MATH. SCAND. 30 (1972), 88–94

ON THE HOMOTOPY GROUPS OF COMPLEX PROJECTIVE ALGEBRAIC MANIFOLDS

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0. Introduction.

In this note we study an algebraic manifold A embedded in some complex-projective space P_n of dimension n, small compared with $\dim A$. In [3] the first author gave a relation between the rational homology $H_*(A,\mathsf{Q})$ of A and the dimension n. This relation provides intermediate results between the well-known properties of hypersurfaces and the elementary fact that $A \subseteqq \mathsf{P}_n$ is connected if $\dim_x A \geqq \frac{1}{2}n$ at all points $x \in A$.

Here we want to generalize these intermediate results to homotopy groups. The best generalization would be the

THEOREM. *If $A \subseteqq \mathsf{P}_n$ is closed algebraic, nonsingular, of dimension a at each of its points, and if $2a \geqq n+s$, then the relative homotopy groups $\pi_i(\mathsf{P}_n,A)$ vanish for $i=1,\ldots,s+1$.*

We do not know whether this theorem holds. Our paper contains only the following two steps towards it:

THEOREM I. *If $A \subseteqq \mathsf{P}_n$ is as above, and if $2a \geqq n+1$, then $\pi_1(A)=0$.*

THEOREM II. *If $A \subseteqq \mathsf{P}_n$ is as above, and if $2a \geqq n+s$, then the relative homotopy groups $\pi_i(\mathsf{P}_n,A)$ are finite for $1 \leqq i \leqq s+1$. In particular, the groups $\pi_3(A),\ldots,\pi_s(A)$ are finite.*

Theorem II is easily reduced to theorem I. Theorem I is proved using Andreotti–Grauert [1] to extend sections in unramified coverings.

1. Preliminaries.

Here we are going to state our notational conventions, and to collect the analytical tools we use.

Received October 30, 1970.

P_n is the complex-projective space of dimension n. We put $G := U(n+1)$, the unitary group, and $G^1 := SU(n+1)$, the special unitary group. G and G^1 operate on P_n by $x \to \sigma x$ for $x \in \mathsf{P}_n$, $\sigma \in G$. The letter A will always denote a connected nonsingular closed algebraic subset of P_n, with dimension at least a. We assume $2a \geqq n+1$. Later on we shall use the maps

$$\varphi\colon G \times A \to \mathsf{P}_n, \quad p_G\colon G \times A \to G, \quad p_A\colon G \times A \to A ,$$

where p_G and p_A are projections and φ is the differentiable fiber bundle defined by $\varphi(\sigma,x) = \sigma x$.

We need some properties of tubular neighbourhoods:

(a) If $B \subseteqq G$ is an open subset containing $1 \in G$, then Bx is open for all $x \in \mathsf{P}_n$.

We choose B in the following way: Since G is compact, there exist local differentiable coordinates l_i for G, centered at 1, such that the function $\sum l_i^2$ is invariant under all inner automorphisms of G. If $c \in \mathsf{R}$, $c > 0$, is small enough, then the set $B_c = \{\sigma \in G \mid \sum l_i^2(\sigma) < c\}$ has the following properties:

i) B_c is connected;

ii) $\sigma B_c \sigma^{-1} = B_c$ for all $\sigma \in G$.

These imply, just as in [2, lemmata 3, 4 and 5]: there exists some $b = b(c)$ such that for all $x \in \mathsf{P}_n$

$$B_c x = \{y \in \mathsf{P}_n \mid \operatorname{dist}(x,y) < b\} ,$$

where dist denotes the usual Fubini-Study metric on P_n. Thus, for small c, the set $B_c A$ is a tubular neighbourhood of A. We fix one such c once for all and put $B := B_c$, $TA := BA$. Then obviously $T\sigma A = \sigma TA$ for $\sigma \in G$ is a tubular neighbourhood of σA.

(b) For every $\sigma \in G$ the set $\varphi^{-1}(A) \cap (B\sigma \times A)$ is connected.

In order to prove (b) it is enough to show that $\varphi^{-1}(A) \cap (\bar{B}_0 \sigma \times A)$ is connected whenever $B_0 \subset\subset B$ is an open connected subset. Now $\varphi^{-1}(A)$ is a closed submanifold of $G \times A$, so $\varphi^{-1}(A) \cap (\bar{B}_0 \sigma \times A)$ contains at most finitely many connected components $K_1, \ldots, K_r$. Since these K_i are compact, their images $p_G(K_i) \subseteqq \bar{B}_0 \sigma$ are closed. Now we use

i) $A \cap \sigma' A$ is never empty for $\sigma' \in G$, since by assumption $2a \geqq n+1$;

ii) $A \cap \sigma' A$ is always connected (cf. [3, prop. 4]).

Property i) shows $\bar{B}_0 \sigma = \bigcup_{i=1}^r p_G K_i$. Since $\bar{B}_0 \sigma$ is connected, $p_G K_i \cap p_G K_j \neq \emptyset$ for some $i \neq j$ if $r \geqq 2$. Then, for $\sigma' \in p_G K_i \cap p_G K_j$, the set

$$\varphi^{-1}(A) \cap (\{\sigma'\} \times A) \cong A \cap \sigma' A$$

must have more than one connected component. So $r \geqq 2$ contradicts ii).

(c) Tubular neighbourhoods are pseudoconcave [2, Satz 3].
We need the following consequence [1, thm. 10] of this fact:
If F is a coherent analytic sheaf over P_n, subject to the condition

$$\operatorname{dih} F > n-a\,,$$

then for every point $q \in \partial(B_d A)$, $d<c$, there exists an arbitrarily small neighbourhood $U \subseteqq \mathsf{P}_n$, such that the restriction

$$H^0(U,F) \to H^0(U \cap (B_d A), F)$$

is bijective.

Obviously, $U_q \cap (B_d A)$ has to be connected if U_q is the connected component of q in $U \cap$(support of F).

2. Reduction of theorem II to theorem I.

Here we assume $\pi_1(A)=0$. There is the general Hurewicz homomorphism for relative groups:

$$\pi_i(\mathsf{P}_n, A) \to H_i(\mathsf{P}_n, A\,;\, \mathsf{Z})\,.$$

If $2a \geqq n+s$, then $H^i(\mathsf{P}_n, A\,;\,\mathsf{R})=0$ for $1 \leqq i \leqq s+1$ [3, thm. III]. Therefore the groups $H_i(\mathsf{P}_n, A\,;\,\mathsf{Z})$ are finite in this range. From [4, thm. 21, p. 511] we deduce, that the Hurewicz homomorphism is an isomorphism modulo the class of abelian torsion groups. This implies that $\pi_i(\mathsf{P}_n, A)$ is finite if $1 \leqq i \leqq s+1$. Next we use the relative homotopy sequence

$$\ldots \to \pi_i(A) \to \pi_i(\mathsf{P}_n) \to \pi_i(\mathsf{P}_n, A) \to \ldots$$

which shows that the kernels of the homomorphisms $\pi_i(A) \to \pi_i(\mathsf{P}_n)$, $1 \leqq i \leqq s$, are finite. It is well known that $\pi_i(\mathsf{P}_n)$ vanishes for $3 \leqq i \leqq 2n$. So theorem II is proved under the assumption $\pi_1(A)=0$.

3. Reformulation of the problem.

By a covering over a complex space S we understand a map $\gamma\colon C \to S$ of a topological space C onto S such that for each $s \in S$ there is a neighbourhood U of s with $\gamma^{-1}(U)$ a disjoint union of open sets on each of which γ is homeomorphic. An isomorphism between two coverings $\gamma_1\colon C_1 \to S$ and $\gamma_2\colon C_2 \to S$ is a bijective map $h\colon C_1 \to C_2$ such that the diagram

$$\begin{array}{ccc} C_1 & \xrightarrow{h} & C_2 \\ & {\scriptstyle\gamma_1}\searrow \quad \swarrow{\scriptstyle\gamma_2} & \\ & S & \end{array}$$

is commutative. We will denote this $h\colon (C_1 \overset{\sim}{\to} C_2)\,|\,S$. A covering is called trivial if it is isomorphic to some projection $S \times J \to S$. If $f\colon T \to S$ is a continuous map and $\gamma\colon C \to S$ is a covering, then we denote by f^*C the fiber product of f and γ, which is a covering over T.

For a connected complex space S, $\pi_1(S)=0$, if and only if every covering over S is trivial.

So to prove theorem I we have to show: every covering $\gamma_1\colon C_1 \to A$ is trivial. If any γ_1 is fixed, we denote the covering $\mathrm{id} \times \gamma_1\colon G \times C_1 \to G \times A$ over $G \times A$ by $\gamma\colon C \to G \times A$. We denote by C_σ the covering $\sigma^{-1*} C_1$ over σA. Because of

$$\varphi\,|\,\{\sigma\} \times A \;=\; \sigma \circ p_A\,|\,\{\sigma\} \times A\ ,$$

we have

$$(\varphi^* C_\sigma \overset{\sim}{\to} C)\,|\,\{\sigma\} \times A\ .$$

Let B and $T\sigma A$ be as in (a) of section 1. Let $\tau\colon T\sigma A \to \sigma A$ be a tubular retraction. Then we have:

Lemma 1. *There exists a unique isomorphism* $(\varphi^* \tau^* C_\sigma \overset{\sim}{\to} C)\,|\,B\sigma \times A$ *extending the natural one over* $\{\sigma\} \times A$.

Proof. Trivial. Existence, because $\tau\varphi$ induces the identity on the fundamental groups. Uniqueness, because the base space is connected.

Now, let us look at $C\,|\,\varphi^{-1}(A) \cap G^1 \times A$ and $\varphi^*(C_1)$. These are isomorphic if restricted to $\{1\} \times A$. If we can show that they are isomorphic all over $\varphi^{-1}(A) \cap G^1 \times A$, then we get theorem I. Because then, C restricted to any fiber $F = \varphi^{-1}(x) \cap G^1 \times A$ for some $x \in A$, and hence any $x \in \mathsf{P}_n$, is trivial.

From the fibering

$$F \overset{\subset}{\longrightarrow} G^1 \times A \overset{\sigma}{\longrightarrow} \mathsf{P}_n$$

we get the exact sequence [4, thm. 10, p. 377]

$$\pi_1(F) \to \pi_1(G^1 \times A) \to \pi_1(\mathsf{P}_n) \;=\; 0\ .$$

So a covering over $G^1 \times A$ is trivial if the restriction to F is trivial. This means that C is trivial, and therefore C_1, with which we started, has to be trivial.

4. An extension lemma.

LEMMA 2. *Let $\sigma \in G$ be arbitrary and $\delta\colon D \to A \cap T\sigma A$ a covering. Then every continuous cross-section $s\colon A \cap \sigma A \to D$ can be uniquely extended to a cross-section over $A \cap T\sigma A$.*

PROOF. The uniqueness part is trivial, since by (b) of section 1 the set $\varphi^{-1}(A) \cap (B\sigma \times A)$ and therefore also

$$\varphi(\varphi^{-1}(A) \cap (B\sigma \times A)) = A \cap \varphi(B\sigma \times A) = A \cap T\sigma A$$

is connected. We put

$$d := \sup\{c' : \text{there exists a cross-section } s_{c'} \text{ over } A \cap (B_{c'}\sigma A) \text{ extending } s\},$$

and have to show $d = c$.

i) $d > 0$: By assumption, $\delta | s(A \cap \sigma A)$ is bijective. We cover $s(A \cap \sigma A)$ by open sets $U_i \subseteqq V_i$ such that V_i is path connected and

a) $\delta | V_i$ is bijective;

b) if $\delta U_i \cap \delta U_j \neq \emptyset$, then $\delta U_i \subseteqq \delta V_j$;

c) $\delta^{-1} A \cap (\cup V_i) = s(A \cap \sigma A)$;

d) $U_i \cap s(A \cap \sigma A) \neq \emptyset$ for all i.

Then δ is bijective on $U = \bigcup U_i$. Otherwise there would exist $p \in U_i$, $q \in U_j$ such that $p \neq q$, but $\delta p = \delta q$. Because of a), we have $i \neq j$. Because of b), $\delta U_i \subseteqq \delta V_j$. So c) and d) show $U_i \cap V_j \neq 0$, and this implies $U_i \subseteqq V_j$. Since $\delta | V_j$ is bijective, we get $p = q$.

ii) There is a cross-section $S\colon A \cap (B_d \sigma A) \to D$ extending s: All sets $A \cap B_{c'}\sigma A$ are connected, since $\varphi^{-1}(A) \cap (B_{c'}\sigma \times A)$ is connected according to (b) of section 1, and

$$\varphi(\varphi^{-1}(A) \cap (B_{c'}\sigma \times A)) = A \cap \varphi(B_{c'}\sigma \times A) = A \cap (B_{c'}\sigma A).$$

So for $c' < d$, the cross-section $s_{c'}$ over $A \cap B_{c'}\sigma A$ is uniquely determined by s. Thus $s_{c'} | B_{c''}\sigma A = s_{c''}$ for $c'' < c'$. This means that the collection $\{s_{c'}\}_{c' < d}$ determines a cross-section S over $A \cap (B_d \sigma A)$.

iii) Denote by R the closure of $S(A \cap B_d \sigma A)$ in D. Then $\delta | R$ is bijective: Since S is a cross-section, $\delta | \mathring{R}$ is bijective. If there are $p_1, p_2 \in R$, $p_1 \neq p_2$, with $q = \delta(p_1) = \delta(p_2)$, then $q \in \partial(B_d \sigma A)$. Now take an open neighbourhood $U_q \subseteqq A$ of q as in (c) of section 1 using O_A for F. This is possible, since by assumption A is nonsingular and so

$$\operatorname{dih} O_A = \dim A = a > n - a.$$

We may assume

$$U_q = \delta U_1 = \delta U_2, \quad U_i \subseteqq D \text{ open},$$

where $p_i \in U_i$ and $\delta | U_i$ is bijective. We may further assume $U_1 \cap U_2 = \emptyset$. So

$$\delta(U_1 \cap \mathring{R}) \cap \delta(U_2 \cap \mathring{R}) = \emptyset ,$$

since $\delta | \mathring{R}$ is bijective. Because of

$$\delta\mathring{R} \cap U = \delta(\mathring{R} \cap \delta^{-1} U) ,$$

the sets $\delta(U_i \cap \mathring{R})$ are connected components of $\delta\mathring{R} \cap U$, which cannot be different in view of (c) of section 1. This contradicts $p_1 \neq p_2$.

iv) In the same way as in i), we show: δ is even bijective on an open neighbourhood of R.

If $d < c$, this would contradict the choice of d. So $d = c$, and the lemma is proved.

5. The extension method.

Here we give a proposition, which is the heart of the proof of theorem I.

We want to extend isomorphisms between coverings. This becomes a special case of extending sections: If $\gamma_i \colon C_i \to S$, $i = 1, 2$ are two coverings, denote by $\mathrm{Isom}(C_1, C_2)$ the sheaf of germs of isomorphisms $(C_1 \xrightarrow{\sim} C_2) | U$, where $U \subseteq S$ is open. Obviously, $\mathrm{Isom}(C_1, C_2)$ is a covering of S, non-empty if γ_1 and γ_2 have the same degree over S.

Proposition. *Let $B \subseteq G$ be an open ball containing* 1 *as in section* 1. *If we are given arbitrarily some $\sigma \in G$ and an isomorphism*

$$i_\sigma \colon (C \xrightarrow{\sim} \varphi^* C_1) | \varphi^{-1}(A) \cap (\{\sigma\} \times A) ,$$

then there exists a unique isomorphism

$$I_\sigma \colon \ (C \xrightarrow{\sim} \varphi^* C_1) | \varphi^{-1}(A) \cap (B\sigma \times A)$$

extending i_σ.

Proof. According to section 3, there exists a covering C_σ over $T\sigma A$ and an isomorphism

$$J \colon \ (C \xrightarrow{\sim} \varphi^* C_\sigma) | B\sigma \times A .$$

Since $\varphi | \{\sigma\} \times A$ is a homeomorphism, we obtain an isomorphism

$$h_\sigma \colon \ (C_\sigma \xrightarrow{\sim} C_1) | A \cap \sigma A$$

with $\varphi^* h_\sigma = i_\sigma \circ J^{-1}$. Now h_σ forms a section over $A \cap \sigma A$ in the covering $\mathrm{Isom}(C_\sigma, C_1)$ of $A \cap T\sigma A$. According to (b) of section 1, $A \cap T\sigma A$ is connected. Using lemma 2 we find that h_σ can be extended to a section H_σ over all of $A \cap T\sigma A$ in the covering $\mathrm{Isom}(C_\sigma, C_1)$. This means that h_σ extends to an isomorphism

$$H_\sigma\colon (C_\sigma \xrightarrow{\sim} C_1) \,|\, A \cap T\sigma A .$$

We put $I_\sigma = (\varphi^* H_\sigma) \circ J \,|\, B\sigma \times A$. Obviously this is an isomorphism between C and $\varphi^* C_1$ over $B\sigma \times A$ extending i_σ. That I_σ is uniquely determined by i_σ follows from the connectedness of $\varphi^{-1}(A) \cap (B\sigma \times A)$.

6. End of proof.

Over $S := \varphi^{-1}(A) \cap (G^1 \times A)$ we have the two coverings C and $\varphi^* C_1$. The covering $\mathrm{Isom}(C, \varphi^* C_1)$ over S is a sheaf of sets, and we can form the direct image sheaf $E := (p_G)_* \,\mathrm{Isom}(C, \varphi^* C_1)$ over G^1. A germ in E_σ, $\sigma \in G^1$, is represented by an isomorphism $(C \xrightarrow{\sim} \varphi^* C_1)$ over a neighbourhood of $\{\sigma\} \times A$.

By the proposition above, for any $\sigma \in G^1$, the natural map

$$\Gamma(B\sigma, E) \to E_\sigma$$

is surjective. This means that the sheaf E is locally constant over G^1. Since G^1 is simply connected E is constant.

The isomorphism $(C \xrightarrow{\sim} \varphi^* C_1) \,|\, \{1\} \times A$ represents (by the proposition) a germ in E_1. This germ can be extended to an element in $\Gamma(G^1, E)$. This means, the isomorphism can be extended to an isomorphism

$$(C \xrightarrow{\sim} \varphi^* C_1) \,|\, \varphi^{-1}(A) \cap (G^1 \times A) .$$

Thus, theorem I is proved.

ADDED IN PROOF: Recently, A. Ogus proved by algebraic methods theorem I for the profinite completion $\hat{\pi}_1(A)$ instead of $\pi_1(A)$. (Thesis, Harvard University, 1972).

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