MATH. SCAND. 30 (1972), 46–50

ON PROJECTION MAPS OF VON NEUMANN ALGEBRAS

ERLING STØRMER

An important class of maps in the theory of von Neumann algebras is the positive linear maps of a von Neumann algebra $\mathscr{R}$ onto a von Neumann subalgebra $\mathscr{M}$ which are the identity on $\mathscr{M}$. Such maps are called projection maps (or projections of norm one, or expectations). Very often such maps will not exist, see e.g. [10]. In the present note we shall show that if $\mathscr{R}$ is of type I and $\mathscr{M}$ contains the center $\mathscr{C}$ of $\mathscr{R}$ then the existence of "sufficiently many" projection maps of $\mathscr{R}$ onto $\mathscr{M}$ is equivalent to $\mathscr{M}$ being of type I with center totally atomic over $\mathscr{C}$.

Following [5] we say a set Λ of projection maps of $\mathscr{R}$ onto $\mathscr{M}$ is *complete* if for each non zero positive operator A in $\mathscr{R}$ there is $\emptyset \in \Lambda$ such that $\emptyset(A) \neq 0$. A positive linear functional ϱ on $\mathscr{R}$ is said to be *singular* if there is no non zero ultra-weakly continuous positive linear functional ψ on $\mathscr{R}$ with $\psi \leqq \varrho$. If $\emptyset$ is a positive linear map from $\mathscr{R}$ to another von Neumann algebra $\mathscr{M}$ then $\emptyset$ is said to be *singular* if its transpose map $\emptyset^*$ carries normal states of $\mathscr{M}$ to singular positive linear functionals on $\mathscr{R}$. As pointed out by Tomiyama [9] singular maps play an important role in the study of projection maps. The author is indebted to J. Tomiyama for some useful comments and the short proof of the following lemma.

LEMMA. *Let $\mathscr{R}$ be a von Neumann algebra of type* I. *Let $\mathscr{C}$ denote the center of $\mathscr{R}$ and suppose $\emptyset$ is a positive singular $\mathscr{C}$-module homomorphism of $\mathscr{R}$ into $\mathscr{C}$. Then $\emptyset(E)=0$ for every abelian projection E in $\mathscr{R}$.*

PROOF. Let E be an abelian projection in $\mathscr{R}$. Suppose $\emptyset(E) \neq 0$. Considering $F\mathscr{R}$ instead of $\mathscr{R}$ for a central projection F in $\mathscr{R}$ we may assume that $\mathscr{C}$ is countably decomposable and $\emptyset(E)$ is invertible in $\mathscr{C}$. Let ω be a faithful normal state of $\mathscr{C}$. Then $\omega \circ \emptyset$ is a positive singular functional of $\mathscr{R}$. Hence by [8] there is a non zero projection $E_1 \leqq E$ in $\mathscr{R}$ such that $\omega \circ \emptyset(E_1)=0$. Since E is abelian there is a central projection G in $\mathscr{R}$ with $E_1 = GE$. Thus $\omega(G\emptyset(E)) = \omega(\emptyset(GE)) = \omega \circ \emptyset(E_1) = 0$. Since ω is faithful $G\emptyset(E)=0$ contradicting the assumption that $\emptyset(E)$ is invertible. Thus $\emptyset(E)=0$. The proof is complete.

Received May 19, 1971.

Let $\mathscr{Z}$ be an abelian von Neumann algebra and $\mathscr{C}$ a von Neumann subalgebra of $\mathscr{Z}$ (containing the identity of $\mathscr{Z}$). A projection E in $\mathscr{Z}$ is said to be *minimal in* $\mathscr{Z}$ *relative to* $\mathscr{C}$ if $E\mathscr{Z} = E\mathscr{C}$, and $\mathscr{Z}$ is said to be *totally atomic over* $\mathscr{C}$ if every non zero projection in $\mathscr{Z}$ majorizes a non zero projection in $\mathscr{Z}$ minimal in $\mathscr{Z}$ relative to $\mathscr{C}$. These two concepts and their generalizations where introduced independently in [3] and [4].

THEOREM. *Let $\mathscr{R}$ be a von Neumann algebra of type* I. *Let $\mathscr{M}$ be a von Neumann subalgebra of $\mathscr{R}$ containing the center $\mathscr{C}$ of $\mathscr{R}$. Then the following five conditions are equivalent.*

1) (resp. 2) *There exists a complete set of normal projection maps of $\mathscr{R}$ onto $\mathscr{M}$ (resp. $\mathscr{M}' \cap \mathscr{R}$).*

3) (resp. 4) *There exists a complete set of projection maps of $\mathscr{R}$ onto $\mathscr{M}$ (resp. $\mathscr{M}' \cap \mathscr{R}$).*

5) *$\mathscr{M}$ is of type* I *and its center is totally atomic over $\mathscr{C}$.*

PROOF. By [7, Lemma 3.4], $\mathscr{M}$ is of type I if and only if $\mathscr{M}' \cap \mathscr{R}$ is of type I. Thus, if we have shown 1) $\Leftrightarrow$ 3) $\Leftrightarrow$ 5) then an application of these equivalences to $\mathscr{M}' \cap \mathscr{R}$ yields the equivalences 2) $\Leftrightarrow$ 4) $\Leftrightarrow$ 5). We shall show 1) $\Rightarrow$ 3) $\Rightarrow$ 5) $\Rightarrow$ 1). Clearly 1) $\Rightarrow$ 3).

3) $\Rightarrow$ 5). Assume there is a complete set Λ of projection maps of $\mathscr{R}$ onto $\mathscr{M}$. Let $\mathscr{Z}$ denote the center of $\mathscr{M}$. We first assume $\mathscr{M}$ is abelian, hence $\mathscr{M} = \mathscr{Z}$. If ω is a normal state of $\mathscr{Z}$ with support E when restricted to $\mathscr{C}$ and A a positive operator in $E\mathscr{Z}$, the functional $B \to \omega(AB)$ on $E\mathscr{C}$ has a Radon–Nikodym derivative $\Phi(A) \in E\mathscr{C}$ with respect to ω, so $\omega(AB) = \omega(\Phi(A)B)$. One easily sees that Φ is a normal projection map of $E\mathscr{Z}$ onto $E\mathscr{C}$, see e.g. [1, p. 635]. Adding up the different Φ's obtained from a separating family of ω's with orthogonal supports when restricted to $\mathscr{C}$, we see there is a complete family Γ of normal projection maps of $\mathscr{Z}$ onto $\mathscr{C}$. Let G denote the group of inner automorphisms $A \to UAU^{-1}$ of $\mathscr{R}$ defined by the unitary operators in $\mathscr{Z}$. Then all the maps in Λ are G-invariant, since if $A \in \mathscr{R}$, $U \cdot U^{-1} \in G$, and $\emptyset \in \Lambda$, then

$$\emptyset(UAU^{-1}) = U\emptyset(A)U^{-1} = \emptyset(A) .$$

Let E be an abelian projection in $\mathscr{R}$. By assumption there is $\emptyset \in \Lambda$ such that $\emptyset(E) \neq 0$. By [9, Cor. 1,1], $\emptyset$ is uniquely decomposed into the sum of a positive singular $\mathscr{Z}$-module homomorphism $\emptyset_S$ and a positive normal $\mathscr{Z}$-module homomorphism $\emptyset_n$ of $\mathscr{R}$ to $\mathscr{Z}$. Then, if $\psi \in \Gamma$, $\psi \circ \emptyset_S$ and $\psi \circ \emptyset_n$ are respectively positive singular and normal $\mathscr{C}$-module homomorphisms of $\mathscr{R}$ to $\mathscr{C}$. Choose ψ such that $\psi(\emptyset(E)) \neq 0$. By the Lemma $\psi \circ \emptyset_S(E) = 0$, hence $\psi \circ \emptyset_n(E) = \psi \circ \emptyset(E) \neq 0$. Let ω be a normal positive linear functional of $\mathscr{C}$ such that $\omega\circ\psi\circ\emptyset_n$ is a normal state of $\mathscr{R}$ with $\omega\circ\psi\circ\emptyset_n(E)\neq 0$. Since $\emptyset_n$ is a $\mathscr{Z}$-module homomorphism of $\mathscr{R}$ to $\mathscr{Z}$, $\omega\circ\psi\circ\emptyset_n$ is G-invariant. Now if A is a non zero positive operator in $\mathscr{R}$ then A majorizes a positive multiple of an abelian projection, hence we have shown the existence of a normal G-invariant state ϱ of $\mathscr{R}$ for which $\varrho(A)\neq 0$. Thus $\mathscr{R}$ is G-finite in the sense of [6]. By [7, Thm. 3.5], $\mathscr{Z}$ is totally atomic over $\mathscr{C}$.

We next consider the general case. If $\mathscr{M}$ is not of type I there is a central projection E in $\mathscr{M}$ such that $E\mathscr{M}E$ has no type I portion. Considering $E\mathscr{R}E$, $E\mathscr{M}E$, and the projections $A \to \emptyset(EAE)$ we have a complete set of projection maps. By [9, Thms. 3 and 4] every projection map from $E\mathscr{R}E$ to $E\mathscr{M}E$ is singular. Now every von Neumann algebra possesses a complete set of normal projections onto its center $\mathscr{B}$. Indeed, it suffices to show that there is a complete set of normal projections of $\mathscr{B}'$ onto $\mathscr{B}$. But by [7, Lem. 4.11] and the remarks following it there is a faithful normal projection of $\mathscr{B}'$ onto a maximal abelian subalgebra $\mathscr{D}$. Compose this projection with a complete set of faithful normal projections from $\mathscr{D}$ onto $\mathscr{B}$ as constructed above to obtain the desired set. We thus obtain a complete set of singular projection maps from $E\mathscr{R}E$ to the center of $E\mathscr{M}E$ and thus to $E\mathscr{C}$. But these projections annihilate all abelian projections in $E\mathscr{R}E$ by the Lemma. Thus every projection map in Λ annihilates every abelian projection majorized by E, hence Λ is not complete, contrary to assumption. Thus $\mathscr{M}$ is of type I.

As shown above there is a complete set Γ of normal projections of $\mathscr{M}$ onto $\mathscr{Z}$. Then the set $\{\psi\circ\emptyset\colon\ \psi \in \Gamma,\ \emptyset \in \Lambda\}$ is a complete set of projection maps of $\mathscr{R}$ onto $\mathscr{Z}$. By the first part of the proof, $\mathscr{Z}$ is totally atomic over $\mathscr{C}$. We have thus shown that 3) $\Rightarrow$ 5).

5) $\Rightarrow$ 1). Assume $\mathscr{M}$ is of type I and its center $\mathscr{Z}$ is totally atomic over $\mathscr{C}$. Then $\mathscr{Z}\supset\mathscr{C}$. Let $\mathscr{B}=\mathscr{Z}'\cap\mathscr{R}$, and let G denote the group of inner automorphisms $A \to UAU^{-1}$ of $\mathscr{R}$ defined by unitaries $U\in\mathscr{Z}$. Then $\mathscr{B}$ is the fixed point algebra of G, and $\mathscr{B}'\cap\mathscr{R}=\mathscr{Z}$ is finite of type I, and its center $(=\mathscr{Z})$ is totally atomic over $\mathscr{C}$. By [7, Thm. 3.5], $\mathscr{R}$ is G-finite, so there is a faithful normal G-invariant projection Φ from $\mathscr{R}$ onto $\mathscr{B}$ [6]. Thus in order to construct a complete set of normal projections from $\mathscr{R}$ to $\mathscr{M}$ it suffices to do this for $\mathscr{R}$ replaced by $\mathscr{B}=\mathscr{Z}'\cap\mathscr{R}$. Therefore we may assume $\mathscr{R}=\mathscr{Z}'\cap\mathscr{R}$, hence $\mathscr{Z}=\mathscr{C}$. If we can construct a complete set of normal projection maps from $E_\alpha\mathscr{R}$ to $E_\alpha\mathscr{M}$ for an orthogonal family of central projections in $\mathscr{R}$ with sum I, then we can add up the different projection maps to obtain a complete set of normal projections from $\mathscr{R}$ to $\mathscr{M}$, see e.g. [5]. Therefore we may assume $\mathscr{R}$ homogeneous, and by cutting down by central projections in $\mathscr{M}$ (so by

projections in $\mathscr{C}$) we may also assume $\mathscr{M}$ is homogeneous. Say $\mathscr{M} = \mathscr{C}\otimes\mathscr{B}(\mathscr{K})$ and $\mathscr{R}=\mathscr{C}\otimes\mathscr{B}(\varkappa)$. Since $\mathscr{M}\subset\mathscr{R}$, and the case when $\varkappa$ is finite dimensional is trivial, we may assume $\varkappa=\mathscr{K}\otimes\mathscr{K}'$ and

$$\mathscr{M} = \mathscr{C}\otimes\mathscr{B}(\mathscr{K})\otimes C_{\mathscr{K}'} \subset \mathscr{C}\otimes\mathscr{B}(\mathscr{K})\otimes\mathscr{B}(\mathscr{K}') = \mathscr{R} .$$

If ω is a normal state of $\mathscr{B}(\mathscr{K}')$ and ι is the identity map of $\mathscr{C}\otimes\mathscr{B}(\mathscr{K})$ onto itself, then $\iota\otimes\omega$ is a normal projection map from $\mathscr{R}$ to $\mathscr{M}$. Indeed, if ϱ is a state of $\mathscr{M}$ and ϱ' its restriction to $\mathscr{C}\otimes\mathscr{B}(\mathscr{K})$, let $A_i \in \mathscr{C}\otimes\mathscr{B}(\mathscr{K})$, $B_i \in \mathscr{B}(\mathscr{K}')$, $i=1,\ldots,n$. Then

$$\varrho(\iota\otimes\omega(\Sigma A_i\otimes B_i)) = \Sigma\varrho(A_i\otimes\omega(B_i)I) = \Sigma\varrho'(A_i)\omega(B_i) = \varrho'\otimes\omega(\Sigma A_i\otimes B_i) .$$

Thus $\varrho\circ(\iota\otimes\omega)$ is a state for each state ϱ of $\mathscr{M}$, hence $\iota\otimes\omega$ is positive. Clearly $\iota\otimes\omega$ is a projection map, and it is normal, for if ϱ is normal then ϱ' is normal, and therefore $\varrho\circ(\iota\otimes\omega)=\varrho'\otimes\omega$ is normal. Since $\iota\otimes\omega$ is a normal projection for each normal state ω of $\mathscr{B}(\mathscr{K}')$, we have obtained a set of normal projection maps from $\mathscr{R}$ onto $\mathscr{M}$, which is easily seen to be complete. This completes the proof of the theorem.

Remark 1. If $\mathscr{R}=\mathscr{B}(\varkappa)$ with $\varkappa$ a separable Hilbert space the theorem was shown by de Korvin [5] by different methods. He conjectured that it was also true for non separable $\varkappa$ when $\mathscr{R}=\mathscr{B}(\varkappa)$. However, the result in this case follows from [9, Thm. 5].

Remark 2. With the assumptions as in the theorem and with G the group of inner automorphisms of $\mathscr{R}$ defined by the unitaries in $\mathscr{M}'\cap\mathscr{R}$ [7, Thm. 3.5] states the equivalence of the following three conditions

i) $\mathscr{R}$ is G-finite,

ii) $\mathscr{M}'\cap\mathscr{R}$ is finite and there exists a faithful normal projection of $\mathscr{R}$ onto $\mathscr{M}$,

iii) $\mathscr{M}'\cap\mathscr{R}$ is finite of type I, and its center is totally atomic over $\mathscr{C}$.

Thus, with a proper definition of G-semi-finite our theorem should be viewed as a G-semi-finite extension of [7, Thm. 3.5].

REFERENCES

1. W. Arveson, *Analyticity in operator algebras*, Amer. J. Math. 89 (1967), 578–642.
2. J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien* (Cahiers Scientifiques 25), Paris, Gauthier–Villars, 1957.
3. A. Guichardet, *Une caractérisation des algèbres de von Neumann discrètes*, Bull. Soc. Math. France 89 (1961), 77–101.
4. R. V. Kadison, *Normalcy in operator algebras*, Duke Math. J. 29 (1962), 459–464.

5. A. de Korvin, *On complete sets of expectations*, Quart.J. Math. Oxford Ser. 22 (1971), 135–143.
6. I. Kovács and J. Szücs, *Ergodic type theorems in von Neumann algebras*, Acta Sci. Math. (Szeged) 27 (1966), 233–246.
7. E. Størmer, *States and invariant maps of operator algebras*, J. Functional Analysis 5 (1970), 44–65.
8. M. Takesaki, *On the singularity of a positive linear functional of an operator algebra*, Proc. Japan Acad. 35 (1959), 365–366.
9. J. Tomiyama, *On the projection of norm one in W^*-algebras* III, Tôhoku Math. J. 11 (1959), 125–129.
10. J. Tomiyama, *The extension property of von Neumann algebras and a class of C^*-algebras associated to them.* To appear.

UNIVERSITY OF OSLO, NORWAY