} \cap \ldots \cap F_{\alpha(y_m)}$$

holds. Then $K_1 \subseteq G_\alpha \subseteq F_\alpha \subseteq \complement K_2$ holds.

4.9 LEMMA (identification of τ-smooth measures). *Let $D \to \mathscr{G}(X) \times \mathscr{F}(X)$ be a latticelike mapping almost separating points and closed sets. Assume that X is regular. Then, for any set-function $\lambda : \mathscr{F}(X) \to \mathsf{R}_+$ which is monotone and τ-smooth at $\emptyset$ we have*

(4.10) $\forall F_1 \cap F_2 = \emptyset \ \forall \varepsilon > 0 \ \exists \alpha \in D\colon \ \lambda(F_1 \setminus G_\alpha) < \varepsilon, \ \lambda(F_2 \cap F_\alpha) < \varepsilon$.

As a corollary we have that if $\mu_1, \mu_2 \in \mathfrak{M}_+(X;\tau)$ *are such that* $\mu_1 G_\alpha \leqq \mu_2 F_\alpha$ *holds for all* $\alpha \in D$, *then* $\mu_1 \leqq \mu_2$ *holds. If, furthermore,* $\mu_1 X = \mu_2 X$, *then* $\mu_1 = \mu_2$ *follows.*

Proof. To establish (4.10), let F_1, F_2 be disjoint closed sets. Consider the class $\mathscr{G} \subseteq \mathscr{G}(X)$ defined by

$$\mathscr{G} = \{G \ : \ G \cap F_2 = \emptyset, \forall \varepsilon > 0 \ \exists \alpha\colon \ G \subseteqq G_\alpha, \ \lambda(F_2 \cap F_\alpha) < \varepsilon\} .$$

First we shall prove that $\mathscr{G}$ is upward filtering, in fact we shall prove that if G_1 and G_2 are in $\mathscr{G}$, then so is $G_1 \cup G_2$. Clearly,

$$(G_1 \cup G_2) \cap F_2 = \emptyset .$$

To any given positive ε, we first choose α_1 and α_2 such that

$$\begin{aligned} G_1 &\subseteqq G_{\alpha_1}, \quad \lambda(F_2 \cap F_{\alpha_1}) < \tfrac{1}{2}\varepsilon , \\ G_2 &\subseteqq G_{\alpha_2}, \quad \lambda(F_2 \cap F_{\alpha_2}) < \tfrac{1}{2}\varepsilon . \end{aligned}$$

We choose α such that

$$G_{\alpha_1} \cup G_{\alpha_2} \subseteqq G_\alpha \subseteqq F_\alpha \subseteqq F_{\alpha_1} \cup F_{\alpha_2} .$$

Then $G_1 \cup G_2 \subseteqq G_\alpha$ and $\lambda(F_2 \cap F_\alpha) < \varepsilon$ hold. We have now seen that

$$G_1 \cup G_2 \in \mathscr{G} .$$

For every $G \in \mathscr{G}$ we have $G \subseteqq \complement F_2$. We aim at proving that $\mathscr{G} \uparrow \complement F_2$ holds. To do this, consider an element $x \in \complement F_2$. Choose an open neighbourhood $N(x)$ of x according to the defining property (4.7) applied to x and F_2. Consider the class $\mathscr{F} \subseteqq \mathscr{F}(X)$ defined by

$$\mathscr{F} = \{F_2 \cap F_\alpha \ : \ \alpha \in D, \ G_\alpha \supseteqq N(x)\} .$$

We shall first prove that $\mathscr{F} \downarrow \emptyset$ holds. To prove that $\mathscr{F}$ is downward filtering, let $F_2 \cap F_{\alpha_1}$ and $F_2 \cap F_{\alpha_2}$ be sets in $\mathscr{F}$. Choose α so that

$$G_{\alpha_1} \cap G_{\alpha_2} \subseteqq G_\alpha \subseteqq F_\alpha \subseteqq F_{\alpha_1} \cap F_{\alpha_2} .$$

It is then easy to check that $F_2 \cap F_\alpha \in \mathscr{F}$; this shows that $\mathscr{F}$ is downward filtering. To prove $\mathscr{F} \downarrow \emptyset$, let $y \in X$ be given. If $y \in F_2$ we choose, by (4.7), $\alpha \in D$ such that $N(x) \subseteqq G_\alpha$ and $y \notin F_\alpha$ hold. Then

$$F_2 \cap F_\alpha \in \mathscr{F} \quad \text{and} \quad y \notin F_2 \cap F_\alpha .$$

If $y \in \complement F_2$, then any set $F_2 \cap F_\alpha$ in $\mathscr{F}$ satisfies $y \notin F_2 \cap F_\alpha$ (and we may assume that $\mathscr{F}$ is non empty). Thus $\mathscr{F} \downarrow \emptyset$ holds.

Since λ is τ-smooth at $\emptyset$ it follows that, for any $\varepsilon > 0$, there exists an $\alpha \in D$ with $\lambda(F_2 \cap F_\alpha) < \varepsilon$ and $G_\alpha \supseteq N(x)$. Looking at the definition of $\mathcal{G}$, we see that $N(x) \in \mathcal{G}$. Hence the desired result:

$$\mathcal{G} \uparrow \complement F_2 .$$

This implies that

$$\{F_1 \setminus G \; : \; G \in \mathcal{G}\} \downarrow \emptyset .$$

Thus, to any $\varepsilon > 0$, we can find $G \in \mathcal{G}$ with $\lambda(F_1 \setminus G) < \varepsilon$. Since $G \in \mathcal{G}$, we can also find $\alpha \in D$ with $G_\alpha \supseteq G$ and $\lambda(F_2 \cap F_\alpha) < \varepsilon$. Then (4.10) is fully proved since we have $\lambda(F_1 \setminus G_\alpha) < \varepsilon$ as well as $\lambda(F_2 \cap F_\alpha) < \varepsilon$.

To prove the remaining part of the lemma, let $\mu_1, \mu_2 \in \mathfrak{M}_+(X\,;\,\tau)$ satisfy $\mu_1 G_\alpha \leqq \mu_2 F_\alpha$ for all α. Consider disjoint closed sets F_1, F_2. Applying (4.10) with $\lambda = \max(\mu_1, \mu_2)$ we find, to $\varepsilon > 0$, an $\alpha \in D$ with

$$\mu_1(F_1 \setminus G_\alpha) < \varepsilon \quad \text{and} \quad \mu_2(F_2 \cap F_\alpha) < \varepsilon .$$

Then we have

$$\begin{aligned} \mu_1 F_1 = \mu_1(F_1 \cap G_\alpha) + \mu_1(F_1 \setminus G_\alpha) &\leqq \mu_1 G_\alpha + \varepsilon \\ &\leqq \mu_2 F_\alpha + \varepsilon \\ &= \mu_2(F_\alpha \cap F_2) + \mu_2(F_\alpha \setminus F_2) + \varepsilon \\ &\leqq 2\varepsilon + \mu_2(\complement F_2) . \end{aligned}$$

We infer that $\mu_1 F_1 \leqq \mu_2(\complement F_2)$. By regularity of μ_1 and μ_2, $\mu_1 \leqq \mu_2$ follows according to P16 of [9].

5. A theorem on projective limits of probability spaces.

5.1 Definition. By $\mathfrak{A}_t$ we denote the category whose *objects* are pairs (X, μ) with X a Hausdorff space and μ a tight probability measure on X, and whose *morphisms* φ: $(X, \mu) \to (Y, \eta)$ are u.s.c. compact-valued and measure preserving correspondences $X \to Y$.

The identity morphism $(X, \mu) \to (X, \mu)$ is the identity correspondence id_X defined by $\mathrm{id}_X(x) = \{x\}$ for all $x \in X$. As composition of morphisms we use composition of correspondences. It is easy to check that $\mathfrak{A}_t$ is a category; for example, let us prove that composition of morphisms is well defined. To do this, let φ_1: $(X, \mu) \to (Y, \eta)$ and φ_2: $(Y, \eta) \to (Z, \zeta)$ be morphisms and consider the correspondence $\varphi_3 = \varphi_2 \varphi_1$: $X \to Z$. The general formulas

$$\varphi_3{}^s B = \varphi_1{}^s(\varphi_2{}^s B) \quad \text{and} \quad \varphi_3 A = \varphi_2(\varphi_1 A)$$

tell us that φ_3 is u.s.c. and compact-valued. Since, for any $G \in \mathscr{G}(Z)$, we have

$$\mu(\varphi_3{}^s G) = \mu(\varphi_1{}^s(\varphi_2{}^s G)) \leqq \eta(\varphi_2{}^s G) \leqq \zeta(G),$$

φ_3 is also measure preserving. Thus φ_3 is a morphism $(X,\mu) \to (Z,\zeta)$.

We shall study projective systems in the category $\mathfrak{A}_t$. A *projective system* consists of an upward directed set $I=(I,\leqq)$, a family $(X_i,\mu_i)_{i\in I}$ of objects in $\mathfrak{A}_t$, and a family $(\varphi_{ij})_{i\leqq j}$ of morphisms in $\mathfrak{A}_t$ indexed by those pairs (i,j) of elements of I for which $i\leqq j$; the morphism φ_{ij} is a morphism $(X_j,\mu_j) \to (X_i,\mu_i)$. We demand that φ_{ii} is, for each $i \in I$, the identity morphism $(X_i,\mu_i) \to (X_i,\mu_i)$ and, furthermore, that the φ_{ij}'s are consistent by which we mean that $\varphi_{ij}\varphi_{jk}=\varphi_{ik}$ holds whenever $i\leqq j\leqq k$. We often speak of "the projective system (X_i,μ_i,φ_{ij})".

Let (X_i,μ_i,φ_{ij}) be a projective system in $\mathfrak{A}_t$. We shall say that (X,μ,φ_i) is a *projective limit* of the given projective system if (X,μ) is an object in $\mathfrak{A}_t$ and if, for each $i \in I$, φ_i is a morphism $(X,\mu) \to (X_i,\mu_i)$ and if, furthermore, the φ_i's are consistent with the φ_{ij}'s by which we mean that the relation $\varphi_i=\varphi_{ij}\varphi_j$ holds whenever $i\leqq j$. The measure μ is often referred to as the *projective limit measure.*

Note that our definition of a projective limit is not the one usually adopted in category theory. What is known in category theory as a projective limit we shall call a universal projective limit. To be precise, we shall say that (X,μ,φ_i) is a *universal projective limit* (of the projective system (X_i,μ_i,φ_{ij})) if (X,μ,φ_i) is a projective limit and if, for any projective limit (Y,η,ψ_i) there exists one and only one morphism $\theta\colon (Y,\eta) \to (X,\mu)$ such that $\psi_i=\varphi_i\circ\theta$ for all $i \in I$.

The terminology introduced above in the specific category $\mathfrak{A}_t$ will later be applied to other categories, in the next section also to a purely topological category.

If (X,μ,φ_i) is a universal projective limit, then (X,μ) is uniquely determined up to an isomorphism (in the categorical sense). It is quite easy to see that (X,μ) and (X',μ') are isomorphic if and only if there exists a measure preserving homeomorphism $\pi\colon X \to X'$.

When we attempt to find a projective limit (X,μ,φ_i), then we shall always (at least in this section) assume that the space X and the correspondences φ_i are given in advance, and we shall then assume that X is a Hausdorff space, that each φ_i is an u.s.c. compact-valued correspondence $X \to X_i$, and that the consistence relation $\varphi_i=\varphi_{ij}\varphi_j$ holds for all $i\leqq j$. We shall refer to X as the *target space,* but most often, when we speak of a target space, we shall in fact have X as well as the φ_i's in mind. If there exists a tight probability measure μ on the target space X

such that (X,μ,φ_i) is a projective limit, then we shall say that a *projective limit can be realized on* X.

If one has applications to stochastic processes in mind, then it is quite clear to us — and perhaps the reader agrees — that the notion of a projective limit is much more important than that of a universal projective limit. For instance, if we consider the usual setup of a stochastic process as a consistent family of finite dimensional distributions, then a universal projective limit gives us a version of the process on the pretty uninformative function space of all functions on the time set in question in the pointwise topology, whereas the possibility to vary the target space allows us to study versions of the process on a variety of concrete function spaces.

Let (X_i,μ_i,φ_{ij}) be a projective system and let X with correspondences (φ_i) be a target space. The only thing we lack in having a projective limit is a certain measure on X. Usually, in attempts to construct such a measure, the class of cylinder sets on X plays a dominant role. This will also be so in our case, but certain difficulties arise since we have two kinds of inverses for correspondences. To find a convenient substitute for the class of cylinder sets, we first define the set D as the set of all pairs $\alpha=(i,E)$ with $i\in I$ and E a subset of X_i (in other words, D is the disjoint sum of the power sets of the X_i). Then we define a mapping $D\to\mathscr{G}(X)\times\mathscr{F}(X)$ by $\alpha\to(G_\alpha,F_\alpha)$ where

$$G_\alpha = \varphi_i{}^s(\mathring{E}),\ F_\alpha = \varphi_i{}^w(\bar{E})\ .$$

This mapping is going to replace the notion of cylinder set. We shall say that the *cylinder sets separate points* (*separate compact sets, separate points and closed sets*, or *almost separate points and closed sets*) if the mapping $D\to\mathscr{G}(X)\times\mathscr{F}(X)$ does so.

5.2 Lemma. *Let (X_i,μ_i,φ_{ij}) be a projective system in the category $\mathfrak{A}_t$ and let X with correspondences (φ_i) be a target space.*

Then the mapping $\alpha\to(G_\alpha,F_\alpha)$ defined above is latticelike, and the mapping separates points if and only if, for any pair (x,y) of distinct points of X, there exists an $i\in I$ such that $\varphi_i(x)$ and $\varphi_i(y)$ are disjoint.

Proof. First we remark that for any $\alpha=(i,E_i)$ in D and any j with $j\geqq i$ there exists $\beta\in D$ of the form $\beta=(j,E_j)$ such that the inclusions

$$G_\alpha \subseteqq G_\beta \subseteqq F_\beta \subseteqq F_\alpha$$

hold. To see this, we just have to put $E_j=\varphi_{ij}{}^s(E_i)$.

To establish (4.2) and (4.3), consider elements α_1 and α_2 in D. Due to the above remark, we may assume that α_1 and α_2 are of the form $\alpha_1=$

(i,E_1), $\alpha_2=(i,E_2)$ with a common i. Then (4.2) and (4.3) follow from the inclusions

$$\varphi_i^s(\mathring{E}_1) \cup \varphi_i^s(\mathring{E}_2) \subseteq \varphi_i^s((E_1 \cup E_2)^\circ) \subseteq \varphi_i^w(\overline{E_1 \cup E_2}) = \varphi_i^w(\bar{E}_1) \cup \varphi_i^w(\bar{E}_2) ,$$

and

$$\varphi_i^s(\mathring{E}_1) \cap \varphi_i^s(\mathring{E}_2) = \varphi_i^s((E_1 \cap E_2)^\circ) \subseteq \varphi_i^w(\overline{E_1 \cap E_2}) \subseteq \varphi_i^w(\bar{E}_1) \cap \varphi_i^w(\bar{E}_2) .$$

If the cylinder sets separate points, then $x \neq y$ clearly implies

$$\varphi_i(x) \cap \varphi_i(y) = \emptyset \quad \text{for some } i .$$

On the other hand, if this condition holds, then the cylinder sets must separate points since, for $x \neq y$, we can find i with $\varphi_i(x) \cap \varphi_i(y) = \emptyset$ and then, by compactness, we can find $E \subseteq X_i$ such that

$$\varphi_i(x) \subseteq \mathring{E} \subseteq \bar{E} \subseteq \complement \varphi_i(y) ,$$

hence $x \in G_\alpha$ and $y \notin F_\alpha$ with $\alpha=(i,E)$.

The same argument shows that the cylinder sets separate compact sets if and only if, for each pair K_1, K_2 of disjoint compact sets, we have

$$\varphi_i(K_1) \cap \varphi_i(K_2) = \emptyset \quad \text{for some } i \in I .$$

Here then comes our main theorem:

5.3 Theorem. *Let $(X_i, \mu_i, \varphi_{ij})$ be a projective system in the category $\mathfrak{A}_t$ and let X with correspondences (φ_i) be a target space. Assume that the cylinder sets separate points.*

Then a projective limit can be realized on X if and only if

$$\sup_{K \in \mathscr{K}(X)} \inf_{i \in I} \mu_i(\varphi_i K) = 1 \tag{5.4}$$

holds, and when this condition is satisfied, the projective limit measure, which is unique, is given by the formula

$$\mu A = \sup_{K \subseteq A} \inf_{i \in I} \mu_i(\varphi_i K) ; \quad A \in \mathscr{B}(X) . \tag{5.5}$$

Proof. Let us first prove that a projective limit measure is unique. Assume that μ_1 and μ_2 are both projective limit measures (on the given target space). Then for any $\alpha=(i,E)$ in D we have

$$\mu_1 G_\alpha = \mu_1(\varphi_i^s \mathring{E}) \leq \mu_i \mathring{E} \leq \mu_i \bar{E} \leq \mu_2(\varphi_i^w \bar{E}) = \mu_2 F_\alpha ,$$

and it follows by Lemma 4.8 that $\mu_1 = \mu_2$.

Then let us prove the "only if" part. This is in fact quite easy, since if μ is a projective limit measure, then, by Lemma 3.9, we must have

$$1 = \sup_K \mu K \leqq \sup_K \inf_{i \in I} \mu_i(\varphi_i K) .$$

We are now faced with the essential part of the proof and assume that (5.4) holds. We shall construct μ in several steps of which only the very last one requires condition (5.4). Firstly,

$$\text{(i)} \qquad i \leqq j,\ K \in \mathscr{K}(X) \ \Rightarrow\ \mu_j(\varphi_j K) \leqq \mu_i(\varphi_i K) .$$

This follows from

$$\mu_i(\varphi_i K) = \mu_i(\varphi_{ij}(\varphi_j K)) \geqq \mu_j(\varphi_j K) .$$

Now define $\lambda\colon \mathscr{K}(X) \to \mathsf{R}_+$ by

$$\lambda K = \inf_{i \in I} \mu_i(\varphi_i K),\ K \in \mathscr{K}(X) .$$

By (i) we can also write

$$\lambda K = \lim_i \mu_i(\varphi_i K) .$$

It follows that λ is subadditive. We now claim:

$$\text{(ii)} \qquad \lambda \text{ is additive} .$$

To prove this, let K_1, K_2 be disjoint sets in $\mathscr{K}(X)$. By Lemma 4.8, the cylinder sets separate compact sets, and we can find $i \in I$ such that

$$\varphi_i K_1 \cap \varphi_i K_2 = \emptyset .$$

Then we also have

$$\varphi_j K_1 \cap \varphi_j K_2 = \emptyset \quad \text{for all } j \geqq i .$$

To $\varepsilon > 0$ we choose $j \geqq i$ so that

$$\lambda(K_1 \cup K_2) \geqq \mu_j(\varphi_j(K_1 \cup K_2)) - \varepsilon .$$

Then

$$\begin{aligned}\lambda(K_1 \cup K_2) &\geqq \mu_j(\varphi_j K_1 \cup \varphi_j K_2) - \varepsilon \\ &= \mu_j(\varphi_j K_1) + \mu_j(\varphi_j K_2) - \varepsilon \geqq \lambda K_1 + \lambda K_2 - \varepsilon .\end{aligned}$$

In connection with the subadditivity of λ this proves (ii). Further we claim that

$$\text{(iii)} \qquad \lambda \text{ is semi-regular} .$$

For the notion of semi-regularity see [5] or Section 2 of [9]. To prove (iii), let $K \in \mathscr{K}(X)$ and $\varepsilon > 0$ be given. Choose i so that

$$\mu_i(\varphi_i K) < \lambda K + \varepsilon.$$

Choose $G_i \supseteq \varphi_i K$ so that

$$\mu_i G_i < \lambda K + \varepsilon$$

Put $G=\varphi_i{}^s G_i$. Then $G \supseteqq K$ and for any $K' \in \mathscr{K}(X)$ with $K' \supseteqq G$ it is easy to see that $\lambda K' < \lambda K + \varepsilon$. This proves (iii).

By (ii) Lemma 2.4 of [9], λ is tight. Hence the formula

$$\mu A = \sup_{K \subseteqq A} \lambda A ; \quad A \in \mathscr{B}(X)$$

defines a measure μ in $\mathfrak{M}_+(X; t)$. By the definition of λ we have, for each i,

(iv) $$K \in \mathscr{K}(X) \Rightarrow \mu K \leqq \mu_i(\varphi_i K) .$$

Lastly, we apply (5.4) and find that $\mu(X)=1$. Thus (X,μ) is an object in our category, and (iv) in connection with Lemma 3.9 shows, that all the φ_i are measure preserving.

Theorem 5.3 is proved.

We shall now indicate in some detail how Theorem 5.3 can be applied in the study of stochastic processes. We consider a stochastic process with values in R over the time set T, given in terms of the finite dimensional distributions $(\mu_i)_{i \in I}$. In other words, the process is given by a projective system $(\mathsf{R}^i, \mu_i, \pi_{ij})$ in $\mathfrak{A}_t$, where i runs over all the finite subsets of T, and π_{ij} for $i \subseteqq j$ denotes the projection from R^j to R^i. We are interested in investigating whether the process can be realized on a given function space $X \subseteqq \mathsf{R}^T$. By π_i we denote the projection $X \to \mathsf{R}^i$ and we put

$$\varphi_i = \bar{\pi}_i \quad \text{and} \quad \varphi_{ij} = \bar{\pi}_{ij}$$

($\bar{\pi}_{ij} = \pi_{ij}$ since π_{ij} is continuous). We assume that the following two conditions are satisfied: Firstly, for every $t \in T$ the mapping $\pi_t \colon X \to \mathsf{R}$ is locally bounded and, secondly, for every pair x, y of distinct functions in X there exist neighbourhoods $N(x)$ and $N(y)$ of x and y, respectively, and a finite subset i of T such that

$$\overline{\pi_i(N(x))} \cap \overline{\pi_i(N(y))} = \emptyset .$$

The first condition ensures that the φ_i's are u.s.c. compact-valued correspondences (cf. Lemma 2.2) and that the consistency relations $\varphi_i = \varphi_{ij}\varphi_j$, $\forall i \leqq j$ hold (cf. Lemma 2.7). The second condition tells us that the cylinder sets separate points. Thus Theorem 5.3 can be applied. Note that the φ_{ij} are all functions, only the φ_i may be correspondences. The "classical" case mentioned in the introduction arises if also the φ_i are functions. In concrete situations we are of course left with two difficult problems, firstly the choice of a function space and, secondly, in order to understand what condition 5.4 involves, we must be able to describe in some detail the compact subsets of X.

Lastly, we shall point out that Theorem 5.3 may also be applied in a somewhat different way. Suppose, we want to study a Markoff process with $T = \mathsf{R}_+$ as time set and with $\{0, 1, 2, \ldots\}$ as the set of possible states. It is natural to suggest that the function space X should consist of all functions $x\colon \mathsf{R}_+ \to \{0, 1, 2, \ldots\}$ that are right continuous and have limits from the left. If $x(t-)$ is finite, it is natural that the induced correspondence φ_t should have the value $\varphi_t(x) = \{x(t-), x(t+)\}$. However, if $x(t-) = \infty$, it is unnatural that $\varphi_t(x) = \{x(t+)\}$ should hold. Therefore, we are forced to compactify the state space, and thus we now consider the extended state space $S = \{0, 1, 2, \ldots\} \cup \{\infty\}$. For $i = \{t_1, \ldots, t_n\}$ we define $\varphi_i(x)$ as the set of all points $(x(t_1 \pm), \ldots, x(t_n \pm))$ in $S^i = S^n$ obtainable by the 2^n possible choices of $+$ and $-$. The weakest topology on X for our theory to work is the weak topology determined by the correspondences φ_i (cf. Section 1). This topology is the same as the weak topology determined by the correspondences φ_t; $t \in \mathsf{R}_+$. With this topology we may continue the investigations; however, we do not yet know how to characterize the compact subsets of X.

6. On the existence of universal projective limits in a purely topological category.

Even though we have remarked that, from some points of view, the problem of universal projective limits in the category $\mathfrak{A}_t$ is not very important, our mathematical curiosity has not been able to withstand the temptation to attack this problem — in fact, we have used a lot of effort to solve it and yet, we do not have a complete solution. However, this much can be said: only very rarely a universal projective limit exists. This circumstance, at least at first sight, seemed surprising since universal projective limits practically always exist in the category of probability spaces connected by continuous functions.

In this section we shall study the purely topological category $\mathfrak{A}$ obtained from $\mathfrak{A}_t$ by simply forgetting the measures. It is hoped that the results of this section are of some interest in their own right.

6.1 Definition. We denote by $\mathfrak{A}$ the category whose objects are Hausdorff spaces X and whose morphisms $\varphi\colon X \to Y$ are u.s.c. compact-valued correspondences.

We demand that all our topological spaces are non empty, thus the objects in $\mathfrak{A}$ are by definition non empty.

Below we shall study a fixed projective system (X_i, φ_{ij}) in the category $\mathfrak{A}$. We shall seek conditions under which a universal projective limit exists. Recall that it is a matter of three or four lines to settle the analogous problem in the category of Hausdorff spaces connected by continuous functions. In $\mathfrak{A}$ the situation is quite different.

We shall denote by X_0 the set of "subconsistent" families $(K_i)_{i\in I}$ of compact non empty subsets of the X_i. By sub-consistency we mean that $\varphi_{ij} K_j \subseteq K_i$ holds for all $i \leqq j$. In short, we can write

$$X_0 = \{(K_i) \; : \; K_i \neq \emptyset, \forall i; \; \varphi_{ij} K_j \subseteqq K_i, \forall i \leqq j\} .$$

For each $i_0 \in I$ we define a correspondence $\varphi_{i_0}: X_0 \to X_{i_0}$ by

$$\varphi_{i_0}((K_i)) = K_{i_0}, \quad (K_i) \in X_0 .$$

In X_0 we introduce an ordering as follows:

$$x_1 \leqq x_2 \Leftrightarrow \forall i: \; \varphi_i(x_1) \subseteqq \varphi_i(x_2);$$

here x_1 and x_2 denote elements of X_0. We define a subset X_{00} of X_0 by

$$X_{00} = \{x \in X_0 \; : \; \forall ij: \; \varphi_{ij}(\varphi_j(x)) = \varphi_i(x)\} ,$$

in other words, X_{00} consists of the consistent families (K_i) in X_0.

By X we denote the set of minimal elements of X_0. In case X is non empty, we provide X with a topology, namely the weak topology determined by the correspondences $\varphi_i: X \to X_i$ (now, of course, φ_i denotes the restriction of the previous φ_i to X). The subsets of X of the form $\varphi_i{}^s(G)$ with $i \in I$ and $G \in \mathscr{G}(X_i)$ constitute a basis for the topology on X.

We shall prove that if there exists a universal projective limit of (X_i, φ_{ij}) at all, then it must be (isomorphic to) X together with the correspondences φ_i (à propos: two objects in $\mathfrak{A}$ are isomorphic if and only if they are homeomorphic).

6.2 Lemma. *For each $i \in I$, let A_i be a subset of X_i (perhaps A_i is empty for some or even for all i). Consider the set*

$$Q = \{q \in X_0 \; : \; \varphi_i(q) \supseteqq A_i, \forall i\} ,$$

and provide Q with the ordering induced from X_0. Then:

(i): *For every $q \in Q$ there exists a minimal element of Q dominated by q.*

(ii): *If q_1 and q_2 are distinct minimal elements of Q, then, for at least one $i \in I$, the sets $\varphi_i(q_1)$ and $\varphi_i(q_2)$ are disjoint.*

(iii): *If each A_i is non empty, and if Q is non empty, then Q contains a smallest element.*

(iv): *If $\varphi_{ij} A_j \supseteqq A_i$ holds for all $i \leqq j$, then every minimal element of Q belongs to X_{00}.*

PROOF. (i): For the given element $q \in Q$ consider the set

$$A = \{a \in Q \ : \ a \leqq q\}.$$

Let B be a totally ordered (non empty) subset of A. For each $i \in I$ define a compact subset K_i of X_i by

$$K_i = \bigcap_{b \in B} \varphi_i(b).$$

A compactness argument tells us that K_i is non empty. Clearly, $\varphi_i(q) \supseteqq K_i \supseteqq A_i$ holds. For $i \leqq j$ we have

$$\varphi_{ij} K_j \subseteqq \bigcap_{b \in B} \varphi_{ij} \varphi_j(b) \subseteqq \bigcap_{b \in B} \varphi_i(b) = K_i.$$

Thus $(K_i) \in Q$. We have seen that every totally ordered subset of A has a minorant in A. Hence, by Zorn's lemma, A has a minimal element.

(ii): Let q_1 and q_2 be minimal elements of Q and assume that $\varphi_i(q_1) \cap \varphi_i(q_2)$ is non empty for all i. Put $K_i = \varphi_i(q_1) \cap \varphi_i(q_2)$. Then (K_i) is an element of Q dominated by q_1 as well as by q_2. By minimality, $(K_i) = q_1$ and $(K_i) = q_2$ follow. Thus $q_1 = q_2$.

(iii): Follows from (i) and (ii).

(iv): Let q be a minimal element of Q. Define, for each $i \in I$, a subset K_i of X_i by

$$K_i = \bigcap_{j \geqq i} \varphi_{ij} \varphi_j(q).$$

We can also write $(\varphi_{ij} \varphi_j(q)) \downarrow K_i$. Since all the $\varphi_{ij} \varphi_j(q)$'s are compact non empty, K_i is compact and non empty as well. Since for $i \leqq j$ we have

$$\varphi_{ij} K_j \subseteqq \bigcap_{k \geqq j} \varphi_{ij} \varphi_{jk} \varphi_k(q) = \bigcap_{k \geqq j} \varphi_{ik} \varphi_k(q) = K_i,$$

(K_i) is an element of X_0. We also have

$$K_i \supseteqq \bigcap_{j \geqq i} \varphi_{ij} A_j \supseteqq A_i,$$

hence $(K_i) \in Q$. Since $(K_i) \leqq q$ and since q is minimal, $(K_i) = q$ follows, that is, we have

$$\bigcap_{j \geqq i} \varphi_{ij} \varphi_j(q) = \varphi_i(q)$$

for all i. The equality $\varphi_{ij} \varphi_j(q) = \varphi_i(q)$ for all $i \leqq j$ follows, hence $q \in X_{00}$.

For the moment we shall only apply the result just proved to the case where all the A_i's are empty. Then $Q = X_0$ and the set of minimal elements of Q is identical to the set X. By (iv), X is a subset of X_{00}. By (i), X is non empty if and only if X_0 is non empty. Let us prove that X is a Hausdorff space (assuming $X \neq \emptyset$). Let x_1 and x_2 be distinct elements of X and choose, according to (ii), $i \in I$ so that

$$\varphi_i(x_1) \cap \varphi_i(x_2) = \emptyset.$$

Then choose disjoint open neighbourhoods G_1 and G_2 of the compact sets $\varphi_i(x_1)$ and $\varphi_i(x_2)$, respectively. Then $\varphi_i{}^s G_1$ and $\varphi_i{}^s G_2$ are disjoint neighbourhoods of x_1 and x_2. Let us put some basic facts on X together:

6.3 LEMMA. *Assume that* $X \neq \emptyset$. *Then* (X, φ_i) *is a projective limit of* (X_i, φ_{ij}) *in the category* $\mathfrak{A}$. *Furthermore, the cylinder sets separate points; they even separate compact sets.*

When we speak of cylinder sets in X we have the mapping $\alpha \to (G_\alpha, F_\alpha)$ from Section 5 in mind. Note that the considerations centering around Lemma 5.2 were purely topological.

We now assume that (X_i, φ_{ij}) admits a universal projective limit (Z, ψ_i). We aim at proving that then (X, φ_i) is a universal projective limit too. First we remark that X must be non empty; to see this, choose an element $z \in Z$; then $(\psi_i(z)) \in X_{00}$ and $X \neq \emptyset$.

6.4 LEMMA. *Let* (Z, ψ_i) *be a universal projective limit.*

(i): *To any* $x \in X_{00}$ *there exists one and only one compact subset* A *of* Z *with* $\psi_i A = \varphi_i x$ *for all* i.

(ii): *If* A *and* A' *are compact non empty subsets of* Z *with* $\psi_i A = \psi_i A'$ *for all* i, *then* $A = A'$.

(iii): *If* A *and* A' *are compact non empty subsets of* Z *with* $\psi_i A \subseteq \psi_i A'$ *for all* i, *then* $A \subseteq A'$.

PROOF. (i): Denote by Y the topological space consisting of the one element x. Then Y is provided in the natural way with morphisms $\varphi_i \colon Y \to X_i$. It is easy to check that (Y, φ_i) is a projective limit of (X_i, φ_{ij}). Hence there exists a uniquely determined morphism $\theta \colon Y \to Z$ such that $\psi_i \theta = \varphi_i$ holds for all i. This proves (i) (put $A = \theta y = \theta(x)$).

(ii) follows from (i), and (iii) follows from (ii) applied to the sets A' and $A \cup A'$.

We now define a function (note, not a correspondence) $f \colon Z \to X_{00}$ by

$$f(z) = (\psi_i(z)), \quad z \in Z .$$

We shall prove in several steps that f is a homeomorphism of Z onto X:

$$f \text{ is one-to-one}, \tag{6.5}$$

$$f(Z) \supseteq X, \tag{6.6}$$

$$f(Z) \subseteq X, \tag{6.7}$$

$$f \text{ is continuous}, \tag{6.8}$$

$$f^{-1} \text{ is continuous}. \tag{6.9}$$

Proof of (6.5): If $f(z)=f(z')$, apply (ii) Lemma 6.4 to the sets $A=\{z\}$ and $A'=\{z'\}$. We conclude that $A=A'$, hence $z=z'$.

Proof of (6.6): Let $x\in X$ be given. Denote by A the set determined from x as explained in (i) Lemma 6.4. Choose $a\in A$. Then $f(a)\leqq x$ holds and, since x is minimal, $f(a)=x$ follows.

Proof of (6.7): Let $x\in f(Z)$, say $x=f(z)$. Let x_1 be an element of X with $x_1\leqq x$ (apply (i) of Lemma 6.2). By (6.6) there exists $z'\in Z$ such that $f(z')=x_1$. By (iii) Lemma 6.4 we have $\{z'\}\subseteqq\{z\}$, hence $z'=z$. Then $x=f(z)=f(z')=x_1\in X$.

Proof of (6.8): This follows from the fact that $\varphi_i f=\psi_i$, $\forall i$ in connection with (i) Lemma 1.11.

Proof of (6.9): Since X is non empty, (X,φ_i) is a projective limit. Thus there exists one and only one morphism $\theta\colon X\to Z$ such that $\psi_i\theta=\varphi_i$, $\forall i$. Let x be an element of X. Then $A=\theta(x)$ is compact and $\psi_i A=\varphi_i(x)$, $\forall i$. The same can be said about the one-point set $f^{-1}(x)$. Thus $A=f^{-1}(x)$. By this argument we see that θ and f^{-1} coincide ($\theta(x)=\{f^{-1}(x)\}$ for all $x\in X$). Since θ is u.s.c., f^{-1} must be continuous.

6.10 Theorem. *If (X_i,φ_{ij}) admits a universal projective limit, then (X,φ_i) is a universal projective limit.*

Proof. Let (Z,ψ_i) be a universal projective limit. What we have seen is that there exists a homeomorphism $f\colon Z\to X$ such that $\varphi_i f=\psi_i$ and $\psi_i f^{-1}=\varphi_i$ for all i. The assertion follows from these facts by a standard argument. Let us write down the details: If (Y,ξ_i) is a projective limit, then there exists $\theta\colon Y\to Z$ such that $\psi_i\theta=\xi_i$, $\forall i$. Put $\theta'=f\theta$. Then θ' is a morphism $Y\to X$ for which $\varphi_i\theta'=\xi_i$, $\forall i$. If $\theta''\colon Y\to X$ is another such morphism, then the morphisms

$$f^{-1}\theta'\colon Y\to Z \quad\text{and}\quad f^{-1}\theta''\colon Y\to Z$$

satisfy

$$\psi_i(f^{-1}\theta') = \xi_i \quad\text{and}\quad \psi_i(f^{-1}\theta'') = \xi_i$$

for all i. Since (Z,ψ_i) is universal, $f^{-1}\theta'=f^{-1}\theta''$ follows. Thus $\theta'=f(f^{-1}\theta') = f(f^{-1}\theta'') = \theta''$.

Somewhat justified by Theorem 6.10 we write $\varprojlim X_i$ instead of X, or instead of (X,φ_i). This notation is not in agreement with the usual notation in category theory since, in a great many cases, $\varprojlim X_i$ will not be a universal projective limit.

We shall now discuss under which conditions X is a universal projective limit. We shall introduce four conditions on the given projective system (X_i, φ_{ij}).

By $[O]$ we denote the condition that X be non empty or, equivalently, that X_0 be non empty.

By $[M]$ we denote a condition somewhat resembling the maximality condition of Bochner. What we demand is, that to any $y \in X_{00}$ and to any $i \in I$ and any $x_i \in \varphi_i(y)$ there exists an element $x \in X$ with $x \leqq y$ and $x_i \in \varphi_i(x)$.

Note that if $y \in X_0$ and if we put $K_i = \varphi_i(y)$, $\forall i$, then (K_i, φ_{ij}) is again a projective system in the category $\mathfrak{A}$. By $[K]$ we denote the condition that for any $y = (K_i) \in X_{00}$, the topological space $\varprojlim K_i$ be compact.

Let $\mathscr{G}_0$ denote the class of sets of the form $\varphi_i{}^s G$ with $i \in I$ and $G \in \mathscr{G}(X_i)$. Denote by $[N]$ the condition introduced earlier in connection with Lemma 1.11; in the present case this condition amounts to the requirement that whenever $K \in \mathscr{K}(X)$ and $N(K)$ is a neighbourhood of K, we can find $G \in \mathscr{G}_0$ with $K \subseteqq G \subseteqq N(K)$.

Note, that $[M]$ holds if and only if

$$\varphi_{i_0}(\varprojlim K_i) = K_{i_0}$$

for any $(K_i) \in X_{00}$ and any $i_0 \in I$.

6.11 Lemma. *If $[K]$ is fulfilled, then $\varprojlim K_i$ is compact for any family (K_i) in X_0.*

Proof. Given $(K_i) \in X_0$, put $A = \varprojlim K_i$. Define for each i a subset of X_i by

$$A_i = \varphi_i(A) .$$

Consider the set Q of Lemma 6.2 associated with the A_i. Since each A_i is contained in K_i, Q is non empty. From Lemma 6.2 we conclude that there exists a smallest element $x \in Q$ and that this x is in X_{00}. Thus there exists $(K_i') \in X_{00}$ such that

$$A_i \subseteqq K_i' \subseteqq K_i, \forall i .$$

We claim that

$$\varprojlim K_i = \varprojlim K_i' .$$

The inclusion "$\supseteqq$" is clear. Assume that $x_0 \in \varprojlim K_i = A$. Then it follows that

$$\varphi_i(x_0) \subseteq \varphi_i(A) = A_i \subseteq K_i'$$

for all i and $x \in \varprojlim K_i'$. Lastly, we just have to remark that $\varprojlim K_i'$ is compact by [K].

6.12 Conjecture. *The fulfillment of* [O], [M], *and* [K] *is necessary and sufficient for* (X_i, φ_{ij}) *to admit a universal projective limit.*

The necessity part of this conjecture is true as will follow from the result below in connection with Theorem 6.10.

6.13 Theorem. *If* $\varprojlim X_i$ *is a universal projective limit, then* [O], [M] *and* [K] *hold.*

Proof. Clearly, [O] is fulfilled.

Let $y \in X_{00}$. Put $K_i = \varphi_i(y)$, $\forall i$. Denote by A the uniquely determined compact subset of X for which $\varphi_i(A) = K_i$ holds for all i (cf. Lemma 6.4). We claim that

$$A = \bigcap_{i \in I} \varphi_i^s(K_i)\,. \tag{6.14}$$

The inclusion "$\subseteq$" follows since $\varphi_i(A) = K_i$. To prove the other inclusion, let $z \in \bigcap_i \varphi_i^s K_i$ be given. By (ii) of Lemma 6.4 we conclude that $A = A \cup \{z\}$, thus $z \in A$. This proves (6.14). By (6.14) we can also write $A = \varprojlim K_i$ (this formula also involves the topology of the two sets). Thus $\varprojlim K_i$ is compact and we have established condition [K].

For the set A above we have $\varphi_i A = K_i$. Therefore, if we consider an $i \in I$ and a point $x_i \in K_i$, we can find $a \in A$ such that $x_i \in \varphi_i(a)$. This argument proves the validity of [M].

6.15 Lemma. *Assume that* [M] *and* [K] *hold. Then, for any* $(K_i) \in X_{00}$ *there exists one and only one compact subset* A *of* X *for which* $\varphi_i A = K_i$ *holds for all* i. *Furthermore, this set is given by the formula*

$$A = \bigcap_{i \in I} \varphi_i^s(K_i) = \bigcap_{i \in I} \varphi_i^w(K_i)\,. \tag{6.16}$$

Proof. Put $A = \bigcap_{i \in I} \varphi_i^s K_i$. Then $A = \varprojlim K_i$, and A compact follows from [K]. Clearly, $\varphi_i A \subseteq K_i$ holds for all i. By [M], we deduce the other inclusion $\varphi_i A \supseteq K_i$. Thus $\varphi_i A = K_i$ holds for all i. This proves existence.

To prove uniqueness, let $A' \in \mathscr{K}(X)$ be another set with $\varphi_i A' = K_i$ for all i. Then $A' \subseteq A$ clearly holds. Assume now that $x \in X \setminus A'$.

By Lemma 6.3 we can find $i \in I$ such that $\varphi_i x \cap \varphi_i A' = \emptyset$, that is,

$$\varphi_i(x) \cap K_i = \emptyset \quad \text{or} \quad x \notin \varphi_i^w K_i .$$

We have seen that

$$\complement A' \subseteq \bigcup_{i\in I} \complement \varphi_i^w K_i$$

holds. Then we have

$$A' \subseteq A = \bigcap_{i\in I} \varphi_i^s K_i \subseteq \bigcap_{i\in I} \varphi_i^w K_i \subseteq A'$$

and conclude that $A = A'$ and that formula (6.16) holds.

6.17 Remark. The above proof shows that if $[M]$ holds, then we have for any $(K_i) \in X_{00}$

$$\varprojlim K_i = \bigcap_{i\in I} \varphi_i^w K_i .$$

In particular, $\varprojlim K_i$ is a closed subset of X. As a simple consequence of this remark and of Lemma 6.11 we see that if each X_i is compact and if $[M]$ holds, then condition $[K]$ is equivalent to the compactness of $X = \varprojlim X_i$.

Assume now that $[O]$, $[M]$ and $[K]$ hold, and let us try to prove that X is a universal projective limit. Then we shall consider some other projective limit, say (Y, ψ_i). It is our task to investigate whether there exists a uniquely determined θ which to any $y \in Y$ assigns a subset $\theta(y)$ of X such that the following conditions hold:

(6.18) θ is a correspondence (that is, $\theta(y) \neq \emptyset$, $\forall y$),

(6.19) θ is u.s.c.,

(6.20) θ is compact-valued,

(6.21) $\varphi_i \theta = \psi_i$, $\forall i$.

By Lemma 6.15 we see that there is only one hope for such a θ viz. the θ given by

(6.22) $$\theta(y) = \bigcap_{i\in I} \varphi_i^s(\psi_i(y)) .$$

By Lemma 6.2 this θ satisfies (6.18), and (6.20) and (6.21) are taken care of by Lemma 6.15. So it only remains to establish (6.19), that is, the u.s.c. of θ. However, due to Lemma 1.11, or rather the remarks to this lemma, we run into difficulties here. Even though we can not prove in general that θ is u.s.c., we can prove something without introducing further conditions:

6.23 Lemma. *Assume that $[O]$, $[M]$ and $[K]$ hold and let (Y, ψ_i) be a projective limit. Define $\theta\colon Y \to X$ by* (6.22).

Then the graph of θ is closed, and for any compact subset K of Y we have

$$\theta K = \bigcap_{i\in I}\varphi_i^s(\psi_i K) = \bigcap_{i\in I}\varphi_i^w(\psi_i K) \,.$$

In particular, $\theta(K)$ is compact for all $K\in\mathscr{K}(Y)$. Thus, by Lemma 1.9, *θ is almost* u.s.c.![1])

Proof. Let (y_α,x_α) be a net on the graph of θ and assume that $y_\alpha\to y$, $x_\alpha\to x$. If (y,x) does not belong to the graph of θ, then there exists an $i\in I$ with $\varphi_i(x)\cap\psi_i(y)=\emptyset$. Choose disjoint neighbourhoods $N(\varphi_i x)$, $N(\psi_i y)$. Since φ_i is u.s.c. and since $x_\alpha\to x$, we have $\varphi_i(x_\alpha)\subseteq N(\varphi_i x)$, eventually. A similar argument shows that $\psi_i(y_\alpha)\subseteq N(\psi_i y)$, eventually. We conclude that there exists an α such that both $\varphi_i(x_\alpha)\subseteq N(\varphi_i x)$ and $\psi_i(y_\alpha)\subseteq N(\psi_i y)$ hold. For this α,

$$\varphi_i(x_\alpha)\cap\psi_i(y_\alpha) = \emptyset$$

holds, in other words, (y_α,x_α) does not belong to the graph of θ. This contradiction proves that the graph of θ is closed.

Now let $K\in\mathscr{K}(Y)$ and put

$$A = \bigcap_{i\in I}\varphi_i^s(\psi_i K) \,.$$

Then A is compact and $\varphi_i A=\psi_i K$, $\forall i$. Since

$$\varphi_i(\theta K) = (\varphi_i\theta)K = \psi_i K, \forall i \,,$$

$\theta K\subseteq\varphi_i^s(\psi_i K)$ for all i, and we have seen that $\theta K\subseteq A$. Since K is compact and the graph of θ closed, the set θK is closed. Hence, as a closed subset of a compact set, θK is compact. Put $A'=\theta K$, apply Lemma 6.15 and conclude that $A=A'$. We then have $\theta K=A$.

If we introduce the further condition $[N]$, then we obtain from the above discussion and from Lemma 1.11 the following theorem.

6.24. Theorem. *If $[O]$, $[M]$, $[K]$ and $[N]$ hold, then $\varprojlim X_i$ is a universal projective limit.*

6.25 Theorem. *If all the X_i's are compact, then the conditions $[O]$, $[M]$ and $[K]$ are sufficient to ensure that $\varprojlim X_i$ is a universal projective limit.*

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[1]) If, for instance, X is a k-space, then θ will be u.s.c.

This result can either be obtained as a corollary to Lemma 6.23 and Lemma 1.9 or as a corollary to Theorem 6.24 (the fact that [N] holds follows since the cylinder sets separate compact sets).

6.26 THEOREM. *Assume that each X_i is metrizable, or, more generally, that to each $i \in I$ and each $K_i \in \mathscr{K}(X_i)$ there exists a countable base for the neighbourhood system of K_i. Assume, furthermore, that I contains a cofinal sequence.*

Then the conditions [O], [M] and [K] are sufficient to ensure that $\varprojlim X_i$ is a universal projective limit.

PROOF. To simplify the proof notationally we shall assume that $I = \{1,2,3,\ldots\}$ in the usual ordering. This assumption also leeds to one or two real simplifications. What we shall prove is that [N] holds (given that [O], [M] and [K] hold). Let $K \in \mathscr{K}(X)$ and $N(K)$, a neighbourhood of K, be given. Put $K_i = \varphi_i K$; $i \in I$. We claim that we can find sets $(G_{im})_{i=1,2,\ldots;\, m=1,2,\ldots}$ such that:

(i) $\forall i,m$: $G_{im} \in \mathscr{G}(X_i)$;
(ii) $\forall i$: $(G_{im}) \downarrow K_i$;
(iii) $\forall i$: $(G_{im})_m$ is a neighbourhood base for K_i;
(iv) $\forall i \leqq j\ \forall m$: $\varphi_{ij}(G_{jm}) \subseteq G_{im}$.

To see this, first find sets (G'_{im}) such that (i), (ii) and (iii) hold. Then define the G_{im} recursively by

$$\begin{aligned} G_{1m} &= G'_{1m}, \forall m; \\ G_{i+1,m} &= G'_{i+1,m} \cap \varphi^{s}_{i,i+1}(G_{im}), \forall m, \quad i=1,2,\ldots, \end{aligned}$$

For the purpose of an indirect proof, assume that whenever $K \subseteq \varphi_i{}^s G_i$ holds (with $G_i \in \mathscr{G}(X_i)$), the inclusion $\varphi_i{}^s G_i \subseteq N(K)$ fails to hold. Therefore we can find a sequence (x_i) such that

$$x_i \in \varphi_i{}^s(G_{ii}) \setminus N(K), \quad i=1,2,\ldots .$$

For $i \leqq j$,

$$\varphi_i(x_j) = \varphi_{ij}\varphi_j(x_j) \subseteq \varphi_{ij}(G_{jj}) \subseteq G_{ij} .$$

Define, for each i, subsets of X_i by

$$A_i = K_i \cup \textstyle\bigcup_{\nu=1}^{\infty} \varphi_i(x_\nu) .$$

The above inclusions in connection with Lemma 7.5 of [9] imply that the sets

$$\textstyle\bigcup_{\nu=i}^{\infty}(K_i \cup \varphi_i(x_\nu)), \quad i=1,2,\ldots$$

are compact. Therefore, each A_i is compact. Since the consistency relations $\varphi_{ij}A_j = A_i$, $i \leqq j$, obviously hold, $(A_i) \in X_{00}$. Put

$$A = \varprojlim A_i .$$

By the remarks to Theorem 6.25, condition $[N]$ holds in the space A. Since $N(K) \cap A$ is a neighbourhood of K in that space, we can find $i \in I$ and $G_i \in \mathscr{G}(X_i)$ such that

$$K \subseteq \varphi_i{}^s(G_i) \cap A \subseteq N(K) \cap A .$$

Determine $j \geqq i$ such that $G_{ij} \subseteq G_i$ holds. Then $x_j \in \varphi_i{}^s(G_i) \cap A$, and $x_j \in N(K)$ follows. This contradiction establishes condition $[N]$.

By three concrete examples we shall now show that none of the conditions $[O]$, $[M]$ or $[K]$ can be omitted in Conjecture 6.12. In these examples, I will be the set of natural numbers in the usual ordering, and all the spaces X_i will be discrete spaces.

6.27 Example. This example is to show that $[M]$ and $[K]$ may hold but $[O]$ fail.

We put $X_i = \{1, 2, \ldots\}$ for each $i = 1, 2, \ldots$, and define the φ_{ij} by the consistency relations and by the requirement

$$\varphi_{i,i+1}(n) = \{n, n+1\}, \quad i \in I, \ n \in X_{i+1} .$$

It is quite easy to see that X_{00} is empty.

6.28 Example. This example is to show that $[O]$ and $[K]$ may hold but $[M]$ fail.

We put $X_i = \{0, 1\}$ for all $i \in I$ and define the φ_{ij} by consistency and by the requirement

$$\varphi_{i,i+1}(0) = \{0\}, \quad \varphi_{i,i+1}(1) = \{0, 1\}, \qquad i \in I .$$

Then X consists of the one element $x = (\{0\})_i$ and X_{00} contains one further element viz. $(X_i)_i$. Clearly, $[M]$ fails but $[O]$ and $[K]$ hold.

6.29 Example. This example is to show that $[O]$ and $[M]$ may hold but $[K]$ fail.

We put $X_i = \{1, 2, \ldots, i\}$ for all $i \in I$. The φ_{ij} are defined by consistency and by the requirements

$$\begin{aligned} \varphi_{i,i+1}(n) &= \{n\}, \quad n \leqq i , \\ \varphi_{i,i+1}(i+1) &= X_i \end{aligned}$$

for all $i \in I$.

X consists of countably many elements $x_1, x_2, \ldots$, where x_{i_0} is defined by

$$\begin{aligned} \varphi_i(x_{i_0}) &= \{i_0\} \quad \text{if } i \geqq i_0\,, \\ &= X_i \quad \text{if } i < i_0\,. \end{aligned}$$

We leave to the reader to verify this; the reader is also asked to verify [M]. That [O] holds, is of course trivial. That [K] fails, follows from the fact that the topology of X is the discrete topology (for $G_i \in \mathscr{G}(X_i)$, defined by $G_i = \{i\}$, we have $\varphi_i{}^s(G_i) = \{x_i\}$).

We remark that we have been unable to find an example where [O], [M] and [K] hold but [N] fails. Due to Theorem 6.26 such an example, if it exists, can not be as simple as the examples above.

Of the conditions introduced for the discussion of our problem, condition [O] is of course quite innocent and we feel that condition [M] is in an acceptable form. However, condition [K] is not in a form which allows one to decide easily, in concrete cases, whether or not it holds. We shall now give a lemma which improves on this situation. The lemma is connected with the usual proof of the Tychonoff compactness theorem, but when we compare the present setup with the setup in the Tychonoff theorem we find that additional difficulties arise. The convergence of sets appearing in the lemma is that of closed topological convergence (cf. P8 of [9]).

6.30 Lemma. *Assume that all the X_i are compact. Let (x_α) be a universal net on $X = \varprojlim X_i$. Then:*

(i): *For each $i \in I$ there exists a compact subset K_i of X_i such that $\varphi_i(x_\alpha) \to K_i$.*

(ii): *For each i, the set K_i is non empty.*

(iii): *For $i \leqq j$, the inclusion $\varphi_{ij} K_j \supseteqq K_i$ holds.*

(iv): *There exists a smallest element $(K_i{}^*)$ of X_0 such that $K_i{}^* \supseteqq K_i$ for all $i \in I$.*

(v): *$(K_i{}^*)$ is an element of X_{00}.*

(vi): *A necessary and sufficient condition that (x_α) converges, is that $(K_i{}^*) \in X$ (in which case $x_\alpha \to (K_i{}^*)$ holds).*

The proof is left to the reader. Parts of it are straight forward, and parts of it follow from Lemma 6.2.

By simple examples it can be seen that $(K_i{}^*)$ may be distinct from

(K_i). Thus take $I=\{1,2\}$, $X_1=\{0,1\}$, $X_2=\{1,2,3,\ldots\}\cup\{\infty\}$ and define φ_{12} by $\varphi_{12}(\infty)=\{0,1\}$, $\varphi_{12}(n)=\{0\}$; consider a universal subnet of the sequence $(n)_{n=1,2,\ldots}$, noting that X and X_2 are to be identified.

One may wonder if the K_i^* can be described explicitly — perhaps by the formula

$$K_i^* = \bigcup_{s\geqq i} \bigcap_{j\geqq s} \varphi_{ij} K_j \,.$$

At last, let us briefly discuss the example in which all the φ_{ij} are functions. By an argument which we shall leave to the reader, it can be seen that X consists of those families $(x_i)_{i\in I}$ of points of the spaces X_i for which $\varphi_{ij}(x_j)=\{x_i\}$ for all $i\leqq j$. All the φ_i are then functions. It is easy to deduce from Lemma 6.30 that [K] holds. Also, it is easy to see that [N] holds. By a well-known argument it can be proved that [M] holds. Thus, a necessary and sufficient condition that X be a universal projective limit is that [O] holds. In case I contains a cofinal sequence or in case all the X_i are compact, condition [O] holds. It is known that [O] does not always hold (cf. [3] or [4]).

7. On the existence of universal projective limits in $\mathfrak{A}_t$.

We shall study a projective system (X_i,μ_i,φ_{ij}) in the category $\mathfrak{A}_t$. By (X,φ_i) we denote the projective limit of (X_i,φ_{ij}) introduced in the previous section (cf. Lemma 6.3).

7.1 Lemma. *Assume that (X_i,φ_{ij}) satisfies conditions [O], [M] and [K] of Section 6. Then a necessary and sufficient condition that a projective limit can be realized on X is that*

$$\sup_{(K_i)\in X_{00}} \inf_{i\in I} \mu_i(K_i) = 1 \,.$$

This result is easily derived from Theorem 5.3 and the results of Section 6.

7.2 Theorem. *Assume that (X_i,φ_{ij}) admits a universal projective limit in the category $\mathfrak{A}$.*

(i): *If a projective limit measure μ can be realized on X, then (X,μ,φ_i) is a universal projective limit of (X_i,μ_i,φ_{ij}).*

(ii): *If (X_i,μ_i,φ_{ij}) has a universal projective limit in the category $\mathfrak{A}_t$, then it can be realized on X.*

Proof. (i): Let (Y,η,ψ_i) be a projective limit of (X_i,μ_i,φ_{ij}). Let

$\theta\colon Y \to X$ denote the uniquely determined morphism in $\mathfrak{A}$ for which $\varphi_i\theta = \psi_i$ holds for all i. Then we have, for $K \in \mathscr{K}(Y)$:

$$\mu(\theta K) = \inf_i \mu_i(\varphi_i\theta K) = \inf_i \mu_i(\psi_i K) \geqq \eta K ,$$

and it follows that θ is measure preserving. Hence θ is a morphism in $\mathfrak{A}_t$.

(ii): Assume that (Z,ζ,ξ_i) is a universal projective limit. Denote by θ the uniquely determined morphism $Z \to X$ in $\mathfrak{A}$ for which $\varphi_i\theta = \xi_i$ holds for all i. We have

$$\begin{aligned} \sup_{K\in\mathscr{K}(X)} \inf_i \mu_i(\varphi_i K) &\geqq \sup_{L\in\mathscr{K}(Z)} \inf_i \mu_i(\varphi_i\theta L) \\ &= \sup_{L\in\mathscr{K}(Z)} \inf_i \mu_i(\xi_i L) \\ &\geqq \sup_{L\in\mathscr{K}(Z)} \zeta(L) = 1 , \end{aligned}$$

and it follows by Theorem 5.3 that we can realize a projective limit measure μ on X. For each compact subset L of Z we have

$$\mu(\theta L) = \inf_i \mu_i(\varphi_i\theta L) = \inf_i \mu_i(\xi_i L) \geqq \zeta L .$$

Hence θ is measure preserving. From this fact and from the universal property of Z, it is easy to deduce that X has the universal property (also employ the identities $\varphi_i\theta = \xi_i$).

8. Variants of the result on projective limits of measure spaces.

We believe that for the problem of projective limits of measure spaces, the category $\mathfrak{A}_t$ is the cleanest and perhaps most natural one. However, it is clear that by changing the category slightly, one can obtain results closely related to our main theorem, Theorem 5.3. For instance, this applies if one works with correspondences for which images of compact sets are measurable or strong inverse images of open sets are measurable. The two results below are not obtained by introducing measurability conditions — we still stick to simple topological conditions on the correspondences. Both results can be seen as an attempt to avoid any compactness assumptions. In the first result we drop the requirement that the correspondences be compact-valued, and in the second we also try to avoid the tightness assumption imposed on the measures.

8.1 Definition. By $\mathfrak{A}_t'$ we denote the category whose objects are pairs (X,μ) with X a Hausdorff space and μ a tight probability measure on X, and whose morphisms $\varphi\colon (X,\mu) \to (Y,\eta)$ are u.s.c. open and measure preserving correspondences $X \to Y$.

8.2 THEOREM. *Let* $(X_i, \mu_i, \varphi_{ij})$ *be a projective system in the category* $\mathfrak{A}_i'$, *and let* X *be a target space with* u.s.c. *open correspondences* $\varphi_i\colon X \to X_i$. *Assume that the cylinder sets separate points.*

Then a projective limit can be realized on X *if and only if*

$$\sup_{K\in\mathscr{K}(X)}\inf_{G\supseteqq K}\inf_i\mu_i(\varphi_i G) = 1 , \tag{8.3}$$

and when this condition is satisfied, the projective limit measure, which is unique, is given by the formula

$$\mu A = \sup_{K\subseteqq A}\inf_{G\supseteqq K}\inf_i\mu_i(\varphi_i G), \quad A \in \mathscr{B}(X) . \tag{8.4}$$

PROOF. Uniqueness of the projective limit measure follows as in the proof of Theorem 5.3. The necessity of (8.3) follows from

$$1 = \mu X = \sup_K\inf_{G\supseteqq K}\mu G \leqq \sup_K\inf_{G\supseteqq K}\inf_i\mu_i(\varphi_i G) .$$

To prove sufficiency of (8.3), consider the functions ν and λ defined on the open and the compact subsets of X, respectively, by

$$\nu G = \inf_i\mu_i(\varphi_i G), \quad G \in \mathscr{G}(X) ,$$
$$\lambda K = \inf_{G\supseteqq K}\nu G, \quad K \in \mathscr{K}(X) .$$

For $j \geqq i$ we have

$$\mu_j(\varphi_j G) \leqq \mu_i(\varphi_i(\varphi_j G)) = \mu_i(\varphi_i G) .$$

Thus we may replace the "inf" occuring in the definition of ν by a "lim", hence ν is subadditive. Since ν is clearly monotone, it follows from the proof of (i), Lemma 2 of [10] that λ will be tight if we merely can prove that

$$\lambda(K_1 \cup K_2) \geqq \lambda K_1 + \lambda K_2$$

holds for all pairs of disjoint compact subsets of X. Let then K_1, K_2 be such a pair. To $\varepsilon > 0$ we choose $G \supseteqq K_1 \cup K_2$ such that

$$\nu G < \lambda(K_1 \cup K_2) + \varepsilon .$$

Then we choose i_0 so that

$$\mu_i(\varphi_i G) < \lambda(K_1 \cup K_2) + \varepsilon$$

holds for all $i \geqq i_0$. Since the cylinder sets separate compact sets (cf. Lemma 4.8), we can find $G_1 \supseteqq K_1$ and $G_2 \supseteqq K_2$ and $i \in I$ such that $\varphi_i G_1$ and $\varphi_i G_2$ are disjoint. We may assume that $i \geqq i_0$ and also that $G_1 \cup G_2 \subseteqq G$. Then we have

$$\begin{aligned}\lambda K_1+\lambda K_2 &\leqq \nu G_1+\nu G_2\\ &\leqq \mu_i(\varphi_i G_1)+\mu_i(\varphi_i G_2)\\ &= \mu_1(\varphi_1 G_1\cup\varphi_i G_2) \leqq \mu_i(\varphi_i G) < \lambda(K_1\cup K_2)+\varepsilon\ ,\end{aligned}$$

from which we obtain the desired conclusion.

Since λ is now known to be tight, we see that μ defined by (8.4) is a tight measure on X, and due to (8.3), $\mu X=1$ holds. For $G\in\mathscr{G}(X)$ and $i\in I$ we have, by (8.4), that $\mu G\leqq\mu_i(\varphi_i G)$. It follows, that for $i\in I$ and $G_i\in\mathscr{G}(X_i)$ we have

$$\mu(\varphi_i{}^s G_i) \leqq \mu_i(\varphi_i(\varphi_i{}^s G_i)) \leqq \mu_i G_i\ ,$$

hence φ_i is measure preserving.

8.3 Definition. By $\mathfrak{A}_\tau$ we denote the category whose objects are pairs (X,μ) with X a regular space and μ a τ-smooth probability measure on X, and whose morphisms $\varphi\colon(X,\mu)\to(Y,\eta)$ are u.s.c. open and measure preserving correspondences $X\to Y$.

8.4 Theorem. *Let (X_i,μ_i,φ_{ij}) be a projective system in the category $\mathfrak{A}_\tau$, and let X be a target space (regular) with* u.s.c. *open correspondences $\varphi_i\colon X\to X_i$. Assume that the cylinder sets almost separate points and closed sets. Consider the set function ν on $\mathscr{G}(X)$ defined by*

$$\nu G = \inf{}_i\mu_i(\varphi_i G)\ .$$

Then sufficient conditions that a projective limit can be realized on X are that ν be τ-smooth at $\emptyset$ with respect to $\mathscr{F}(X)$ (cf. P12 *of* [9]) *and that $\mu_i(\varphi_i X)=1$ holds for all $i\in I$. Under these conditions the projective limit measure, which is unique, is given by*

$$\mu A = \sup{}_{F\subseteqq A}\inf{}_{G\supseteqq F}\nu G,\quad A\in\mathscr{B}(X)\ . \tag{8.5}$$

Proof. Define $\lambda\colon\mathscr{F}(X)\to\mathsf{R}_+$ by

$$\lambda F = \inf{}_{G\supseteqq F}\nu G\ .$$

Then λ is monotone and τ-smooth at $\emptyset$, and ν is monotone and subadditive (subadditive since $\nu G=\lim\mu_i(\varphi_i G)$). If we can establish the implication

$$F_1\cap F_2=\emptyset \Rightarrow \lambda(F_1\cup F_2)\geqq\lambda F_1+\lambda F_2\ ,$$

then it will follow from results in [10] (Lemma 2 and Theorem 2) that μ defined by (8.5) is a τ-smooth measure on X.

Therefore, let F_1, F_2 in $\mathscr{F}(X)$ with $F_1 \cap F_2 = \emptyset$ be given. Given is also $\varepsilon > 0$. Choose $G \supseteqq F_1 \cup F_2$ such that

$$\nu G < \lambda(F_1 \cup F_2) + \varepsilon .$$

Then choose i_0 such that

$$\mu_i(\varphi_i G) < \lambda(F_1 \cup F_2) + \varepsilon$$

holds for all $i \geqq i_0$. Since the cylinder sets almost separate points and closed sets we can, by Lemma 4.9, find $\alpha = (i, E)$ such that

$$\lambda(F_1 \setminus G_\alpha) < \varepsilon \quad \text{and} \quad \lambda(F_2 \cap F_\alpha) < \varepsilon .$$

We may and do assume that $i \geqq i_0$. Put

$$G_1 = G \cap G_\alpha, \quad G_2 = G \setminus F_\alpha .$$

Then

$$\varphi_i G_1 \cap \varphi_i G_2 = \emptyset$$

holds. We claim that the inequalities

$$\lambda F_1 \leqq \lambda(F_1 \setminus G_\alpha) + \nu G_1, \quad \lambda F_2 \leqq \lambda(F_2 \cap F_\alpha) + \nu G_2$$

hold. To prove the first one, let H be open such that $H \supseteqq F_1 \setminus G_\alpha$. Then $F_1 \subseteqq H \cup G_1$, and

$$\lambda F_1 \leqq \nu(H \cup G_1) \leqq \nu H + \nu G_1$$

follows. Now the first inequality follows from

$$\lambda F_1 \leqq \inf_{H \supseteqq F_1 \setminus G_\alpha} (\nu H + \nu G_1) = \lambda(F_1 \setminus G_\alpha) + \nu G_1 .$$

The second inequality is proved in the same way. Now we have

$$\begin{aligned} \lambda F_1 + \lambda F_2 &\leqq \lambda(F_1 \setminus G_\alpha) + \nu G_1 + \lambda(F_2 \cap F_\alpha) + \nu G_2 \\ &\leqq 2\varepsilon + \nu G_1 + \nu G_2 \\ &\leqq 2\varepsilon + \mu_i(\varphi_i G_1) + \mu_i(\varphi_i G_2) \\ &= 2\varepsilon + \mu_i(\varphi_i G_1 \cup \varphi_i G_2) \\ &\leqq 2\varepsilon + \mu_i(\varphi_i G) \leqq 3\varepsilon + \lambda(F_1 \cup F_2) . \end{aligned}$$

This argument establishes the desired implication and we have seen that $\mu \in \mathfrak{M}_+(X; \tau)$. We also have $\mu X = 1$ since $\mu_i(\varphi_i X) = 1$ for all i. That the φ_i are measure preserving follows by a known argument.

The uniqueness of μ follows from Lemma 4.9.

Perhaps, the conditions of Theorem 8.4 are necessary as well.

The reason why we have worked in Theorem 8.4 with the condition that the cylinder sets almost separate points and closed sets, and not with the simpler condition that the cylinder sets separate points and closed sets is that the latter condition is far too restrictive. For instance, if the target space is $C[0,1]$, then the cylinder sets will not separate points and closed sets, but it is easy to see that the cylinder sets almost separate points and closed sets. A similar statement can be made in case the target space is $D[0,1]$, however, it becomes quite difficult to prove that the cylinder sets almost separate points and closed sets.

9. Weak convergence of measures on a projective limit space.

Let (X_i, φ_{ij}) be a projective system in the category $\mathfrak{A}$ and let (X, φ_i) be a projective limit. We shall find conditions under which a given net (μ_α) on $\mathfrak{M}_+^1(X;t)$ converges weakly. The conditions will be of the usual type, a compactness condition and a condition ensuring that sufficiently many of the nets obtained by "projection" on the "coordinate spaces" converge.

9.1 Theorem. *Let (X_i, φ_{ij}) be a projective system in $\mathfrak{A}$ and (X, φ_i) a projective limit of (X_i, φ_{ij}). Let I' be an upward directed subset of I and assume that the cylinder sets based on indices in I' separate points (that is, for $x \neq y$ we can find $\alpha = (i, E)$ with $i \in I'$ such that $x \in G_\alpha$ and $y \notin F_\alpha$). Assume that all the X_i are completely regular.*

Let (μ_α) be a compact net on $\mathfrak{M}_+^1(X;t)$ and assume that for each $i \in I'$ there exists a convergent net $(\eta_{i\alpha})$ on $\mathfrak{M}_+^1(X_i;t)$, say $\eta_{i\alpha} \to_w \eta_i$, such that $\eta_{i\alpha} \in \varphi_i(\mu_\alpha)$ holds for all α.

Then (μ_α) converges in $\mathfrak{M}_+^1(X;t)$: $\mu_\alpha \to_w \mu$ and the limit measure can be identified by the formula

$$\mu A = \sup_{K \subseteq A} \inf_{i \in I'} \eta_i(\varphi_i K), \quad A \in \mathscr{B}(X). \tag{9.2}$$

Proof. Let $(\mu_{\alpha'})$ and $(\mu_{\alpha''})$ be convergent subnets of (μ_α):

$$\mu_{\alpha'} \to_w \mu', \quad \mu_{\alpha''} \to_w \mu''.$$

Let K_1 and K_2 be disjoint compact subsets of X. Then we can find $i \in I'$ such that

$$\varphi_i K_1 \cap \varphi_i K_2 = \emptyset.$$

Since X_i is completely regular, we can find $E \subseteq X_i$ such that

$$\varphi_i K_1 \subseteq \mathring{E}, \quad \varphi_i K_2 \cap \bar{E} = \emptyset, \quad \eta_i(\partial E) = 0.$$

Then we have

$$\begin{aligned}\mu' K_1 &\leqq \mu'(\varphi_i{}^s \mathring{E}) \leqq \liminf_{\alpha'} \mu_{\alpha'}(\varphi_i{}^s \mathring{E}) \\ &\leqq \liminf_{\alpha'} \eta_{i\alpha'}(\mathring{E}) \leqq \limsup_{\alpha} \eta_{i\alpha}(\bar{E}) \\ &\leqq \eta_i(\bar{E}) = \eta_i(\mathring{E}) \leqq \liminf_{\alpha} \eta_{i\alpha}(\mathring{E}) \\ &\leqq \limsup_{\alpha''} \eta_{i\alpha''}(\bar{E}) \leqq \limsup_{\alpha''} \mu_{\alpha''}(\varphi_i{}^w \bar{E}) \\ &\leqq \mu''(\varphi_i{}^w \bar{E}) \leqq \mu''(\complement K_2) .\end{aligned}$$

We now easily deduce that μ' and μ'' are identical. Since all limit points of (μ_α) are identical, (μ_α) converges, say $\mu_\alpha \to_w \mu$.

By Theorem 3.13 we see that $\eta_i \in \varphi_i(\mu)$ holds for all $i \in I'$. From this fact it follows that

$$\mu K \leqq \inf_{i \in I'} \eta_i(\varphi_i K)$$

for all $K \in \mathscr{K}(X)$. To prove the reverse inequality, let $K_1 \in \mathscr{K}(X)$ be given and, to $\varepsilon > 0$, choose $K_2 \in \mathscr{K}(X)$ such that

$$K_1 \cap K_2 = \emptyset \quad \text{and} \quad \mu K_1 + \mu K_2 > 1 - \varepsilon .$$

Then choose $i_0 \in I'$ such that

$$\varphi_{i_0} K_1 \cap \varphi_{i_0} K_2 = \emptyset .$$

Then we have

$$\begin{aligned}\inf_{i \in I'} \eta_i(\varphi_i K_1) \leqq \eta_{i_0}(\varphi_{i_0} K_1) &\leqq 1 - \eta_{i_0}(\varphi_{i_0} K_2) \\ &\leqq 1 - \mu K_2 \leqq \mu K_1 + \varepsilon .\end{aligned}$$

Thus, (9.2) is proved for compact sets, and the general validity of (9.2) then follows by tightness.

Probably, the hypothesis that the X_i be completely regular is superfluous.

The result proved can be viewed as an abstract version of Theorem 2 of [8].

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REFERENCES

1. H. Bauer, *Mesures avec une image donnée*, Rev. Roumaine Math. Pures Appl. 11 (1966), 747–752.
2. N. Bourbaki, *Éléments de mathématique* Fasc. 35, Livre VI, *Intégration*, Chapitre 9, Intégration sur les espaces topologiques séparés (Act. Sci. Ind. 1343), Hermann et Cie, Paris, 1969.
3. L. Henkin, *A problem on inverse mapping systems*, Proc. Amer. Math. Soc. 1 (1950), 224–225.
4. G. Higman and A. H. Stone, *On inverse systems with trivial limits*, Proc. Lond. Math. Soc. (2) 29 (1954), 233–236.
5. J. Kisyński, *On the generation of tight measures*, Studia Math. 30 (1968), 141–151.
6. C. L. Scheffer, *Sur l'existence de la limite projective dans la catégorie des espaces de probabilité tendus*. C. R. Acad. Sci. Paris 269 (1969), 205–207.
7. V. Strassen, *Messfehler und Information*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 2 (1964), 273–305.
8. F. Topsøe, *A criterion for weak convergence of measures with an application to convergence of measures on $D[0,1]$*, Math. Scand. 25 (1969), 97–104.
9. F. Topsøe, *Topology and measure* (Lecture Notes in Mathematics 133) Springer-Verlag, Berlin, 1970.
10. F. Topsøe, *Compactness in spaces of measures*, Studia Math. 36 (1970), 195–212.

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