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DIFFERENTIABLE MOTIONS OF UNKNOTTED, UNLINKED CIRCLES IN 3-SPACE[1]

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The study of braids was originated by E. Artin [2], [4], [5]. Recently, R. H. Fox has interpreted braids as motions of n discrete points in Euclidean 2-space and has generalized this concept to include the study of motions of submanifolds inside larger manifolds. In [9] one of Fox's students, D. M. Dahm, calculated the group of (ambient) topological motions of unknotted, unlinked circles in Euclidean 3-space. The purpose of this paper is to carry out the same computation in the differentiable category.

1. Preliminaries.

For convenience we briefly sketch the Braid Theory of n points in Euclidean 2-space. A complete exposition may be found in [2], [4], [5], [10], [11], and [13].

1.1 Definition. Let R^n denote Euclidean n-space and let I denote the closed unit interval $[0,1]$. Let P denote a set of n distinct points $p_1,p_2,\ldots,p_n$ in R^2. Then a *braid on n strands* is a set of n paths $w_1,w_2,\ldots,w_n\colon I \to \mathsf{R}^2$ such that:

(a) $w_i(0) = p_i$ for each i.

(b) $w_i(t) \neq w_j(t)$ for each $i \neq j$ and $0 \leqq t \leqq 1$.

(c) $w_i(1) \in P$ for each i.

Braids are generally pictured in R^3 by looking at the paths $\overline{w}_1,\overline{w}_2,\ldots,\overline{w}_n\colon I \to \mathsf{R}^3$ defined by $\overline{w}_i(t)=(w_i(t),t)$.

Two braids $w_1,w_2,\ldots,w_n$ and $v_1,v_2,\ldots,v_n$ may be combined to obtain a new braid $(w*v)_1,(w*v)_2,\ldots,(w*v)_n$ given by the following:

$$\begin{aligned}(w*v)_i(t) &= w_i(2t) &&\text{for}\quad 0\leqq t\leqq \tfrac12\,,\\ &= v_j(2t-1) &&\text{for}\quad \tfrac12\leqq t\leqq 1\,,\end{aligned}$$

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where $w_i(1) = p_j = v_j(0)$.

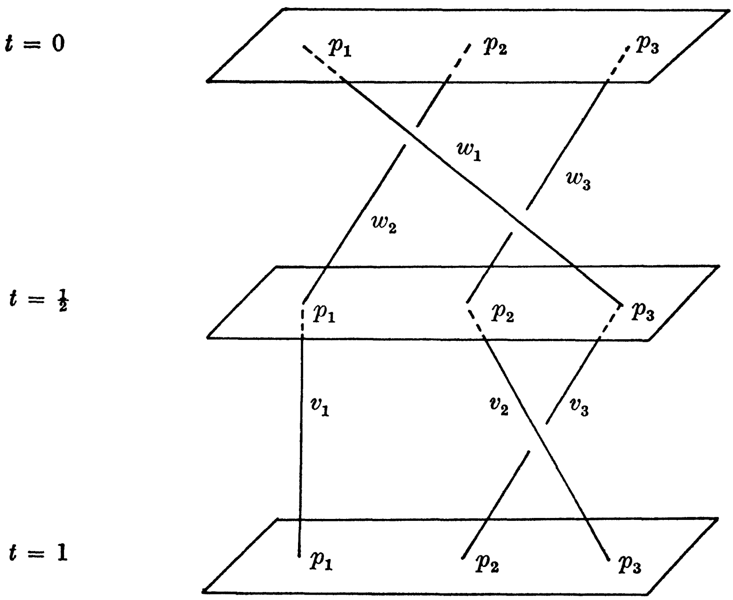


Fig. 1.1

Two braids $w_1,w_2,\ldots,w_n$ and $v_1,v_2,\ldots,v_n$ are *isotopic* provided there is an isotopy of braids between them. That is, provided there is a set of mappings $W_1,W_2,\ldots,W_n\colon I \times I \to \mathsf{R}^2$ such that:

$$
\begin{array}{lll}
\text{(a)}\ W_i(t,0) & = w_i(t), & 0 \leqq t \leqq 1\,. \\
\text{(b)}\ W_i(t,1) & = v_i(t), & 0 \leqq t \leqq 1\,. \\
\text{(c)}\ W_i(t,s) & \neq W_j(t,s), & 0 \leqq t,\, s \leqq 1,\, i \neq j\,. \\
\text{(d)}\ W_i(0,s) & = p_i = w_i(0) = v_i(0), & 0 \leqq s \leqq 1\,. \\
\text{(e)}\ W_i(1,s) & = w_i(1) = v_i(1), & 0 \leqq s \leqq 1\,.
\end{array}
$$

Braid classes are then defined to be isotopy classes of braids. The operation $*$ is, in fact, a group operation on the set of braids. Hence the nth *Braid Group* $B_n(\mathsf{R}^2)$ is defined to be the set of braid classes on n-strands together with the operation $*$.

1.2 Definition. Suppose that M is a manifold. Then the nth *configuration space of* M, $F_n(M)$, is defined by

$$F_n(M) = \{(x_1,x_2,\ldots,x_n) \in M \times M \times \ldots \times M \mid i \neq j \Rightarrow x_i \neq x_j\}\,.$$

The symmetric group on n letters S_n acts freely on $F_n(M)$ by permuting coordinates. That is, if $s \in S_n$ then

$$s(x_1,x_2,\ldots,x_n) = (x_{s(1)},x_{s(2)},\ldots,x_{s(n)}) .$$

If x and y are two elements of $F_n(M)$ then we say $x \sim y$ provided there is an element s of S_n such that $x=s(y)$. This obviously defines an equivalence relation on $F_n(M)$ and since S_n acts freely on $F_n(M)$, we define the space $G_n(M)$ to be $F_n(M)/\sim$ and obtain a natural covering map $p\colon F_n(M) \to G_n(M)$ with fiber S_n.

Now, if $w_1,w_2,\ldots,w_n$ is a braid on n-strands we define a path $w\colon I \to F_n(\mathsf{R}^2)$ by

$$w(t) = (w_1(t),w_2(t),\ldots,w_n(t)) .$$

Then the mapping $pw\colon I \to G_n(\mathsf{R}^2)$ is a loop in $G_n(\mathsf{R}^2)$. In fact, the *nth braid group of a manifold M, $B_n(M)$*, may now be defined to be $\pi_1(G_n(M))$. For $M=\mathsf{R}^2$ this agrees with the classical definition given above. Using this definition of the group $B_n(M)$ as motivation we consider the group of differentiable motions of a submanifold N inside a larger manifold M as follows.

1.3 Definition. Suppose that M is a smooth connected manifold without boundary and that N is a compact smooth submanifold of M. By "smooth" throughout this paper we mean C^∞. In general, N will have a finite number of components $N_1,N_2,\ldots,N_n$. Let $E(N,M)$ denote the set of smooth embeddings of N into M with the C^∞ topology. In particular, if $N=\{p_1,p_2,\ldots,p_n\}$ is a finite set of points then $E(N,M)= F_n(M)$.

Let e denote the inclusion mapping $N \subsetneq M$ and let P denote the subset of $E(N,M)$ consisting of those embeddings of N into M which agree setwise with e. That is, $i \in P$ if and only if $i(N)=e(N)=N$.

Define $\bar{B}(M,N)$ to be the set $\pi_1(E(N,M),P,e)$. Intuitively, $\bar{B}(M,N)$ is the set of differentiable motions of N in M. (See, for example, [22] for a definition of relative homotopy groups $\pi_1(A,B,x)$.) This set can, in fact, be given a group structure. The analogous construction in the topological case admits "pathological" motions like the "snapping" of a knotted circle in R^3 which is pictured in Fig. 1.2.

Hence, we proceed as follows in order to provide a definition which will be applicable to the topological case. Let $D(M)$ denote the group of diffeomorphisms of M with the C^∞ topology. Let I_M denote the identity in $D(M)$. Let $D(M,N)$ denote the subgroup of $D(M)$ consisting of those diffeomorphisms which leave N setwise fixed. That is,

$$f \in D(M,N) \iff f(N) = N .$$

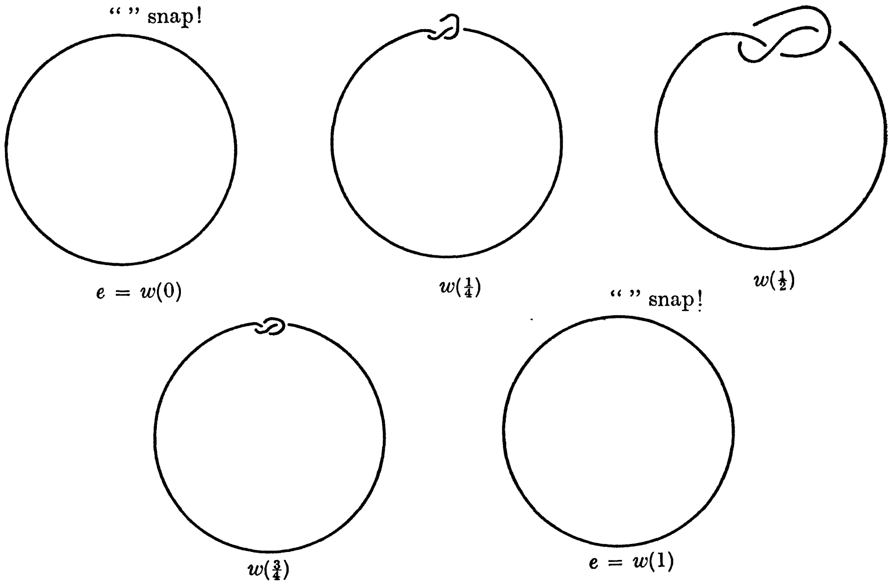


Fig. 1.2

Define the *group of motions of N in M*, $B(M,N)$, to be

$$\pi_1(D(M),\, D(M,N),\, I_M)\,.$$

Notice that $B(M,N)$ inherits a group structure from $D(M)$ since $D(M,N)$ is a subgroup of $D(M)$. The following lemma shows that this group is isomorphic to the intuitive set of motions $\bar{B}(M,N)$.

1.4 LEMMA. *Suppose that the mapping $r\colon D(M) \to E(N, M)$ is defined by $r(f)=fe$. Then $r_*\colon \pi_1(D(M), D(M,N)) \to \pi_1(E(N, M), P)$ is an isomorphism.*

PROOF. The conclusion follows immediately from Palais' result that the mapping r is a locally trivial fiber map [17, Theorem C].

1.5 DEFINITION. Suppose that M is a topological manifold without boundary and that N is a compact submanifold of M. Let e denote the inclusion map $N \subsetneq M$. Let $H(M)$ denote the group of homeomorphisms of M with the compact open topology, and let I_M denote the identity in $H(M)$. Let $H(M,N)$ denote the subgroup of $H(M)$ consisting of those homeomorphisms which leave N setwise fixed. That is, $f \in H(M,N) \Leftrightarrow f(N)=N$. The *topological group of motions of N in M*. $B_{\mathrm{top}}(M,N)$, is defined to be $\pi_1(H(M), H(M,N), I_M)$.

2. Motions with compact support.

Throughout the remainder of this paper we will restrict our attention to the motions of a submanifold N in R^k, although much of the preliminary work goes through in more general situations. In [9] Dahm computes the topological group of motions of unknotted, unlinked circles in R^3 based on paths and isotopies of paths in $H(\mathsf{R}^3)$ with compact support. Let $H_{\mathrm{cpt}}(\mathsf{R}^k)$ and $H_{\mathrm{cpt}}(\mathsf{R}^k,N)$ denote the subgroups of $H(\mathsf{R}^k)$ and $H(\mathsf{R}^k,N)$, respectively, consisting of those homeomorphisms with compact support. A minor technical problem arises if these groups are given the compact open topology since there then are continuous paths $w\colon I \to H_{\mathrm{cpt}}(\mathsf{R}^k)$ for which there is no compact subset $K \subsetneqq \mathsf{R}^k$ such that for every $t \in I$, the support of $w(t)$ is contained in K. There are several ways around this problem.

2.1 Definition. For each integer $n \geqq 1$ let $H_n(\mathsf{R}^k)$ denote the subgroup of $H(\mathsf{R}^k)$ consisting of those homeomorphisms of $H(\mathsf{R}^k)$ which have support contained in the closed ball of radius n centered at the origin. That is,

$$H_n(\mathsf{R}^k) = \{f \in H(\mathsf{R}^k) \mid \|x\| \geqq n \Rightarrow f(x)=x\}.$$

Notice that $H_{\mathrm{cpt}}(\mathsf{R}^k) = \bigcup_n H_n(\mathsf{R}^k)$. One possible solution to the technical problem above is to give $H_{\mathrm{cpt}}(\mathsf{R}^k)$ the direct limit topology. Then the following lemma shows that that situation cannot occur.

2.2 Lemma. *Suppose that K is a compact set and that $f\colon K \to H_{\mathrm{cpt}}(\mathsf{R}^k)$ is a continuous mapping with $H_{\mathrm{cpt}}(\mathsf{R}^k)$ having the direct limit topology. Then there is an integer n such that the image of f is contained in $H_n(\mathsf{R}^k)$.*

Proof. The proof is completely straightforward.

Alternatively, we can use the usual compact open topology on $H_{\mathrm{cpt}}(\mathsf{R}^k)$ and make use of the following lemma.

2.3 Lemma. *Every element of $\pi_1(H_{\mathrm{cpt}}(\mathsf{R}^k), H_{\mathrm{cpt}}(\mathsf{R}^k,N))$ can be represented by a path $w\colon I \to H_{\mathrm{cpt}}(\mathsf{R}^k)$ such that for some compact set K the support of $w(t)$ is contained in K for every $t \in I$.*

Proof. Let m be any integer such that N is contained in the ball of radius $m-1$ centered at the origin. Let $h\colon [0,\infty) \to [0,m+1)$ be any diffeomorphism such that $h(x)=x$ for $x \leqq m$. Let B denote the open ball of radius $m+1$ and define $H\colon \mathsf{R}^k \to B$ by

$$H(x) = h(\|x\|)x/\|x\| .$$

Notice that $H(x)=x$ for $x \in N$ and that H is a diffeomorphism.

Now, if v represents an element of $\pi_1(H_{\mathrm{cpt}}(\mathsf{R}^k), H_{\mathrm{cpt}}(\mathsf{R}^k, N))$ define $w\colon I \to H_{m+1}(\mathsf{R}^k)$ by

$$\begin{aligned} w(t)(x) &= Hv(t)H^{-1}(x) && \text{for } x \in B , \\ &= x && \text{otherwise} . \end{aligned}$$

It is completely straightforward to verify that w is the desired path.

The differentiable case is handled identically. We have now introduced several different groups of motions. In the topological category we have Dahm's group $\pi_1(H_{\mathrm{cpt}}(\mathsf{R}^3), H_{\mathrm{cpt}}(\mathsf{R}^3, N))$, where N is a finite set of unknotted, unlinked circles, as well as our original $\pi_1(H(\mathsf{R}^3), H(\mathsf{R}^3, N))$ based on motions without compact support. It appears to be a very difficult question whether these two are isomorphic. In the differentiable category we again have, a priori, two possibilities,

$$\pi_1(D_{\mathrm{cpt}}(\mathsf{R}^3), D_{\mathrm{cpt}}(\mathsf{R}^3, N)) \quad \text{and} \quad \pi_1(D(\mathsf{R}^3), D(\mathsf{R}^3, N))$$

where $D_{\mathrm{cpt}}(\mathsf{R}^3)$ and $D_{\mathrm{cpt}}(\mathsf{R}^3, N)$ are defined in the obvious way. However, in the differentiable category these two groups are immediately seen to be isomorphic via the following lemma together with Lemma 1.4.

2.4 LEMMA. *Using the notation of Section* 1 *suppose that the mapping* $\pi\colon D_{\mathrm{cpt}}(\mathsf{R}^k) \to E(N, \mathsf{R}^k)$ *is given by* $\pi(f)=fe$. *Then*

$$\pi_*\colon \pi_1(D_{\mathrm{cpt}}(\mathsf{R}^k), D_{\mathrm{cpt}}(\mathsf{R}^k, N)) \to \pi_1(E(N, \mathsf{R}^k), P)$$

is an isomorphism.

PROOF. Immediate from Palais' results [17, Theorem C], noting that Palais is working there with diffeomorphisms having compact support.

From now on we will work with the groups.

$$B_{\mathrm{cpt}}(\mathsf{R}^k, N) = \pi_1(D_{\mathrm{cpt}}(\mathsf{R}^k), D_{\mathrm{cpt}}(\mathsf{R}^k, N))$$

and

$$B_{\mathrm{ctop}}(\mathsf{R}^k, N) = \pi_1(H_{\mathrm{cpt}}(\mathsf{R}^k), H_{\mathrm{cpt}}(\mathsf{R}^k, N)) .$$

3. The Dahm homomorphism *d*.

Let S^3 denote the 3-sphere viewed as the one point compactification of R^3, and let $H_*(S^3)$ denote the group of homeomorphisms of S^3 which leave ∞ fixed. Suppose that

$$w\colon (I,0,1) \to (H_{\mathrm{cpt}}(\mathsf{R}^3), 1, H_{\mathrm{cpt}}(\mathsf{R}^3,N))$$

represents an element of $B_{\mathrm{ctop}}(\mathsf{R}^3,N)$ or of $B_{\mathrm{cpt}}(\mathsf{R}^3,N)$. Then define $\overline{w}\colon S^3 \to S^3$ by

$$\begin{aligned} \overline{w}(x) &= w(1)(x) \quad \text{if } x \in \mathsf{R}^3 \\ \overline{w}(\infty) &= \infty \, . \end{aligned}$$

Since $w(1)(N)=N$, $\overline{w}$ restricts to a mapping $w\colon S^3 \setminus N \to S^3 \setminus N$, which in turn induces an automorphism $w'\colon \pi_1(S^3 \setminus N, \infty) \to \pi_1(S^3 \setminus N, \infty)$.

3.1 Definition and lemma. The mapping

$$d\colon B_{\mathrm{ctop}}(\mathsf{R}^3,N) \to \mathrm{Auto}(\pi_1(S^3 \setminus N, \infty))$$

or

$$B_{\mathrm{cpt}}(\mathsf{R}^3,N) \to \mathrm{Auto}(\pi_1(S^3 \setminus N, \infty))$$

defined by $d([w])=w'$ is a well-defined homomorphism called the *Dahm homomorphism.*

The proof is straightforward.

In particular, if N is a set of n unknotted, unlinked circles in R^3, then $\pi_1(S^3 \setminus N)$ is the free group F_n on n generators and the Dahm automorphism goes from $B_{\mathrm{ctop}}(\mathsf{R}^3,N)$ or $B_{\mathrm{cpt}}(\mathsf{R}^3,N)$ into $\mathrm{Auto}(F_n)$. The main result of Dahm's thesis [9] is that in the topological case the mapping $d\colon B_{\mathrm{ctop}}(\mathsf{R}^3,N) \to \mathrm{Auto}(F_n)$ is an isomorphism into. The present paper obtains the same result in the differentiable category.

3.2 Remarks. The mapping d can be constructed equally well in the case of motions which do not have compact support. In his thesis [9] Dahm actually constructs a map

$$d\colon B_{\mathrm{ctop}}(M,N) \to \mathrm{Auto}(\pi_1(M \setminus N))$$

whenever M is a noncompact manifold. He accomplishes this by choosing a "sliding" basepoint for $\pi_1(M \setminus N)$ as follows. Let $w\colon [0,\infty) \to M$ be a one-to-one mapping such that $\overline{w([0,\infty))}$ is not compact. Then the path w provides natural isomorphisms between $\pi_1(M \setminus N, w(s))$ and $\pi_1(M \setminus N, w(t))$ for $0 \leqq t,\ s \leqq \infty$. In the present case $\pi_1(\mathsf{R}^3 \setminus N) \approx \pi_1(S^3 \setminus N)$, so this complication is unnecessary.

4. The differentiable motions of a single unknotted circle in R^3.

The purpose of this section is to show that the group of differentiable motions of the circle,

$$S^1 = \{(x_1, x_2, 0) \mid x_1{}^2 + x_2{}^2 = 1\},$$

in R^3 is isomorphic to Z_2. From now on we will be working strictly in the differentiable category where it makes no difference whether we use diffeomorphisms with or without compact support. Hence, we may use the simpler notation $B(\mathsf{R}^3, N)$ rather than $B_{\mathrm{cpt}}(\mathsf{R}^3, N)$ although in our computations we will often use diffeomorphisms with compact support.

Let SO_3 denote the group of orientation preserving orthogonal transformations of R^3 and let $\pi\colon SO_3 \to S^2$ be defined by $\pi(f) = f(0,0,1)$. Thus π is a locally trivial fiber map. Notice that $SO_3 \cap H(\mathsf{R}^3, S^1) \approx O_2$, the group of orthogonal transformations of R^2. The map π induces an isomorphism:

$$\pi_*\colon \pi_1(SO_3, O_2) \approx \pi_1(S^2, \{e_1, e_2\}) \approx \mathsf{Z}_2$$

where $e_1 = (0,0,1)$ and $e_2 = (0,0,-1)$. We immediately obtain a commutative diagram.

$$\begin{tikzcd}
\pi_1(SO_3, O_2) \arrow[r, "\bar{d}"] \arrow[d, "i_*"'] & \mathrm{Auto}(\pi_1(S^3 \setminus S^1)) \approx \mathsf{Z}_2 \\
\pi_1(D(\mathsf{R}^3), D(\mathsf{R}^3, S^1)) = B(\mathsf{R}^3, S_1), \arrow[ur, "d"'] &
\end{tikzcd}$$

where the mapping $\bar{d}$ is constructed analogously to d and is seen to be an isomorphism by inspection. The remainder of this section is devoted to demonstrating that the map i_* induced by the inclusion

$$i\colon (SO_3, O_2) \to (D(\mathsf{R}^3), D(\mathsf{R}^3, S^1))$$

is also an isomorphism and hence the Dahm homomorphism d is an isomorphism.

Suppose that $w\colon (I, 0, 1) \to (D(\mathsf{R}^3), 1, D(\mathsf{R}^3, S^1))$ represents an element of $B(\mathsf{R}^3, S^1)$. Then $w(1)$ is a mapping in $D(\mathsf{R}^3, S^1)$ which restricts to a mapping of S^1 onto itself which has degree plus or minus one. We will show in the following pages that if it has degree plus one then the motion $[w]$ is in the image of i_*. It follows immediately that i_* is an isomorphism. The proof proceeds in five steps:

a. $[w]$ may be represented by a path $v\colon (I, 0, 1) \to (D(\mathsf{R}^3), 1, D(\mathsf{R}^3, S^1))$ such that $v(1)$ is the identity on S^1.

b. $[v]$ may be represented by a path $u\colon (I,0,1) \to (D(\mathsf{R}^3), 1, D(\mathsf{R}^3, S^1))$ such that $u(1)$ is the identity on a tubular neighborhood D of S^1 in R^3.
c. $[u]$ may be represented by a path $t\colon (I,0,1) \to (D(\mathsf{R}^3), 1, D(\mathsf{R}^3, S^1))$ such that $t(1)$ is the identity on the 2-disk

$$D^2 = \{(x_1,x_2,0) \mid x_1{}^2+x_2{}^2 \leqq 1\}.$$

d. $[t]$ may be represented by a path $s\colon (I,0,1) \to (GL(3), 1, GL(2))$, where $GL(n)$ denotes the general linear group on R^n.
e. $[s]$ may be represented by a path $r\colon (I,0,1) \to ((SO(3), 1, 1)$ which is actually the trivial path.

4.1 Lemma. *Suppose that $f\colon S^1 \to S^1$ is a diffeomorphism of degree one. Then there is a path $g\colon I \to D(S^1)$ (where $D(S^1)$ denotes the diffeomorphism group of S^1 with the C^∞ topology) such that $g(0)=f$ and $g(1)$ is the identity on S^1.*

Proof. Let S^1 be considered as $\mathsf{R}^1/\approx$, where $x\approx y$ if $x-y$ is an integral multiple of 2π. We have a differentiable covering map $\pi\colon \mathsf{R}^1 \to S^1$ and the mapping f lifts to give us the diagram below:

$$\begin{array}{ccc} \mathsf{R}^1 & \xrightarrow{\bar{f}} & \mathsf{R}^1 \\ {\scriptstyle\pi}\Big\downarrow & & \Big\downarrow{\scriptstyle\pi} \\ S^1 & \xrightarrow[f]{} & S^1 \end{array}.$$

Define the mapping $\bar{h}\colon \mathsf{R}^1 \times I \to \mathsf{R}^1$ by the equation

$$\bar{h}(x,t) = tx+(1-t)\bar{f}(x).$$

Since $\bar{h}$ respects the relation $\approx$, it induces a mapping $h\colon S^1\times I \to S^1$ such that $h\pi=\pi\bar{h}$. The desired mapping g is defined by $g(t)(x)=h(x,t)$.

4.2 Corollary. *The motion $[w]$ may be represented by a path $v\colon (I,0,1) \to (D(\mathsf{R}^3), 1, D(\mathsf{R}^3, S^1))$ such that $v(1)$ is the identity on S^1.*

Proof. Apply the previous lemma to obtain a function $g\colon I \to D(S^1)$ such that $g(0)=w(1)$ and $g(1)=1$ on S^1. By Palais' result [17] g can be extended to a path $G\colon I \to D(\mathsf{R}^3, S^1)$ such that $G(t)=g(t)$ on S^1 for $0\leqq t \leqq 1$, and $G(1)=I$ on R^3. Let $H(t)=G(1-t)$. Then H represents the trivial motion in $B(\mathsf{R}^3, S^1)$ and thus $v(t)=w(t)H^{-1}(t)$ represents the same motion as w. The path v has the desired properties.

4.3 LEMMA. *Suppose that* $f\colon \mathsf{R}^3 \to \mathsf{R}^3$ *is a diffeomorphism such that* f *is the identity on* S^1*. Then there is a tubular neighborhood* D *of* S^1 *in* R^3 *and a diffeotopy* $g\colon \mathsf{R}^3 \times I \to \mathsf{R}^3$ *such that*:

$$\begin{aligned} &1)\ g(x,1) = f(x) && \textit{for each } x \textit{ in } \mathsf{R}^3, \\ &2)\ g(x,0) = x && \textit{for each } x \textit{ in } D, \\ &3)\ g(x,s) = x && \textit{for each } x \textit{ in } S^1,\ s \textit{ in } I. \end{aligned}$$

PROOF. Let D' be a solid torus around S^1. We identify D' with $S^1 \times E$ where E is a two disk of radius 1. By the uniqueness theorem for tubular neighborhoods and the diffeotopy extension theorem (see, for example, C.T.C. Wall [21], or S. Lang [14] and R. Palais [17]) we may assume that f restricted to D' is an $SO(2)$-bundle equivalence. That is, if h denotes f restricted to D' then $h(s,e) = (s, k(s)(e))$ where $k\colon S^1 \to SO(2)$ is a smooth map of S^1 into the special orthogonal group $SO(2)$. The lemma is proven by showing that k is homotopically trivial and thus can be extended to a smooth mapping $K\colon D^2 \to SO(2)$ where D^2 is the 2-disk of radius 1, centered at $(0,0)$. We can assume that $K((0,0))$ is the identity. We will denote points of D^2 with polar coordinates (s,t), $s \in S^1$, $t \in I$. Then the mapping

$$g\colon \mathsf{R}^3 \times I \to \mathsf{R}^3$$

is defined as follows. Let $\lambda\colon I \to I$ be a smooth monotonic function such that:

$$\begin{aligned} \lambda(x) &= 0, \quad 0 \leqq x \leqq \tfrac{1}{2}, \\ \lambda(x) &= 1, \quad \tfrac{3}{4} \leqq x \leqq 1. \end{aligned}$$

If $x \in D' \times I$, then $x = (s,e,t) \in S^1 \times E \times I$. Define

$$g(s,e,t) = \big(s, K\big(s, \lambda(\|e\|) + (1-\lambda(\|e\|))t\big)(e)\big).$$

If $(x,t) \in (\mathsf{R}^3 \setminus D') \times I$, define

$$g(x,t) = f(x).$$

Notice:

(i) If $x = (s,e,t) \in D'$ such that $\|e\| \geqq \frac{3}{4}$, then

$$g(x) = (s, K(s,1)(e)) = (s, k(s)(e)) = f(x),$$

so g is smooth.

(ii) If $x = (s,e,t) \in D'$ such that $t = 1$,

$$g(x) = \big(s, K(s,1)(e)\big) = \big(s, k(s)(e)\big) = f(x)$$

so that if $x = (y,1) \in \mathsf{R}^3 \times I$, then $g(y,1) = f(y)$.

(iii) Let $D = \{(s,e) \in D' \mid \|e\| \leqq \frac{1}{2}\}$. Then if $(s,e) \in D$,

$$g(s,e,0) = \big(s, K(s,0)(e)\big) = (s,e)\,.$$

Hence if the mapping K can be found, the lemma is proved.

In order to show that $k\colon S^1 \to SO(2)$ is homotopically trivial consider the circle $C = \{(x_1,x_2,0) \mid x_1{}^2 + x_2{}^2 = 1-\varepsilon\}$. By choosing ε small enough C is contained in D', and so $f(C)$ is contained in D', also. The circle C is identified with $S^1 \times \{x_0\}$ for some $x_0 \in E$.

Next consider the commutative diagram

$$\begin{tikzcd}
 & & & SO(2) \arrow[d, "e"] \\
C \arrow[rrru, "\bar{k}"] \arrow[r, "f"] \arrow[rrr, bend right, "r"'] & D \setminus S^1 \arrow[r, "p", "\approx"'] & S^1 \times S^1 \arrow[r, "\pi"] & S^1
\end{tikzcd}$$

e = evaluation map at x_0

where π is the restriction of the projection $S^1 \times E \to E$, $\bar{k}(s,x_0) = k(s)$, and p is the obvious homotopy equivalence. Since e induces an isomorphism $\pi_1(SO(2)) \to \pi_1(S^1)$, k is homotopically trivial if and only if r is. Since C is homotopically trivial in $\mathsf{R}^3 \setminus S^1$, $f(C)$ must be homotopically trivial in $\mathsf{R}^3 \setminus S^1$. But the diagram

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[rrr, bend left, "r"] \arrow[rrrd, "f"'] & D \setminus S^1 \arrow[r, "\approx"'] & S^1 \times S^1 \arrow[r, "\pi"] & S^1 \arrow[d, "j"] \\
 & & & R^3 \setminus S^1
\end{tikzcd}$$

commutes up to homotopy and $j_*\colon \pi_1(S^1) \to \pi_1(\mathsf{R}^3 \setminus S^1)$ is an isomorphism. Therefore, r and hence k must be homotopically trivial which completes the proof.

In what follows we will want to be able to say that $D^2 \cup u(1)(D^2)$ is an embedded two sphere. In order to do this we must twist the tubular neighborhood D around S^1 in order to smooth out the intersection (S^1) of D^2 and $u(1)(D^2)$.

4.4 Corollary. *The motion* $[v]$ *may be represented by a path*

$$u\colon (I,0,1) \to (D(\mathsf{R}^3),\, 1,\, D(\mathsf{R}^3,S^1))$$

such that, on a tubular neighborhood D *of* S^1 *in* R^3, $u(1)$ *is given by the formula*

$$u(1)(s,e) = (s,A(e))\,,$$

where $(s,e) \in D = S^1 \times E$ *and* A *is the* 180 *degree twist of* E.

Proof. Use the same proof as Lemma 4.3 only assume that $K(0,0)$ is A. Then apply the technique of Corollary 4.2.

The next step is to further modify the motion so that at 1 it is the identity on the 2-disk D^2 spanned by S^1 in R^3. Working with $u(1)(D^2)$ and D^2 we first deform $u(1)(D^2)$ so that it intersects D^2 transversally in a finite number of circles. These intersections are then removed one by one, making use of the Schoenflies Theorem to enclose each intersection in turn in a 3-ball. When there are no intersections left, the Schoenflies theorem is again used to make $D^2 \cup u(1)(D^2)$ the boundary of a 3-ball. Then $u(1)(D^2)$ is finally deformed so that it matches D^2.

4.5 Lemma. *The mapping* $u(1)$ *can be deformed keeping* S^1 *fixed to obtain a new mapping* $f\colon (\mathsf{R}^3,S^1) \to (\mathsf{R}^3,S^1)$ *which has all the following properties*:

1) $f(D^2)$ *intersects* D^2 *transversally*,
2) f *agrees with* $u(1)$ *on a tubular neighborhood of* S^1,
3) $f(D^2) \cap D^2$ *is a finite number of circles* $C_1, C_2, \ldots, C_k$ *plus the original* S^1.

Proof. This all follows immediately from general transversality theory and Palais' result [17], [20], [21].

4.6 Definition. We define an ordering on the circles of intersection $C_1, C_2, \ldots, C_k$ as follows:
$C_i < C_j$ if and only if C_i and S^1 are in different components of $D^2 \setminus C_j$.

4.7 Lemma. *The ordering* $<$ *defined above is a partial ordering.*

Proof. We must show two things, antisymmetry and transitivity.

(i) Antisymmetry. Suppose that $C_i < C_j$. Then let ω be a path connecting C_i and S^1 such that $\omega(t) \in C_i$ if and only if $t=0$. Since $C_i < C_j$ the path ω must intersect C_j at some time s, but then the portion of ω between s and 1 is a path in $D^2 \setminus C_i$ from C_j to S^1 so that C_j cannot be $< C_i$. This shows that $<$ is antisymmetric.

(ii) Transitivity. Suppose that $C_i < C_j$ and $C_j < C_k$ but that $C_i \nless C_k$. We shall show that this leads to a contradiction. Since $C_i \nless C_k$ there is a path ω in $D^2 \setminus C_k$ from C_i to S^1. But since $C_i < C_j$ the path ω must intersect C_j at some time s. But then the portion of ω between s and 1 is a path in $D^2 \setminus C_k$ which connects C_j and S^1. This contradicts the fact that $C_j < C_k$ and completes the proof.

4.8 Definition. A circle C_i is called *minimal* if there is no circle C_j such that $C_j < C_i$.

4.9 Lemma. *Each circle C_i is either itself minimal or there is a circle $C_j < C_i$ such that C_j is minimal.*

Proof. Clear.

4.10 Lemma. *Suppose that $C = C_i$ is a minimal circle of intersection of $f(D^2) \cap D^2$.*

Then there is a diffeotopy from f to a map g which removes the intersection C without introducing any new intersections, and which keeps S^1 fixed throughout the diffeotopy.

Proof. Let E_1 be the component of $D^2 \setminus C$ which does not contain S^1. Similarly, let E_2 be the component of $f(D^2 \setminus C)$ which does not contain S^1. Although D^2 may intersect E_2, E_1 does not intersect E_2 since C was chosen to be minimal.

For convenience we assume that E_2 "starts" below D^2 as shown in Fig. 4.1. By transversality theory we can choose an ε small enough so that the disk $D^2 + \varepsilon$ intersects $f(D^2)$ transversally in circles C_i' just above C_i as shown. Let C' be the circle on $D^2 + \varepsilon$ just above C. Let E_1' be the component of $(D^2 + \varepsilon) \setminus C'$ not containing S^1 and let E_2' be the component of $f(D^2) \setminus C'$ which does not contain S^1. Smooth out $E_1' \cup E_2'$ by a small diffeotopy of f with support close to C' so that together they give a smooth imbedding of S^2 in R^3 as shown in Fig. 4.1.

By the Schoenflies Theorem [16] this imbedding of S^2 can be extended to the 3-ball D^3. In D^3 there is a diffeotopy taking the bottom half sphere E_2' as close as we want to the top half sphere E_1' keeping E_2' fixed in a neighborhood of its boundary C'. In particular there is a diffeotopy as described above which pulls E_2' free of the disk D^2. This diffeotopy extends to a diffeotopy of $f(D^2)$ which is fixed outside of the 3-ball D^3. This diffeotopy in turn by the diffeotopy extension theorem extends to a diffeotopy of R^3 with compact support which leaves the

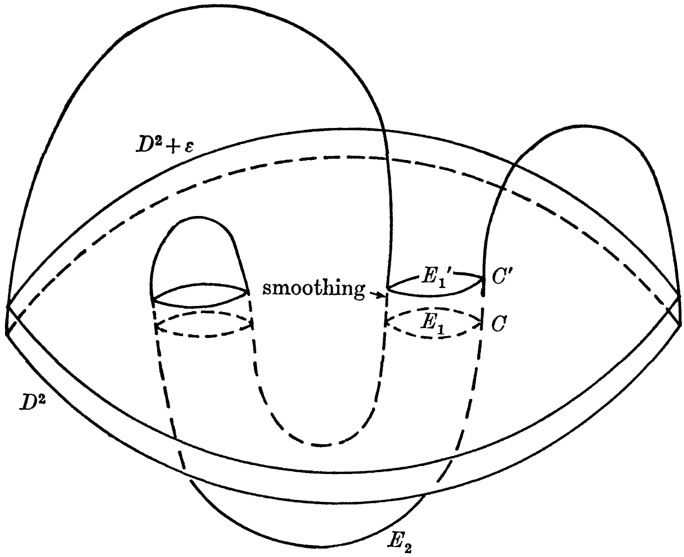


Fig. 4.1

portion of $f_1(D^2)$ outside of the 3-ball D^3 and thus S^1 pointwise fixed and which pulls E_2' off of D^2 eliminating the intersection C. No new intersections are introduced and, in fact, any other intersections of E_2 with D^2 are eliminated. This completes the proof.

By repeated applications of Lemma 4.10 we remove all the intersections other than S^1 of D^2 and $f(D^2)$. This enables us to bring $f(D^2)$ down to match D^2 by the following lemma.

4.11 Lemma. *Suppose we have the situation given by the conclusion of Lemma* 4.5 *and that, in addition,* $f(D^2) \cap D^2 = S^1$. *Then there is a diffeotopy of* R^3 *fixing* S^1 *which takes* f *onto a mapping which is the identity on* D^2.

Proof. The embeddings D^2 and $f(D^2)$ of D^2 in R^3 give a differentiable embedding of the 2-sphere S^2 in R^3:

$$S^2 = D^2{}_1 \cup_1 D^2{}_2 \xrightarrow[1 \cup f]{} \mathsf{R}^3 .$$

By the Schoenflies Extension Theorem this embedding can be extended to an embedding of the 3-ball D^3 into R^3. In the 3-ball $D^2{}_2$ is diffeotopic to $D^2{}_1$, keeping S^1 fixed. This diffeotopy can be extended by Palais' result (or the diffeotopy extension theorem) to all of R^3. This extension is the desired diffeotopy.

4.12 COROLLARY. *The motion* $[u]$ *may be represented by a path* $t\colon (I,0,1) \to (D(\mathsf{R}^3), 1, D(\mathsf{R}^3, S^1))$ *such that* $t(1)$ *is the identity on* D^2.

PROOF. The conclusion follows immediately from Lemmas 4.5–4.11 noting that each of them keeps S^1 fixed.

4.13 LEMMA. *The motion* $[t]$ *may be represented by a path*

$$s\colon (I,0,1) \to (GL(3), 1, GL(2)) .$$

PROOF. Define the mapping $H\colon I \times I \to D(\mathsf{R}^3)$ by

$$\begin{aligned} H(y,z)(x) &= t(y)((1-z)x)/(1-z) \qquad \text{if } z \neq 1 , \\ H(y,1)(x) &= A(y)(x) , \end{aligned}$$

where $A(y)$ is the element of $GL(3)$ corresponding to the derivative of $t(y)$ at the origin. Define $s\colon I \to GL(3)$ by

$$s(y) = H(y,1) = A(y) \in GL(3) .$$

t and s represent the same motions since $s(y) = H(y,0)$ and

$$H(0,z)(x) = t(0)((1-z)x)/(1-z) = x ,$$

and for each x in S^1,

$$H(1,z)(x) = t(1)((1-z)x)/(1-z) = x$$

since $t(1)$ is the identity on D^2.

4.14 LEMMA. *The motion* $[s]$ *may be represented by a path*

$$r\colon (I,0,1) \to (SO(3), 1, SO(2))$$

which is homotopically trivial.

PROOF. The first part of the conclusion follows from the standard deformation of $GL(k)$ into $O(k)$. The triviality of r follows from the isomorphisms

$$\begin{tikzcd} & \pi_1(S^2, \{e_1, e_2\}) \approx \mathbf{Z}_2 \\ \pi_1(SO(3), O(2)) \arrow[ur, "\approx", "\pi_*"'] \arrow[dr, "\partial", "\approx"'] & \\ & \pi_0(O(2)) \approx \mathbf{Z}_2 . \end{tikzcd}$$

This completes the program outlined at the beginning of this section giving us the following theorem.

4.15 THEOREM. *The mapping* $i_*: \pi_1(SO(3), O(2)) \to \pi_1(D(\mathrm{R}^3), D(\mathrm{R}^3, S^1))$ *induced by the inclusion mapping* i *is an isomorphism. Hence, the Dahm homomorphism* $d: B(\mathrm{R}^3, S^1) \to \mathrm{Auto}(\pi_1(S^3 \setminus S^1)) \approx Z_2$ *is an isomorphism.*

5. Motions of n unknotted, unlinked circles in R^3.

In this section we turn our attention to the group

$$B(\mathrm{R}^3, S^1{}_1 \cup S^1{}_2 \cup \ldots \cup S^1{}_n) .$$

The circles $S^1{}_i$ are embedded in the xy-plane as circles of unit radius centered at the points $(5i-4, 0, 0)$.

Let x_i denote $(5i-5, 0, 0)$ which we will use as a "basepoint" on the circle $S^1{}_i$. Then we define an injection

$$\nu: B(\mathrm{R}^3, \{x_1, x_2, \ldots, x_n\}) \to B(\mathrm{R}^3, S^1{}_1 \cup S^1{}_2 \cup \ldots \cup S^1{}_n)$$

as follows.

5.1 DEFINITION. Suppose that $w: I \to F_n(\mathrm{R}^3)$ is a path in $F_n(\mathrm{R}^3)$ given by $w(t) = (w_1(t), w_2(t), \ldots, w_n(t))$ representing an element of

$$B(\mathrm{R}^3, \{x_1, x_2, \ldots, x_n\}) .$$

Hence, $w_i(0) = x_i$ and $w_i(1) \in \{x_1, x_2, \ldots, x_n\}$ for each i. Define $h_w: I \to \mathrm{R}$ by

$$h_w(t) = \tfrac{1}{5}\mathrm{Min}\{\|w_i(t) - w_j(t)\| \mid i \neq j\} .$$

Then $h_w(0) = h_w(1) = 1$. Define $v_w: I \to E(S^1{}_1 \cup S^1{}_2 \cup \ldots \cup S^1{}_n, \mathrm{R}^3)$ by

$$v_w(t)(s) = w_i(t) + (s - x_i)h_w(t) \quad \text{if } s \in S^1{}_i .$$

Then $v_w(0)$ is the inclusion map, and $v_w(1)$ is a permutation of

$$S^1{}_1 \cup S^1{}_2 \cup \ldots \cup S^1{}_n .$$

By Palais' result [17], v_w can be lifted to a mapping $V(w): I \to D(\mathrm{R}^3)$ such that $V(w)(0)$ is the identity and $V(w)(t) = v_w(t)$ on $S^1{}_1 \cup S^1{}_2 \cup \ldots \cup S^1{}_n$ for each t. This construction defines a homomorphism

$$\nu: B(\mathrm{R}^3, \{x_1, x_2, \ldots, x_n\}) \to B(\mathrm{R}^3, S^1{}_1 \cup S^1{}_2 \cup \ldots \cup S^1{}_n) .$$

5.2 LEMMA. $B(\mathrm{R}^3, \{x_1, x_2, \ldots, x_n\})$ *is isomorphic to the symmetric group on* n *letters* S_n.

PROOF. This follows from the fact that $\pi_1(F_n(\mathrm{R}^3))$ is trivial and hence $\pi_1(G_n(\mathrm{R}^3))$ is isomorphic to the group of covering transformations of the covering space $S_n \to F_n(\mathsf{R}^3) \to G_n(\mathsf{R}^3)$. The reader is referred to Fadell and Neuwirth [11] for a complete exposition.

We also obtain a mapping

$$\mu\colon B(\mathsf{R}^3, S^1{}_1 \cup S^1{}_2 \cup \ldots \cup S^1{}_n) \to B(\mathsf{R}^3, \{x_1, x_2, \ldots, x_n\})$$

as follows:

5.3 Definition. Suppose

$$w\colon (I,0,1) \to (D(\mathsf{R}^3), 1, D(\mathsf{R}^3, S^1{}_1 \cup S^1{}_2 \cup \ldots \cup S^1{}_n))$$

represents a motion in $B(\mathsf{R}^3, S^1{}_1 \cup S^1{}_2 \cup \ldots \cup S^1{}_n)$. Let $c_w \in S_n$ be the permutation defined by $w(1)(S^1{}_i) = S^1{}_{c_w(i)}$. We may assume that $w(1)(x_i) = x_{c_w(i)}$ for each i, since if $w(1)(x_i) \neq x_{c_w(i)}$, there is an isotopy of $S^1{}_{c_w(i)}$ taking $w(1)(x_i)$ onto $x_{c_w(i)}$ and hence w can be modified keeping

$$S^1{}_1 \cup S^1{}_2 \cup \ldots \cup S^1{}_n$$

setwise fixed so that $w(1)(x_i) = x_{c_w(i)}$. The mapping $m(w)\colon I \to F_n(\mathsf{R}^3)$ is defined by

$$m(w)(t) = \big(w(t)(x_1), w(t)(x_2), \ldots, w(t)(x_n)\big)\,.$$

Clearly, $m(w)$ defines a motion in $B(\mathsf{R}^3, \{x_1, x_2, \ldots, x_n\})$ and this construction gives us a mapping

$$\mu\colon B(\mathsf{R}^3, S^1{}_1 \cup S^1{}_2 \cup \ldots \cup S^1{}_n) \to B(\mathsf{R}^3, \{x_1, x_2, \ldots, x_n\})$$

such that $\mu\nu$ is the identity on $B(\mathsf{R}^3, \{x_1, x_2, \ldots, x_n\})$. Hence, ν is injective.

Let F_n denote $\pi_1(S^3 \setminus (S^1{}_1 \cup S^1{}_2 \cup \ldots \cup S^1{}_n))$ the free group on n generators. Let $\{a_1, a_2, \ldots, a_n\}$ be a free basis for F_n given by the loops shown in Fig. 5.1.

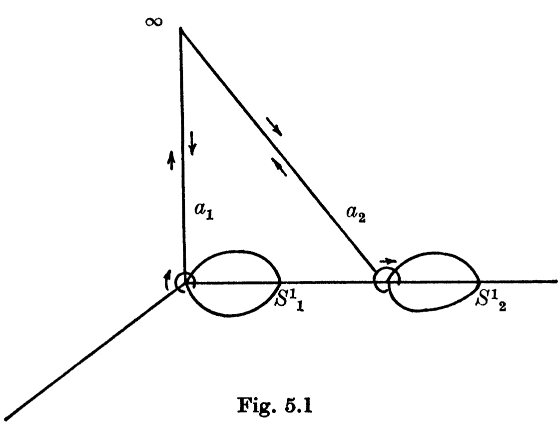


Fig. 5.1

There are three kinds of motions which are immediately evident. First the "flipping" motion α_i which flips the circle $S^1{}_i$ around the x-axis through an angle of 180 degrees in a manner similar to the non-trivial motion in $B(\mathsf{R}^3, S^1)$. The corresponding automorphisms $A_i = d(\alpha_i)$ of F_n are given by

$$\begin{aligned} A_i(a_j) &= a_j \quad \text{for } j \neq i\,, \\ A_i(a_i) &= a_i^{-1}\,. \end{aligned}$$

The second kind of motion is the permutation of circles which comes from a motion in $B(\mathsf{R}^3, \{x_1, x_2, \ldots, x_n\})$. If c is a permutation in

$$S_n \approx B(\mathsf{R}^3, \{x_1, x_2, \ldots, x_n\})$$

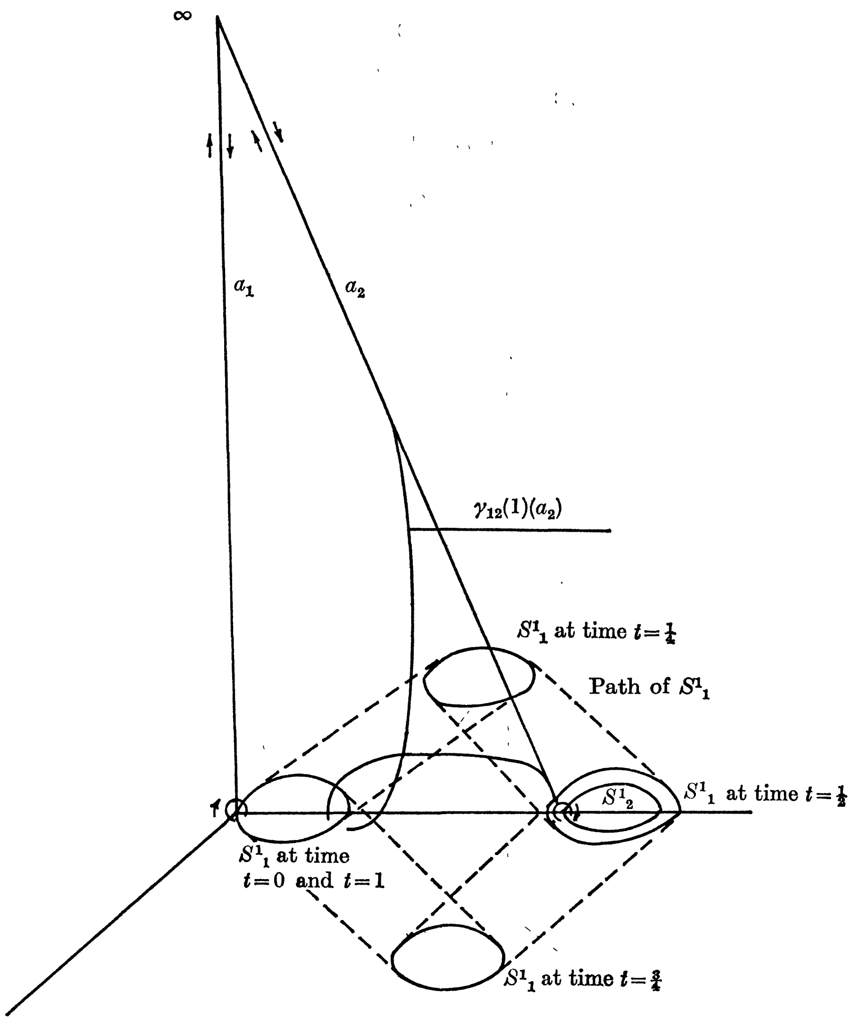


Fig. 5.2

and β_c denotes the motion $\nu(c)$ in $B(\mathrm{R}^3, S^1{}_1 \cup S^1{}_2 \cup \ldots \cup S^1{}_n)$ then the corresponding automorphism $B_c = d(\beta_c)$ of F_n is given by

$$B_c(a_i) = a_{c(i)} .$$

The third type of motion γ_{ij} is the motion which loops the circle $S^1{}_i$ around the circle $S^1{}_j$ as illustrated in Fig. 5.2. The corresponding automorphism $C_{ij} = d(\gamma_{ij})$ is given by

$$\begin{aligned} C_{ij}(a_k) &= a_k \qquad \text{for } k \neq j , \\ C_{ij}(a_j) &= a_i^{-1} a_j a_i . \end{aligned}$$

5.4 Definition. Let F_n be as above and let $T(F_n)$ denote the subgroup of $\mathrm{Auto}(F_n)$ consisting of those automorphisms $r\colon F_n \to F_n$ of the form

$$r(a_i) = q_i a_{c(i)}{}^{e(i)} q_i^{-1} ,$$

where $e(i)$ is plus or minus one and $c \in S_n$. The following lemma due to Dahm [9] is identical in both the topological and differentiable case.

5.5 Lemma. *The Dahm homomorphism*

$$d\colon B(\mathrm{R}^3, S^1{}_1 \cup S^1{}_2 \cup \ldots \cup S^1{}_n) \to \mathrm{Auto}(F_n)$$

is onto $T(F_n)$.

Proof. It is clear from Fig. 5.2 that d goes into $T(F_n)$. To show that d goes onto $T(F_n)$ it suffices to show that $T(F_n)$ is generated by the automorphisms A_i, B_c, and C_{ij}. In fact, it suffices to show that if $r \in T(F_n)$ is of the form

$$r(a_i) = q_i a_i q_i^{-1} ,$$

then r is a product of the C_{ji}'s. We may assume that the $q_i a_i q_i^{-1}$ are reduced words in the a_j's. We prove that r is a product of the C_{ji}'s by induction on the total length of the q_i's. Since r is an automorphism there are elements b_i of F_n such that $r(b_i) = a_i$. We can write the b_i's as reduced words:

$$b_i = \textstyle\prod_{j=1}^{m(i)} a_{n(i,j)}{}^{e(i,j)} \quad \text{where } e(i,j) = +1 \text{ or } -1 .$$

Then

$$a_i = \textstyle\prod_{j=1}^{m(i)} q_{n(i,j)} a_{n(i,j)}{}^{e(i,j)} q_{n(i,j)}{}^{-1} .$$

Cancellations may occur only at the junctions $q_{n(i,j)}{}^{-1} q_{n(i,j+1)}$ since the $q_i a_i q_i^{-1}$ are already in reduced form. Consider such a junction:

$$\ldots q_s a_s{}^e q_s{}^{-1} q_t a_t{}^d q_t{}^{-1} \ldots .$$

One of the following cases must occur:

$$q_s^{-1} = ha_t^{-1}q_t^{-1} \tag{1}$$
$$q_s^{-1} = ha_tq_t^{-1} \tag{2}$$
$$q_t = q_sa_sh \tag{3}$$
$$q_t = q_sa_s^{-1}h\,, \tag{4}$$

where h is a reduced word. Equations (3) and (4) are the same as (1) and (2) with a change of indices and taking inverses. In both cases (1) and (2), since we started with reduced words, the length of q_s must be less than the length of h plus the length of q_t.

For case (1) consider the automorphism rC_{ts},

$$rC_{ts}(a_k) = q_ka_kq_k^{-1} \quad \text{for } k \neq s\,,$$
$$rC_{ts}(a_s) = r(a_t^{-1}a_sa_t) = q_ta_t^{-1}q_t^{-1}q_sa_sq_s^{-1}q_ta_tq_t^{-1} = q_th^{-1}a_shq_t^{-1}\,.$$

Since the total length of the "q_i's" for rC_{ts} is less than for r, the induction works for case (1). The same method works for case (2) completing the proof.

The rest of this section is devoted to showing that d is monic and hence $B(\mathsf{R}^3, S^1{}_1 \cup S^1{}_2 \cup \ldots \cup S^1{}_n)$ is isomorphic to $T(F_n)$.

5.6 Lemma. *Suppose w: $(I,0,1) \to (D(\mathsf{R}^3), 1, D(\mathsf{R}^3, S^1{}_1 \cup S^1{}_2 \cup \ldots \cup S^1{}_n))$ represents a motion in $B(\mathsf{R}^3, S^1{}_1 \cup S^1{}_2 \cup \ldots \cup S^1{}_n)$ such that $d([w])$ is the trivial automorphism. Then w may be represented by a path*

$$u\colon (I,0,1) \to (D(\mathsf{R}^3), 1, D(\mathsf{R}^3, S^1{}_1 \cup S^1{}_2 \cup \ldots \cup S^1{}_n))$$

such that $u(t)(x_i) = x_i$ for all t and each basepoint x_i.

Proof. As in 5.3 we may assume that $w(1)(x_i) = x_i$ for each i. Consider the path $v = V(m(w))$ constructed in 5.1 and 5.3. Since $d([w]) = 1$, $m(w) = 1$ so that v represents the trivial motion in $B(\mathsf{R}^3, S^1{}_1 \cup S^1{}_2 \cup \ldots \cup S^1{}_n)$, but

$$v(t)(x_i) = w(t)(x_i) \quad \text{for each } t \text{ and } i\,.$$

Let $u(t) = v^{-1}(t)w(t)$. Then $[u] = [v^{-1}][w] = [w]$ and $u(t)(x_i) = x_i$ for each t and i.

5.7 Lemma. *Suppose that*

$$u\colon (I,0,1) \to (D(\mathsf{R}^3), 1, D(\mathsf{R}^3, S^1{}_1 \cup S^1{}_2 \cup \ldots \cup S^1{}_n))$$

is as obtained above. Then $[u]$ may be represented by a path

$$p\colon (I,0,1) \to (D(\mathsf{R}^3), 1, D(\mathsf{R}^3, S^1{}_1 \cup S^1{}_2 \cup \ldots \cup S^1{}_n))$$

satisfying the following properties:

1) $p(t)(x_i) = x_i$ *for each* t *and* i.
2) $p(1)$ *restricted to tubular neighborhoods* D_i *of the* $S^1{}_i$ *looks like* $u(1)$ *in Corollary* 4.4.
3) *Let* $D^2{}_i$ *denote the disk spanned by* $S^1{}_i$. *Then for each* j, $p(1)(D^2{}_j)$ *intersects the original disks* $D^2{}_i$ *transversally in a finite number of circles* $C_{j,1}, C_{j,2}, \ldots, C_{j,k(j)}$ *plus the circle* $S^1{}_j$.

PROOF. The proof is identical to that of Lemmas 4.3–4.5 with the following additional observation. At first glance it appears that the disk $p(1)(D^2{}_j)$ might intersect $D^2{}_i$ on the boundary of $D^2{}_i$ for some $i \neq j$. However, this is impossible since $p(1)$ is a diffeomorphism and condition (2) specifies what it looks like near the boundary of $D^2{}_i$.

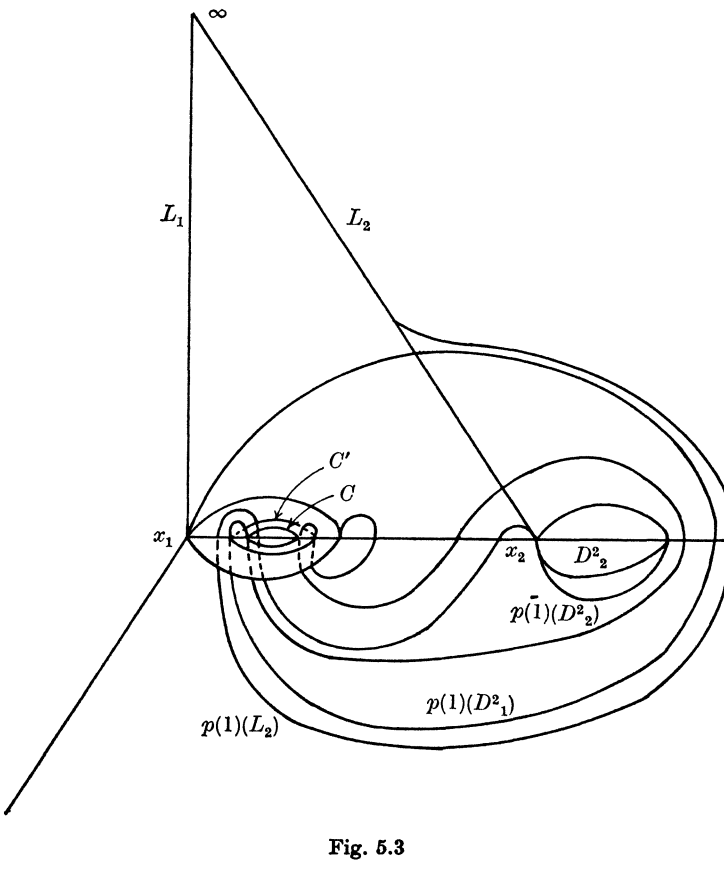


Fig. 5.3

Next we would like to use the techniques of Lemmas 4.10 and 4.11 to modify $p(1)$ by a diffeotopy keeping the $S^1{}_i$'s fixed so that $p(1)$ is the identity on all the disks $D^2{}_i$. Fig. 5.3 shows that these techniques will not work directly since the disks might interfere with each other. Notice that if we attempted to use Lemma 4.10 to remove the minimal circle C, we would run into difficulty since the circle $S^1{}_2$ is trapped inside the 2-sphere that would be constructed for the circle C.

Let L_i denote the line running down from ∞ to x_i so that a_i is represented by a loop coming down L_i to a point almost to x_i within D_i running around $S^1{}_i$ inside D_i and then running back out L_i. We avoid the impasse pictured in Fig. 5.3 by first modifying $p(1)$ by a diffeotopy which keeps the circles $S^1{}_i$ fixed and moves the lines $p(1)(L_i)$ so that they do not intersect any of the disks $D^2{}_j$. Notice that since $p(1)$ is a diffeomorphism the lines $p(1)(L_i)$ do not intersect any of the disks $p(1)(D^2{}_j)$ either.

Fig. 5.4 shows the result of pulling $p(1)(L_2)$ off $D^2{}_1$ in the situation in Fig. 5.3 near the circles C and C'.

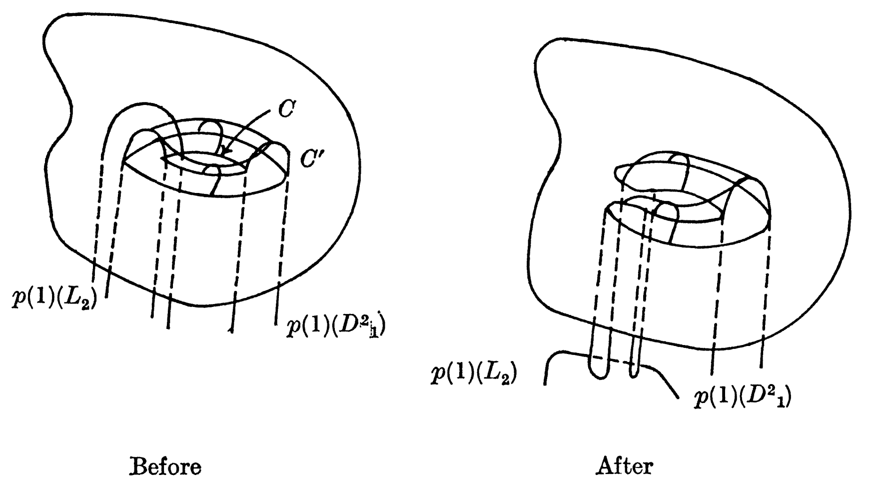


Fig. 5.4

Notice that the two intersections C and C' have been merged into a single intersection which can be removed by the technique of Lemma 4.10. In order to pull the lines $p(1)(L_i)$ off the disks $D^2{}_j$ we follow Dahm [9] and choose a projection of R^3 onto R^2 in the spirit of knot theory as shown in Fig. 5.5. In particular, we can assume that all the intersections of the projections of the circles and lines are transversal and

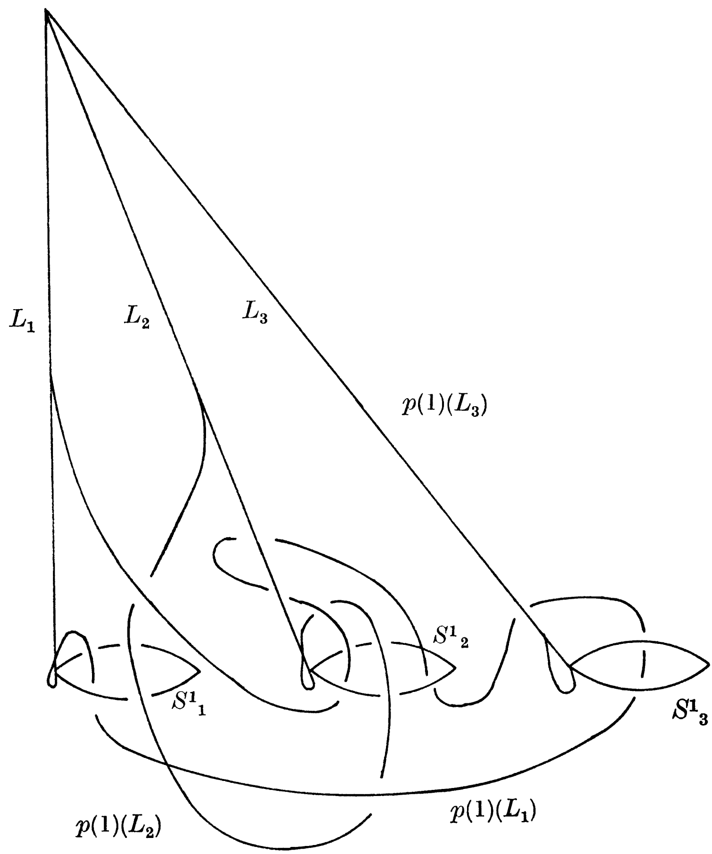


Fig. 5.5

that there are no double intersections, that is, no places where three or more of the lines and circles intersect.

For each of the lines $p(1)(L_i)$ we construct a sequence of the form $b_1 b_2 b_3 \ldots b_k$ where each of the b_j's is either o_j or u_j. This sequence is constructed corresponding to the overpasses and underpasses of the line $p(1)(L_i)$ with the circles $S^1{}_j$, starting at the top of $p(1)(L_i)$ at ∞ and terminating on the circle $S^1{}_i$ at x_i. In Fig. 5.5 we obtain the sequences $u_2 o_2 o_2 u_2 u_3 u_3 u_1 o_1$ and $o_1 o_1 o_2 o_2$ corresponding to the lines $p(1)(L_1)$ and $p(1)(L_2)$ respectively. The empty sequence corresponds to the line $p(1)(L_3)$.

Observe first of all that since each of the circles $S^1{}_i$ disconnects the projection onto R^2, each of sequences corresponding to $p(1)(L_j)$ must consist of subsequences of the form $b_1 b_2 b_3, \ldots, b_k$ where each b_h is an overpass or underpass for the same circle $S^1{}_j$ and k is even. For example, the sequence $u_1o_1o_1u_1o_2o_2u_3o_3$ might occur, but the sequence $u_1o_1u_1u_3u_3$ cannot occur because the line would get stuck inside $S^1{}_1$.

We must show first that if something of the form u_iu_i or o_io_i occurs in one of the sequences then it can be eliminated by a diffeotopy which does not disturb anything which is important. Notice that if we apply this kind of reduction to the line $p(1)(L_2)$ in the picture on the preceding page, we effectively pull $p(1)(L_2)$ free of all the disks $D^2{}_i$.

5.8 Lemma. *Suppose that the combination u_iu_i or o_io_i occurs in the sequence corresponding to the line $p(1)(L_j)$ where j may or may not equal i. Then there is a diffeotopy of $\mathbb{R}^3$ which fixes all of the circles $S^1{}_i$ as well as all the lines $p(1)(L_k)$ for $k \neq j$, and which eliminates the combination u_iu_i or o_io_i without introducing any new overpasses or underpasses.*

Proof. The proof is given for the case o_io_i. The other case is identical. The combination o_io_i represents two overpasses of the line $p(1)(L_j)$ over the circle $S^1{}_i$ with no other overpasses or underpasses with any other circle in between. However, there may be some overpasses or underpasses of that portion of the line $p(1)(L_j)$ between the two overpasses represented by o_io_i with some of the other lines $p(1)(L_k)$ or even with another part of the line $p(1)(L_j)$ as in Fig. 5.6.

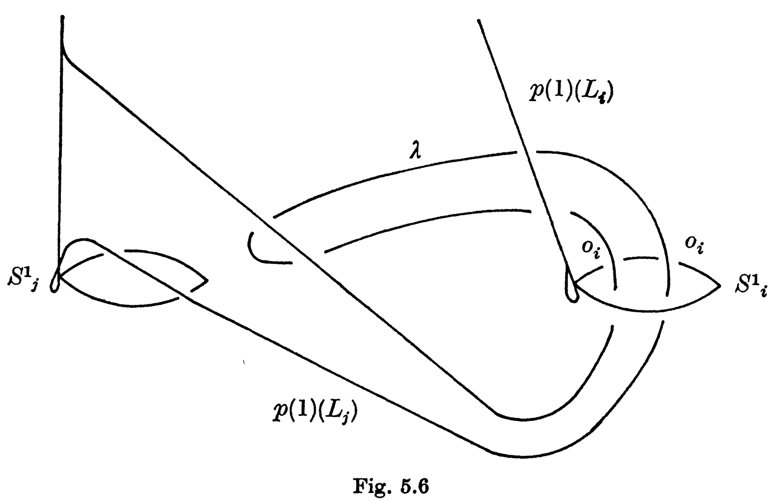


Fig. 5.6

Let λ denote the portion of the line $p(1)(L_j)$ in between the two overpasses corresponding to o_io_i. If λ does not pass under any of the lines $p(1)(L_k)$ for $k \neq j$, or $p(1)(L_k) \setminus \lambda$, then there is no difficulty at all. We simply lift λ up and pull it free of $S^1{}_i$. However, if there are any such underpasses, this operation might create some new underpasses or overpasses. Thus we must first eliminate any underpasses of λ under any of the $p(1)(L_k)$ for $k \neq j$ or $p(1)(L_j) \setminus \lambda$.

Suppose that λ passes under $p(1)(L_k)$. Then from the side we have Fig. 5.7a. First, we pull a small portion of λ up right next to $p(1)(L_k)$ so that we obtain Fig. 5.7b. Then we pull λ along $p(1)(L_k)$ until we reach $S^1{}_k$, keeping very close to $p(1)(L_k)$ so as not to interfere with anything else. This gives us Fig. 5.7c. Finally, λ is pulled around $S^1{}_k$ following $p(1)(D^2{}_k)$ very closely and then pulled back above its original position to obtain Fig. 5.7d. This replaces the original underpass by an overpass without creating any new under or overpasses.

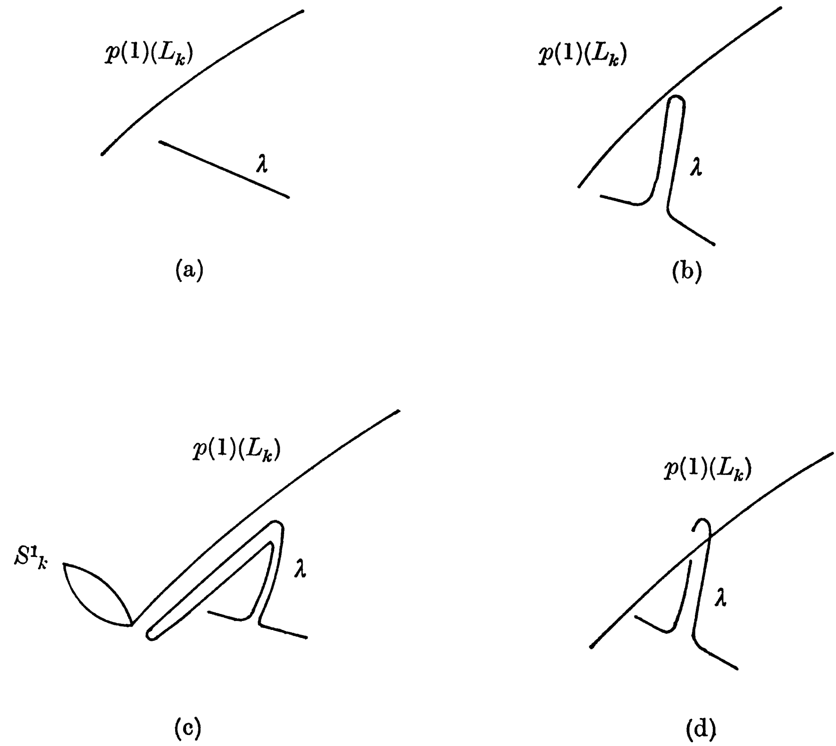


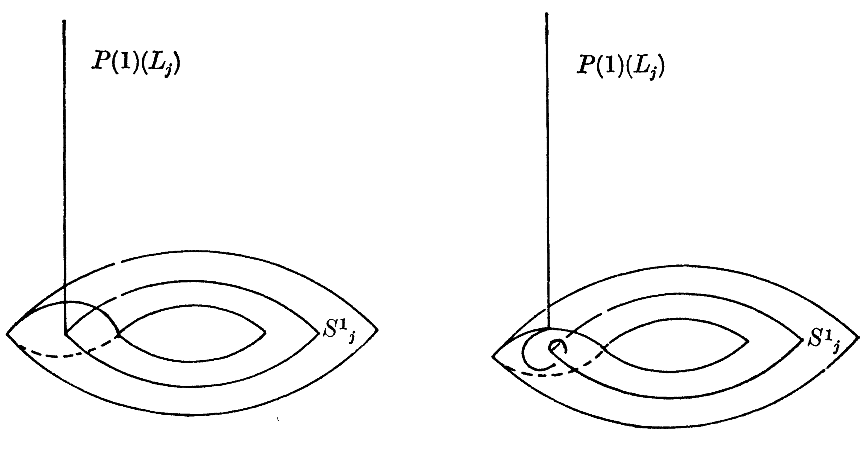
Fig. 5.7

$p(1)(L_j) \setminus \lambda$ consists of two parts: L containing ∞ and L' containing $S^1{}_j$. If λ passes under L' the preceding technique goes through without any changes. However, if λ passes under L the above technique won't work directly since λ would run into itself and get stuck. In this case we use the same method but instead of pulling λ up to L and following $p(1)(L_j)$ around, we pull L down to λ and follow $p(1)(L_j)$ around. Using this technique each of the underpasses of λ is eliminated. Then the two overpasses corresponding to $o_i o_i$ are removed by pulling λ up and free of $S^1{}_k$. These motions of λ are extended to R^3 keeping everything important fixed. This completes the proof of lemma 5.8.$\rfloor$

Lemma 5.8 also enables us to eliminate any overpasses or underpasses of $p(1)(L_j)$ with the circle $S^1{}_j$ which might appear at the very end of $p(1)(L_j)$ by the following corollary.

5.9 COROLLARY. *Suppose the sequence for the line $p(1)(L_j)$ terminates in $o_j u_j$ (or $u_j o_j$). Then there is a diffeotopy of R^3 which fixes all of the circles $S^1{}_i$ and all of the lines $p(1)(L_i)$ except for $p(1)(L_j)$, which eliminates the combination $o_j u_j$ (or $u_j o_j$) and does not create any new under or overpasses.*

PROOF. The proof is identical in either case. Hence, we just look at the case $o_j u_j$. By twisting the torus D_j surrounding $S^1{}_j$ around $S^1{}_j$ as in Fig. 5.8 we add the combination $u_j o_j$ so that the sequence now ends in $o_j u_j u_j o_j$ which is eliminated by two applications of lemma 5.8.



Before After

Fig. 5.8

Now for each of the lines $p(1)(L_i)$ we may assume that we have a sequence of the form $d_1 b_1 d_2 b_2 \ldots d_q b_q$ where if $d_j = o_k$ then $b_j = u_k$ and if $d_j = u_k$ then $b_j = o_k$, and $b_q \neq o_i$ or u_i. For each line $p(1)(L_i)$ we define a word $W_i{}'$ in the generators a_k of F_n as follows:

5.10 DEFINITION. We define $W_i = a_{n(1)}^{e(1)} a_{n(2)}^{e(2)} \ldots a_{n(q)}^{e(q)}$ where $n(j) = k$ if $d_j = o_k$ or $d_j = u_k$ and $e(j) = +1$ if $d_j = o_k$ and $e(j) = -1$ if $d_j = u_k$.

The following lemma is immediate from the preceding lemma and definitions.

5.11 LEMMA. *W_i is a reduced word in the a_j's and $d([p])(a_i) = W_i a_i W_i^{-1}$.*

PROOF. W_i is a reduced word since a subsequence of W_i of the form $a_j a_j^{-1}$ would correspond to $o_j u_j u_j o_j$ which was eliminated by Lemma 5.8. The second part of the conclusion follows from the fact that the loop that was chosen to represent a_i ran in along L_i almost to x_i, then around x_i inside the tubular neighborhood D_i around $S^1{}_i$, and finally back out L_i.

5.12 COROLLARY. *Suppose that p is as above. Then there are no underpasses or overpasses of the lines $p(1)(L_i)$ with any of the circles $S^1{}_j$. Thus, none of the lines $p(1)(L_i)$ intersects any of the disks $D^2{}_j$.*

PROOF. Since $d([p])$ is the trivial automorphism, $W_i a_i W_i^{-1}$ must equal a_i for each i. However, since W_i is a reduced word this can only occur if W_i is a power of a_i or is 1. But by Corollary 5.9 W_i does not end in a_i or a_i^{-1}. Hence $W_i = 1$ so that there are no over or underpasses of $p(1)(L_i)$ with any of the circles $S^1{}_j$. This completes the proof.

We are now in a position to apply the technique of Section 4 to show that $[p]$ can be represented by a path q such that $q(1)$ is the identity on the disks $D^2{}_i$.

5.13 THEOREM. *Suppose that $[p]$ is a motion in $B(\mathsf{R}^3, S^1{}_1 \cup S^1{}_2 \cup \ldots \cup S^1{}_n)$ such that $d([p])$ is the identity. Then $[p]$ may be represented by a path*

$$q\colon\ (I, 0, 1) \to (D(\mathsf{R}^3),\ 1,\ D(\mathsf{R}^3, S^1{}_1 \cup S^1{}_2 \cup \ldots \cup S^1{}_n))$$

such that $q(1)(x) = x$ for each $x \in D^2{}_1 \cup D^2{}_2 \cup \ldots \cup D^2{}_n$.

PROOF. By the preceding corollary we may assume that none of the lines $p(1)(L_i)$ intersect any of the disks $D^2{}_j$. It is possible that the preceding lemmas destroyed the transversality condition for the disks $p(1)(D^2{}_i)$ and $D^2{}_j$. However, this can be rectified without introducing any new intersections of the lines $p(1)(L_i)$ with the disks $D^2{}_j$ by the usual transversality arguments. From here on the proof is identical to that of Theorem 4.12 with the following two observations. First, the disk

$p(1)(D^2_i)$ may very well intersect D^2_j with $j \neq i$, but this causes no difficulty at all. Second, in both Lemmas 4.10 and 4.11 we must make sure that when we extend the mapping of the 2-sphere which is obtained to a mapping of the 3-ball, that the 3-ball does not hit any of the circles S^1_i or any of the disks $p(1)(D^2_i)$ except for that part of $p(1)(D^2_j)$ which forms part of the boundary of the 3-ball. This follows from the fact that $p(1)(L_i) \cup p(1)(D^2_i)$ is a connected set part of which (∞) is outside of the 3-ball, and none of which (except the boundary of the portion of $p(1)(D^2_j)$ which forms part of the boundary of the 3-ball) intersects the boundary of the 3-ball.

These remarks conclude the proof of Theorem 5.13. We can now complete the proof that d is an isomorphism onto $T(F_n)$.

5.14 THEOREM. *Suppose that* $[q]$ *is a motion in* $B(\mathrm{R}^3, S^1_1 \cup S^1_2 \cup \ldots \cup S^1_n)$ *such that* $d([q])$ *is the identity. Then* $[q]$ *is the trivial motion. Hence,* d *is an isomorphism.*

PROOF. By Theorem 5.12 we may assume that $q(1)(x) = x$ for each $x \in D^2_1 \cup D^2_2 \cup \ldots \cup D^2_n$. Now consider the mapping

$$q' \colon (D^2_1 \cup D^2_2 \cup \ldots \cup D^2_n) \times I \to \mathrm{R}^3$$

defined by

$$q'(x,t) = q(t)(x) .$$

Since $(D^2_1 \cup D^2_2 \cup \ldots \cup D^2_n) \times I$ is a compact set there is a positive ε such that if $\|x-y\| \leqq \varepsilon$, then $\|q'(x,t) - q'(y,t)\| \leqq 1$. Recall that by Lemma 5.6 we can assume that $q'(x_i,t) = x_i$ for all t and i. By changing the parametrization of q we may also assume that q has the following two properties:

1) $q(t)(x) = x$ for all $x \in \mathrm{R}^3$ and all $t \leqq \frac{1}{4}$,
2) $q(t)(x) = q(1)(x)$ for all $x \in \mathrm{R}^3$ and all $t \geqq \frac{3}{4}$.

Let h denote a path in $D(\mathrm{R}^3)$ which shrinks each of the circles S^1_i down to radius ε keeping the basepoints x_i all fixed in time t running from 0 to $\frac{1}{4}$; then stays constant until time $t = \frac{3}{4}$; then expands the circles back to their original size between time $t = \frac{3}{4}$ and $t = 1$. So that, in particular, $h(t) = h(1-t)$. Then h clearly defines a trivial motion. Now look at the motion $[q] * [h]$ which is the same motion as $[q]$. For each value of t the image of each of the circles S^1_i is contained in the ball of radius 2 around the point x_i. Consider the path $q(t)h(t) | S^1_1 \cup S^1_2 \cup \ldots \cup S^1_n$ as a path in $E(S^1_1 \cup S^1_2 \cup \ldots \cup S^1_n, \mathrm{R}^3)$. By Palais' work this path can be lifted to a path $Q \colon I \to D(\mathrm{R}^3)$ with the following properties:

(i) $Q(t)(x) = q(t)h(t)(x)$ for each t and each $X \in S^1{}_1 \cup S^1{}_2 \cup \ldots \cup S^1{}_n$,
(ii) $Q(t)(x) = x$ for each t and each x such that x is not within $2\frac{1}{4}$ of one of the x_i's.

By the first property Q represents the same motion as q. By the second property we can look at Q as n disjoint motions of (R^3, S^1) isolated from each other, each of which is trivial by Theorem 4.15 and Remark 2.6.

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