MATH. SCAND. 30 (1972), 51–61

ON A FORMULA FOR HAAR MEASURE IN COMPACT GROUPS

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Let G be a compact abelian group, H a locally compact abelian group and $\varphi \colon H \to G$ a continuous homomorphism such that $\varphi(H)$ is dense in G. Let μ,ν denote Haar measures on G, H, respectively. Hewitt and Ross [7, 26.17] have shown that if $\{H_n\}_{n=1}^{\infty}$ is a "Bohr sequence" in H and U is open in G with $\mu(U)=\mu(\overline{U})$, then

$$(*) \qquad \mu(U) = \lim_{n\to\infty} \frac{1}{\nu(H_n)} \int_{H_n} \chi_U \circ \varphi \, d\nu .$$

where χ_U is the indicator of U. They give several applications concerning equidistributions. We point out that the theorem extends to arbitrary compact groups G and arbitrary "Bohr nets" in H. Often the limit (*) is uniform with respect to translation in H. We specify many of the corresponding U's. These sets form a base for the topology of G and any measurable set may be approximated by such a U. Examples are given. From these ideas we obtain an extension to groups, and a strengthening, of a classical theorem due to Besicovitch and Bohr concerning the almost periodicity of the integral translation numbers of an almost periodic function. As another application we give a new characterization of the Weyl almost periodic functions.

Let us make some notational remarks. If H is a locally compact group (=LC group), $\alpha(H)$ denotes the set of continuous Bohr–von Neumann AP functions on H. We use the term "Bohr net" as in [4], [5]: Let $(D, \geq)$ be a directed set and for each $d \in D$ let H_d be a Borel set in H. Let ν_d be a totally finite non-zero Borel measure on H_d. $\Phi = (H_d, \nu_d, d \in D, \geq)$ is called a *Bohr net* in H if for every $f \in \alpha(H)$

$$\lim_{d\in D} \frac{1}{\nu_d(H_d)} \int_{H_d} f \, d\nu_d = Mf .$$

Here Mf is the mean value of f. If $(D, \geq)$ is the set of all positive integers

Received March 16, 1971.

Work performed under the auspices of the U.S. Atomic Energy Commission.

with natural order we say Φ is a *Bohr sequence*. If each ν_d is restriction of left Haar measure to U_d we say Φ is *homogeneous*. Examples of Bohr nets are given in [4]. All LC groups have (many) Bohr nets.

Let ν denote left Haar measure on H. By $L_{1,\mathrm{loc}}(H)$ we denote the set of complex-valued Borel measurable functions f on H such that $\int_E |f|\,d\nu < \infty$ for all compact sets $E \subset H$. If $f \in L_{1,\mathrm{loc}}(H)$ we write

$$M_\Phi f = \lim_{d\in D} \frac{1}{\nu_d(H_d)} \int_{H_d} f\, d\nu_d$$

whenever this exists. Also define

$$\bar{M}_\Phi |f| = \|f\|_\Phi = \limsup_{d\in D} \frac{1}{\nu_d(H_d)} \int_{H_d} |f|\, d\nu_d\,,$$

$$\underline{M}_\Phi |f| = \liminf_{d\in D} \frac{1}{\nu_d(H_d)} \int_{H_d} |f|\, d\nu_d\,,$$

$$M_\Phi^W |f| = \|f\|_\Phi^W = \limsup_{d\in D} \sup_{x,y\in H} \frac{1}{\nu_d(H_d)} \int_{H_d} |{}_xf_y|\, d\nu_d\,.$$

Here ${}_xf_y(t) = f(xty)$ for all $t \in H$. The Besicovitch and Weyl almost periodic functions on H are obtained by closing $\alpha(H)$ in $L_{1,\mathrm{loc}}(H)$ via the norms $\|\cdot\|_\Phi$, $\|\cdot\|_\Phi^W$, respectively. The corresponding spaces are denoted $B_\Phi(H)$ and $W_\Phi(H)$. See [4], [5] for further discussion of these spaces. For Borel sets $E \subset H$ we write $m_\Phi(E) = \|\chi_E\|_\Phi$. By $C(H)$ we denote the set of continuous complex-valued functions on H. The identity element of a group is denoted e.

THEOREM 1. *Let G be a compact group, H an* LC *group and $\varphi: H \to G$ a continuous homomorphism such that $\varphi(H)$ is dense in G. Let μ denote Haar measure in G and let $\Phi = (H_d, \nu_d, d \in D, \geq)$ be a Bohr net in H. Let U, F be, respectively, open and closed sets in G such that $U \subset F$ and $\mu(U) = \mu(F)$. Then*

$$(*) \qquad \mu(U) = \lim_{d\in D} \frac{1}{\nu_d(H_d)} \int_{H_d} \psi_U \circ \varphi\, d\nu_d\,.$$

U may be replaced by F in (*). Also $\chi_U \circ \varphi$, $\chi_F \circ \varphi \in B_\Phi(H)$.

PROOF. (*) is proven as in 26.17 of [7]. To show, for example, that $\chi_F \circ \varphi \in B_\Phi(H)$, take $g_n \in C(G)$ such that $\chi_F \leqq g_{n+1} \leqq g_n$ and $\int g_n\, d\mu \to \mu(F)$ as $n \to \infty$. Then $g_n \circ \varphi \in \alpha(H)$ and

$$\begin{aligned}\|g_n\circ\varphi-\chi_F\circ\varphi\|_\Phi &= \overline{M}_\Phi[g_n\circ\varphi-\chi_F\circ\varphi]\\ &\leqq \overline{M}_\Phi(g_n\circ\varphi)-\underline{M}_\Phi(\chi_F\circ\varphi)\\ &= \int g_n d\mu-\mu(F)\to 0 \quad \text{as } n\to\infty .\end{aligned}$$

If U,F satisfy the hypothesis of theorem 1 and $x,y\in H$, then $U'=\varphi(x)U\varphi(y)$, $F'=\varphi(x)F\varphi(y)$ also satisfy the hypothesis of theorem 1. As $\mu(U')=\mu(U)$, $\mu(F')=\mu(F)$ and as $x,y\in H$ are arbitrary we get that under the hypothesis of theorem 1

$$(\dagger)\qquad \mu(U) = \lim_{d\in D}\frac{1}{\nu_d(H_d)}\int_{H_d} {}_x(\chi_U\circ\varphi)_y\, d\nu_d$$

for any $x,y\in H$. However this limit may not be uniform in x,y.

NOTATION. In the terminology of theorem 1, let $\mathscr{E}_0$ denote the Borel sets $U\subset G$ such that (†) holds for every $x,y\in H$. Let $\mathscr{E}_1$ be the sets $U\in\mathscr{E}_0$ such that (†) holds uniformly in $x,y\in H$.

Theorems 2 and 4, below, indicate that $\mathscr{E}_0$ and $\mathscr{E}_1$ are plentiful.

THEOREM 2. *Assume the notation of theorem* 1. *Take* $f\in C(G)$ *and* $y\in G$. *For all but a countable set of* $\eta>0$ *the sets*

$$\{x\in G : |f(x)-f(y)|<\eta\},\qquad \{x\in G : |f(x)-f(y)|\leqq\eta\}$$

are in $\mathscr{E}_0$.

PROOF. Letting U_η, F_η be the two sets above, we apply theorem 1.

$$F_\eta \sim U_\eta \subset \{x\in G : |f(x)-f(y)|=\eta\} = A_\eta .$$

$\mu(A_\eta)=0$ for all but a countable set of $\eta>0$ because the A_η are pairwise disjoint and $\mu(G)<\infty$. For these η, $U_\eta, F_\eta\in\mathscr{E}_0$.

NOTATION. For $f\in C(G)$, $\eta>0$ define $D_\eta(f)=\{x\in G : \|f_x-f\|_\infty\leqq\eta\}$ and $E_\eta(f)=\varphi^{-1}(D_\eta(f))$. Here $\|\cdot\|_\infty$ denotes the supremum norm and φ is as in theorem 1.

The theorems below are also true if we had defined $D_\eta(f)$ to be

$$\{x\in G : \|f_x-f\|_\infty<\eta\}\,,$$
$$\{x\in G : \|{}_xf-f\|_\infty<\eta\}$$

or

$$\{x\in G : \|{}_xf-f\|_\infty\leqq\eta\}\,.$$

THEOREM 3. *Assume the hypothesis of theorem* 1. *Let* M *be a locally compact subgroup of* H *with Bohr net* Ψ. *Let* μ_1 *denote Haar measure on* $\varphi(M)^-$. *For* $f\in C(G)$ *and* $\eta>0$ *define*

$$\bar{E}_\eta = \bar{E}_\eta(f) = E_\eta(f) \cap M .$$

There is a set $T_M(f)$ containing all but a countable set of the positive numbers such that for all $\eta \in T_M(f)$

(1) $$\mu_1(D_\eta(f) \cap \varphi(M)^-) = m_\Psi(\bar{E}_\eta)$$

and

(2) $$\chi_{\bar{E}_\chi} \in W_\Psi(M) .$$

Furthermore, for all $\eta \in T_M(f)$

(3) $$\lim_{\varrho \downarrow 0,\, \eta+\varrho \in T_M(f)} m_\Psi(\bar{E}_{\eta+\varrho}) = m_\Psi(\bar{E}_\eta) .$$

Finally,

$$T_M(f) = \{\eta > 0 \colon \mu_1(\{x \in \varphi(M)^- : \|f_x - f\|_\infty = \eta\}) = 0\} .$$

PROOF. (1) is proven by applying theorem 1 and the methods of theorem 2. Note that, since f is uniformly continuous on G, $\|f_x - f\|_\infty$ is a continuous function of $x \in G$. By the methods of theorem 2 $T_M(f)$, defined as indicated above, contains all but a countable set of the positive numbers. For $\eta \in T_M(f)$ we apply theorem 1 letting $\varphi(M)^-$ play the role of G and $F = D_\eta(f) \cap \varphi(M)^-$. This yields (1) along with the fact that $\chi_{\bar{E}_\eta} \in B_\varphi(M)$.

(3) follows from (1) since $\bar{E}_{\eta+\varrho} \downarrow \bar{E}_\eta$ and μ_1 is continuous.

We next show that for $\eta \in T_M(f)$

(4) $$\|\chi_{\bar{E}_\eta}\|_\Psi^W \leqq m_\Psi(\bar{E}_\eta) ,$$

from which it follows that

(5) $$\|\chi_{\bar{E}_\eta}\|_\Psi^W = m_\Psi(\bar{E}_\eta) .$$

Take $\eta \in T_M(f)$ and arbitrary $\varepsilon > 0$. By (1) and (3) we may take $\varrho > 0$ such that $\eta + 2\varrho \in T_M(f)$ and

(6) $$m_\Psi(\bar{E}_{\eta+2\varrho}) \leqq m_\Psi(\bar{E}_\eta) + \varepsilon .$$

Since $D_\varrho(f)$ is a neighborhood of e in G and $\varphi(M)$ is compact, there exist $w_1, \ldots, w_n,\ w_1', \ldots, w_m' \in M$ such that

$$M = \bigcup_{i=1}^{n} w_i \bar{E}_\varrho = \bigcup_{i=1}^{m} \bar{E}_\varrho w_i' .$$

Write $\Psi = (V_l, \lambda_l, l \in L, >)$. By the remark following theorem 1, we may take $l_0 \in L$ such that

(7) $$\frac{1}{\lambda_l(V_l)} \int_{V_l} {}_{w_i'}(\chi_{\bar{E}_{\eta+2\varrho}})_{w_j} d\lambda_l \leqq m_\Psi(\bar{E}_{\eta+2\varrho}) + \varepsilon$$

for all $l>l_0$ and all $1\leqq i\leqq m$, $1\leqq j\leqq n$. Take arbitrary $x,y\in M$. For some p,q, then, $x=w_pu$, $y=vw_q'$ where $u,v\in \bar{E}_\varrho$. For $l>l_0$

$$\frac{1}{\lambda_l(V_l)}\int_{V_l} {}_y(\chi_{\bar{E}_\eta})_x\,d\lambda_l = \frac{1}{\lambda_l(V_l)}\int_{V_l} {}_{w_q'}({}_v(\chi_{\bar{E}_\eta}))_{w_p}\,d\lambda_l$$

$$\leqq \frac{1}{\lambda_l(V_l)}\int_{V_l} {}_{w_q'}(\chi_{\bar{E}_{\eta+2\varrho}})_{w_p}\,d\lambda_l$$

$$\leqq m_\Psi(\bar{E}_{\eta+2\varrho})+\varepsilon,\ \text{by (7)}$$

$$\leqq m_\Psi(\bar{E}_\eta)+2\varepsilon,\ \text{by (6)}\,.$$

As $x,y\in M$ are arbitrary we get that for all $l>l_0$

$$\sup_{x,y\in M}\frac{1}{\lambda_l(V_l)}\int_{V_l} {}_x(\chi_{\bar{E}_\eta})_y\,d\lambda_l \leqq m_\Psi(\bar{E}_\eta)+2\varepsilon\,,$$

whence

$$\|\chi_{\bar{E}_\eta}\|_\Psi^W \leqq m_\Psi(\bar{E}_\eta)+2\varepsilon\,.$$

As $\varepsilon>0$ is arbitrary, (4), and, hence, (5) follow.

We now show that (2) holds. Take $\eta\in T_M(f)$. As

$$\mu_1(\{x\in\varphi(M)^- : \|f_x-f\|_\infty=\eta\}) = 0$$

and as μ_1 is regular, there exist $f_n\in C(\varphi(M)^-)$ such that

$$0 \leqq f_n \leqq f_{n+1} \leqq \chi_{D_\eta(f)\cap\varphi(M)^-}$$

and

$$\int f_n\,d\mu_1 \to \mu_1(D(f)\cap\varphi(M)^-)\,.$$

Let $\varphi_1=\varphi/M$. Then $f_n\circ\varphi_1\in\alpha(M)$. Also

$$\|\chi_{\bar{E}_\eta}-f_n\circ\varphi_1\|_\Psi^W = \limsup_{l\in L}\sup_{x,y\in M}\frac{1}{\lambda_l(V_l)}\int_{V_l}[{}_x(\chi_{\bar{E}_\eta})_y - {}_x(f_n\circ\varphi_1)_y]\,d\lambda_l$$

$$\leqq \|\chi_{\bar{E}_\eta}\|_\Psi^W - \liminf_{l\in L}\inf_{x,y\in M}\frac{1}{\lambda_l(V_l)}\int_{V_l} {}_x(f_n\circ\varphi_1)_y\,d\lambda_l$$

$$= m_\Psi(\bar{E}_\eta)-M(f_n\circ\varphi_1)\,,$$

by (5) and the fact that the mean value of a Bohr–Von Neumann AP function is obtained via a Bohr net uniformly with respect to translation (see proof of 2.1 in [3]). Thus by (1)

$$\|\chi_{\bar{E}_\eta} - f_n \circ \varphi_1\|_\Psi^W \leqq \mu_1(D_\eta(f) \cap \varphi(M)^-) - \int f_n \, d\mu_1 \to 0 \quad \text{as } n \to \infty .$$

This proves the theorem.

It is useful to point out the following.

PROPOSITION. *Let $\Phi = (H_d, \nu_d, d \in D, \geq)$ be a Bohr net in the locally compact group H. Let $f \in W_\Phi(H)$. Then*

$$M_\Phi f = \lim_{d \in D} \frac{1}{\nu_d(H_d)} \int_{H_d} {}_x f_y \, d\nu_d$$

uniformly in $x, y \in H$.

PROOF. $M_\Phi f$ exists, as is pointed out in [4]. For $h \in L_{1,\mathrm{loc}}(H)$ write

$$w_d(h) = \frac{1}{\nu_d(H_d)} \int_{H_d} h \, d\nu_d .$$

If $h \in \alpha(H)$,

$$w_d({}_x h_y) \to Mh \text{ as } d \text{ gets large in } D ,$$

uniformly in $x, y \in H$ by the proof of 2.1 in [3]. Take $\varepsilon > 0$ and $h \in \alpha(H)$ such that $\|f - h\|_\Phi^W < \varepsilon$. Then

$$|M_\Phi f - Mh| \leqq \|f - h\|_\Phi^W < \varepsilon .$$

Take $d_0 \in D$ such that

$$|w_d({}_x h_y) - Mh| < \varepsilon, \quad w_d|_x(h-f)_y| < \varepsilon$$

for all $d \geq d_0$, $x, y \in H$. Then for $d \geq d_0$, $x, y \in H$

$$|w_d({}_x f_y) - M_\Phi f| \leqq w_d|_x(h-f)_y| + |w_d({}_x h_y) - Mh| + |Mh - M_\Phi f| < 3\varepsilon ,$$

proving the proposition.

COROLLARY. *If $f \in C(G)$ and $\eta \in T_H(f)$, then $D_\eta(f) \in \mathscr{E}_1$.*

PROOF. Apply (1), (2) of theorem 3 with $M = H$ to the above proposition.

The following shows that the members of $\mathscr{E}_1$ are plentiful. Let Δ denote symmetric difference.

THEOREM 4. *Assume the hypothesis of theorem 1. Then $\mathscr{E}_1$ contains a closed (and an open) base for the topology of G. If $A \subset G$ is Haar measurable and $\varepsilon > 0$, then there exists $E \in \mathscr{E}_1$ such that $\mu(A \Delta E) > \varepsilon$. E may be taken to be either open or closed.*

PROOF. To show that $\mathscr{E}_1$ contains a base for the topology of G it suffices to show that it contains a local base at e because $\mathscr{E}_1$ is closed under translation by members of $\varphi(H)$ and $\varphi(H)^- = G$. If W is an arbitrary neighborhood of e and $f \in C(G)$ is chosen so that $f \equiv 1$ on $G \sim W$, $f(G) \subset [0,1]$, and $f(e) = 0$, then

$$e \in \{x \in G : \|f_x - f\|_\infty < \eta\} \subset D_\eta(f) \subset W$$

for all $0 < \eta < 1$. It has been shown that for most such η $D_\eta(f) \in \mathscr{E}_1$ and by a similar argument the same is true for $\{x \in G : \|f_x - f\|_\infty < \eta\}$. These sets are respectively closed and open. Thus $\mathscr{E}_1$ contains the required local base.

To prove the last assertion note that since μ is regular it suffices to prove the following: For every closed $F \subset G$ and open $U \subset G$ with $F \subset U$ there exists a closed (open) set $E \in \mathscr{E}_1$ such that $F \subset E \subset U$. We give the proof for the case of a closed set E. The open case is handled by considering sets of the form $\{x \in G : \|f_x - f\|_\infty < \eta\}$ instead of the sets $D_\eta(f)$. As each $D_\eta(f)$ is a neighborhood of e, as the translates of the $D_\eta(f)$'s by elements of $\varphi(H)$ are a base for the topology of G, and as F is compact, there exists $x_1, \ldots, x_n \in H$, $f_1, \ldots, f_n \in C(G)$, $\eta_1, \ldots, \eta_n > 0$ such that $\eta_i \in T_H(f_i)$, $1 \leqq i \leqq n$, and

$$F \subset E \subset U, \quad \text{where } E = \bigcup_{i=1}^{n} \varphi(x_i) D_{\eta_i}(f_i) .$$

E is closed and, since $\eta_i \in T_H(f_i)$ for each i, $\mu(E \sim E^0) = 0$, because

$$\mathrm{bd}(E) \subset \bigcup_{i=1}^{n} \mathrm{bd}[\varphi(x_i) D_{\eta_i}(f_i)]$$

and each of the summands has μ-measure zero due to the way $T_H(f_i)$ is defined. By theorem 1, $\mu(E) = m_\Phi(\varphi^{-1}(E))$. Hence, to show that $E \in \mathscr{E}_1$ it suffices to show that $\chi_{\varphi^{-1}(E)} \in W_\Phi(H)$ by the proposition. As $\chi_{x_i E_{\eta_i}(f_i)} \in W_\Phi(H)$ for each i, it suffices to show the following:

$$\text{If } \chi_{A_i} \in W_\Phi(H),\ i = 1,2, \text{ then } \chi_{A_1 \cup A_2} \in W_\Phi(H) .$$

Now $W_\Phi(H)$ is a vector space closed under the formation of absolute values, since if $f \in W_\Phi(H)$ and $\|f - g_n\|_\Phi^W \to 0$ with $g_n \in \alpha(H)$, then

$$\| |f| - |g_n| \|_\Phi^W \leqq \|f - g_n\|_\Phi^W \underset{n}{\to} 0 .$$

As

$$\chi_{A_1 \cup A_2} = \tfrac{1}{2}(\chi_{A_1} + \chi_{A_2} + |\chi_{A_1} - \chi_{A_2}|) ,$$

it follows that $\chi_{A_1 \cup A_2} \in W_\Phi(H)$ when χ_{A_i} does, $i = 1,2$. This proves the theorem.

COROLLARY. *The sets of the form $D_\eta(f)$, where $f \in C(G)$ and $\eta \in T_H(f)$, are a local base for the neighborhood system of e. $\mathscr{E}_1$ contains all finite unions of the translates of these sets by elements in $\varphi(H)$ as well as the symmetric difference of two such translates.*

EXAMPLES. a) Let G be an arbitrary compact group with Haar measure μ. Let $H=G$ with the discrete topology and let φ be the identity map. Let D be the set of all ordered pairs $(\{\alpha_r\}_{r=1}^n, \{a_r\}_{r=1}^n)$ where $\alpha_r > 0$ for $1 \leqq r \leqq n$, $\sum \alpha_r = 1$ and $a_r \in G$, $1 \leqq r \leqq n$ (n may differ for different members of D). For

$$d_1 = (\{\alpha_r\}_1^n, \{a_r\}_1^n) \quad \text{and} \quad d_2 = (\{\beta_t\}_1^m, \{b_t\}_1^m)$$

in D define $d_1 d_2 \in D$ by

$$d_1 d_2 = (\{\alpha_r \beta_t\}_{r,t=1}^{n,m}, \{a_r b_t\}_{r,t=1}^{n,m}) \,.$$

For $d, d' \in D$ define $d \leq d'$ if and only if there exist $d_1, \ldots, d_k, e_1, \ldots, e_l \in D$ such that $d' = d_1 \ldots d_k\, d\, e_1 \ldots e_l$. $(D, \geq)$ is a directed set. Let us denote a typical element $d \in D$ by

$$d = (\{\alpha_r^d\}_{r=1}^{n(d)}, \{a_r^d\}_{r=1}^{n(d)}) \,.$$

For $d \in D$ define $U_d = \{a_r^d\}_{r=1}^{n(d)}$. Let ν_d be a measure on the subsets of U_d determined by the requirement that $\nu_d(\{a_r{}^d\}) = \alpha_r{}^d$ for each $a_r{}^d \in U_d$. Then $(U_d, \nu_d, d \in D, \geq)$ is a Bohr net (cf., [4, 3.1c]). For $E \in \mathscr{E}_1$ we have

$$\mu(E) = \lim_{d \in D} \sum \{\alpha_r{}^d : a_r{}^d \in (xEy) \cap U_d\}$$

uniformly in $x, y \in G$.

b) Let G be an arbitrary compact group with Haar measure μ. Let H, φ be as in a). Haar measure in H is counting measure and for finite $F \subset H$ we let $|F|$ be the cardinal of F. Let $(U_d, d \in D, \geq)$ be a homogeneous Bohr net in H. This exists by theorem 3.4 of [3]. Then for $E \in \mathscr{E}_1$

$$\mu(E) = \lim_{d \in D} |(xEy) \cap U_d| / |U_d|$$

uniformly in $x, y \in G$.

c) Let G be a separable compact group with Haar measure μ. Then there is a denumerable subgroup H dense in G. There is a sequence $A_n = \{a_{n1}, \ldots, a_{nk_n}\}$, $n = 1, 2, \ldots$, of finite sets in H such that $A_1 \subset A_2 \subset \ldots$, $\bigcup_{n=1}^\infty A_n = H$, and for every $E \in \mathscr{E}_1$

$$\mu(E) = \lim_{n \to \infty} |(xEy) \cap A_n| / k_n \,,$$

uniformly in $x, y \in H$. In the abelian case one can require that $k_n = |A_n|$. This follows from 4.2 of [3] or Theorem 1 of [6], and 18.14 of [7] for the abelian case. It is striking to contrast the plentifulness of $\mathscr{E}_1$ (theorem 4) with the thinness of H.

More examples occur in 26.20 of [7].

NOTATION. Let R be the additive group of real numbers (usual topology), Z the additive group of integers (discrete topology), and $\overline{\mathsf{R}}$ the Bohr compactification of R. Suppose that, in the context of theorem 3, $\mathsf{R}=H$. Then we shall always assume $G=\overline{\mathsf{R}}$ and $M=\mathsf{Z}$. We shall consider R to be a subset of $\overline{\mathsf{R}}$ so that φ may be considered the identity map. The map $\alpha(\mathsf{R}) \to C(\overline{\mathsf{R}})$ by $f \to \bar{f}$, where $\bar{f}$ is the continuous extension of f to $\overline{\mathsf{R}}$, is an isomorphism. Thus for $f \in \alpha(\mathsf{R})$ it makes sense to write $\bar{E}_\eta(f)$ for $\bar{E}_\eta(\bar{f})$. In fact $\bar{E}_\eta(f)$ is the set of integral η-translation numbers of f.

The following classical definition and theorem are due to Besicovitch and Bohr [1]. Let $|F|$ denote the cardinal of F.

DEFINITION. Let $f \in \alpha(\mathsf{R})$. Take $\varepsilon, \eta > 0$. We say $\bar{E}_\varepsilon(f) = \bar{E}_\varepsilon$ *is almost periodic with error* $\leqq \eta$ if there exists $\varrho < \varepsilon$, $I > 0$ such that for all $a, b \in \mathsf{Z}$ with $b - a > I$ we have

$$\frac{|[(\bar{E}_\varepsilon \cap [a,b]) + s] \cap (\mathsf{Z} \sim \bar{E}_\varepsilon)|}{b-a} \leqq \eta$$

whenever $s \in \bar{E}_\varrho$. We say $\bar{E}_\varepsilon$ is *almost periodic* if for every $\eta > 0$, $\bar{E}_\varepsilon$ is almost periodic with error $\leqq \eta$.

THEOREM (Besicovitch and Bohr). *Let $f \in \alpha(\mathsf{R})$. Then $\bar{E}_\varepsilon(f)$ is almost periodic for almost all $\varepsilon > 0$.*

In the original Besicovitch–Bohr definition of being almost periodic with error $\leqq \eta$ it is not required that $a, b \in \mathsf{Z}$ but only that $a, b \in \mathsf{R}$. Also the open interval (a, b) rather than $[a, b]$ is considered. However it is not hard to show that the two definitions are equivalent. We now use theorem 3 to extend the Besicovitch–Bohr theorem, in a strengthened form, to groups.

THEOREM 5. *Assume the hypothesis of theorem 3. Let λ denote left Haar measure on M and assume that $\Psi = (V_l, l \in L, >)$ is homogeneous. Take $f \in C(G)$, ε in $T_M(f)$, and $\eta > 0$. Then there exists $\varrho < \varepsilon$, $l_0 \in L$ such that*

$$\frac{\lambda\{[s(\bar{E}_\varepsilon \cap [x V_l y])] \cap (M \sim \bar{E}_\varepsilon)\}}{\lambda(V_l y)} \leqq \eta$$

for all $l > l_0$, $x, y \in M$, $s \in \bar{E}_\varrho$.

PROOF. By theorem 3 there exists $\varrho < \varepsilon$ such that $(\varepsilon + \varrho) \in T_M(f)$ and

$m_\Psi(\bar{E}_{\varepsilon+\varrho} \sim \bar{E}_\varepsilon) < \frac{1}{2}\eta$. As in the proof to theorem 4, $\chi_{\bar{E}_{\varepsilon+\varrho} \sim \bar{E}_\varepsilon} \in W_\Psi(M)$. By the proposition following theorem 3, there exists $l_0 \in L$ such that

$$\sup_{u,v \in M} \frac{\lambda[\bar{E}_{\varepsilon+\varrho} \cap (uV_l v) \cap (M \sim \bar{E}_\varepsilon)]}{\lambda(V_l v)} < \eta$$

for all $l > l_0$. Take any $s \in \bar{E}_\varrho$ and any $x, y \in M$. Then

$$\frac{\lambda\{[s(\bar{E}_\varepsilon \cap [xV_l y])] \cap (M \sim \bar{E}_\varepsilon)\}}{\lambda(V_l y)} \leqq \frac{\lambda[\bar{E}_{\varepsilon+\varrho} \cap (sxV_l y) \cap (M \sim \bar{E}_\varepsilon)]}{\lambda(V_l y)} < \eta$$

for all $l > l_0$ by the above. This proves the theorem.

Corollary. *Let $f \in \alpha(R)$. Then $\bar{E}_\varepsilon(f)$ is almost periodic for all but a countable set of $\varepsilon > 0$.*

Proof. We apply theorem 5 letting Ψ be the Bohr sequence

$$(\{i\}_{i=1}^n,\ n \in \{1,2,\ldots\},\ \geq) \text{ in } \mathbf{Z}.$$

Theorem 1 has been applied in [2] to give a new characterization of the Besicovitch almost periodic functions (cf., 2.7, 2.8 and 2.11 of [2]). In a similar fashion theorem 3 yields a new characterization of the Weyl almost periodic functions. In the context of theorem 3 let $H = \mathbf{R}$ and let $\Phi = ((-T,T), T \in (0,\infty), \geq)$ be the usual Bohr net. We omit writing Φ for this case. Thus, for example, if $f \in L_{1,\mathrm{loc}}(R)$ we have

$$(\sharp) \qquad \|f\|^W = \lim_{T\to\infty}\left[\sup_{x\in R} \frac{1}{2T}\int_{-T}^{T} |{}_x f|\, d\varrho\right].$$

Here ϱ denotes Lebesgue measure. (It is well-known that in this setting writing "lim" on the right side of ($\sharp$) is equivalent to writing "lim sup".) For a measurable set $E \subset R$ define

$$\bar{\varrho}^W(E) = \|\chi_E\|^W.$$

The characterization of $B(R)$ given in [2] is in terms of five conditions (Ai), $1 \leqq i \leqq 4$, and (B). Let (WAi), (WB) be the conditions obtained by making the following symbolic changes in (Ai), (B), $1 \leqq i \leqq 4$:

$$\|\cdot\| \to \|\cdot\|^W, \quad BE(\varepsilon,f) \to WE(\varepsilon,f), \quad \bar{M} \to M^W, \quad \bar{\mu} \to \bar{\varrho}^W.$$

Thus, for example, we have

(WA1) f is $\| \ \|^W$-normal,

(WB) For all but a countable set of $\varepsilon > 0$,

$$M_x{}^W M_w{}^W[|f(x+w) - f(x)|\chi_{WE(\varepsilon,f)}(w)] \leqq \varepsilon\bar{\varrho}^W(WE(\varepsilon,f)).$$

THEOREM 6. *Let* $f \in L_{1,\mathrm{loc}}(R)$. *Then* $f \in W(R)$ *if and only if* f *satisfies* (WA1) *and* (WB). *Condition* (WA1) *may be replaced by any of the equivalent conditions* (WA2), (WA3) *or* (WA4).

PROOF. In all of section 2 of [2] except for the proof of 2.7 make the following symbolic changes, additional to those above:

$$(\mathrm{Ai}) \to (\mathrm{WAi}), \quad (\mathrm{B}) \to (\mathrm{WB}), \quad \{B-AP\} \to W(R), \quad \mu \to \varrho .$$

For the proof of 2.7 apply theorem 3, above.

The function given in section 3 of [2] may be used to show that the width requirement of (WA2) is necessary. Finally, it should be mentioned that by using theorem 1 instead of theorem 3 it is possible to obtain slightly different characterizations of $W(R)$. Namely, (WB) in theorem 6 may be replaced by either of

(WB1) For all but a countable set of $\varepsilon > 0$

$$M_x{}^W \bar{M}_w[|f(w+x)-f(x)|\chi_{WE(\varepsilon,f)}(w)] \leqq \varepsilon\,\bar{\varrho}(WE(\varepsilon,f)) ,$$

or

(WB2) For all but a countable set of $\varepsilon > 0$

$$M_x{}^W \bar{M}_w[|f(w+x)-f(x)|\chi_{BE(\varepsilon,f)}(w)] \leqq \varepsilon\,\bar{\varrho}(BE(\varepsilon,f)) .$$

Here $\bar{\varrho}(E) = \bar{M}(\chi_E)$. In the case that (WB2) is used, the conditions (WAi) in theorem 6 may be replaced by (Ai), $1 \leqq i \leqq 4$. This is proven by adjusting the arguments of [2] in a straightforward way.

Among the questions suggested by the above are: What is a fuller description of $\mathscr{E}_0$, $\mathscr{E}_1$? Is it necessary to have an exceptional set in the Besicovitch–Bohr theorem and in the condition (WB)?

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