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INITIAL VALUE PROBLEMS IN L_p FOR SYSTEMS WITH VARIABLE COEFFICIENTS

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1. Introduction.

In this note, which should be regarded as a sequel to [1], we will consider the Cauchy problem

$$(1)\qquad \begin{aligned} \partial u/\partial t &= P(x,D)u, \\ u(x,0) &= u_0(x)\ , \end{aligned} \qquad x \in \mathsf{R}^n,\ 0 \leqq t \leqq T\ ,$$

where $P(x,D)$ is an $N \times N$-matrix of pseudo-differential operators, and where u and u_0 are N-vector functions. Here the pseudo-differential operator $P(x,D)$ is defined by

$$(2)\qquad P(x,D)\,u(x) = \int \exp(-2\pi i\langle x,y\rangle)\, P(x,y)\, \hat{u}(y)\, dy\ .$$

We assume that for y fixed, $P(x,y) \in \mathscr{C}^\infty$ and denote the principal part of P by P_d, where $d>0$ is the exact order of P. For details, see section 2 below.

Let S^N denote the set of N-vectors with components in S, the space of rapidly decreasing C^∞-functions (again, see section 2). We say that (1) is well posed in L_p if $P(x,D)$ is the generator of a C_0 semi-group of solution operators $E(t)$ in L_p, that is

$$E(t+s) = E(t)E(s), \qquad t \geqq 0,\ s \geqq 0\ ,$$

and

$$(3)\qquad \|E(t)u_0\|_p \leqq C(T)\,\|u_0\|_p, \qquad 0 \leqq t \leqq T,\ u_0 \in S^N\ ,$$

and

$$(3)'\qquad \|h^{-1}(E(t+h)-E(t))u_0 - P(\cdot,D)E(t)u_0\|_p \to 0 \quad \text{as } h \to 0,\ u_0 \in S^N\ .$$

Let ${L_p}^N$ denote the set of N-vectors with components in L_p and let ${FL_p}^N$ denote the corresponding set of Fourier transforms. By ${M_p}^{N,N}$ we denote the set of multipliers on ${FL_p}^N$, and we write ${M_p}^{N,N}(\cdot)$ for the natural norm on ${M_p}^{N,N}$. For details, see section 2.

We can now formulate the main theorem of this note.

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THEOREM 1. *If* (1) *is well posed in* L_p, *then*

$$\sup_{x_0 \in \mathbf{R}^n} M_p^{N,N}\big(\exp(P_d(x_0,\cdot))\big) < +\infty . \tag{4}$$

For $p=2$, and for differential operators, this result is due to Strang [5]. The proof which we will give here is different from that of [5] also for $p=2$.

If we let Theorem 1 take the place of Lemma 5.2 in [1] in the proofs of Theorems 5.1 and 5.2 there, we get the following corollaires:

COROLLARY 1. *Let* $p \neq 2$. *Assume that the eigenvalues of* $P_d(x,y)$ *are real on* $\mathbf{R}^n \times \mathbf{R}^n$, *and that* $P(x,D)$ *is a differential operator. If* (1) *is well posed in* L_p, *then*

$$P_d(x,D) = \sum_{j=1}^{n} A_j(x)\, \partial/\partial x_j ,$$

where $A_1,\ldots,A_n$ *are commuting, diagonalizable matrices with real eigenvalues.*

COROLLARY 2. *Let* $p \neq 2$, $n>1$. *Assume that the eigenvalues of* $P_d(x,y)$ *are real on* $\mathbf{R}^n \times \mathbf{R}^n$, *and that for fixed* $x \in \mathbf{R}^n$, $P_d(x,y) \in \mathscr{C}^{N+\nu}(\mathbf{R}^n \setminus \{0\})$, *for some* $\nu \geqq 1$. *If* (1) *is well posed in* L_p, *then*

$$P_d(x,D) = \sum_{k=1}^{n} \sum_{j=1}^{n} a_{jk}(x) E_k(x,D)\, \partial/\partial x_j$$

where $a_{jk}(x)$ *are real functions and where* $E_k(x,y)$ *are idempotent* $N \times N$-*matrices with sum* E, *which are homogeneous of degree* 0 *in* y, *and which belong, for fixed* x, *to* $M_p^{N,N} \cap \mathscr{C}^{\nu+1}(\mathbf{R}^n \setminus \{0\})$ *in* y.

We give some of the basic definitions in section 2, mostly referring to [1], [2], [3], and [4]. The proof of Theorem 1 is given in section 3. In section 4 some extensions are considered.

Finally we use this opportunity to refer the reader to the following paper [6] in which corrections to two earlier papers of ours on related subjects are given. The article [6] is placed immediately after the present paper.

2. Some definitions.

In this section we will review some basic definitions and notations concerning multipliers and pseudo-differential operators. For details and detailed references, see [1], [2].

For complex N-vectors, $\langle u,v \rangle$ shall denote the scalar product and $|v|$ the Euclidean norm. The norm of an $N \times N$ matrix A will be the operator norm $|A| = \sup\{|Av|;\ |v| \leqq 1\}$.

By $\mathscr{C}^\nu(B)$ we will denote the set of N-vectors, and occasionally $N \times N$-matrices, with elements in $\mathscr{C}^\nu(B)$. If $g \in C^\infty(\mathsf{R}^n) = C^\infty$, and if

$$\sup\{|x|^m |D^k g(x)|;\ x \in \mathsf{R}^n\} < +\infty$$

for $m = 0, 1, \ldots$ and for any multiindex $k = (k_1, \ldots, k_n)$, $|k| = k_1 + \ldots + k_n$, we say that $g \in S$. Here

$$D^k = (-2\pi i)^{-|k|}(\partial/\partial x_1)^{k_1} \ldots (\partial/\partial x_n)^{k_n} .$$

We give the linear space S the topology defined by the above family of semi-norms. The set of N-vector functions with components in S is denoted S^N. The dual space S' of S is the space of tempered distributions. The convolution $\mu * g$ between an $N \times N$-matrix μ with elements in S' and a $g \in S^N$ is defined in the obvious way. The Fourier transform $\hat{\mu}$ of a tempered distribution μ is defined by $\hat{\mu}(f) = \mu(\hat{f})$, $f \in S$, where

$$\hat{f}(y) = \int_{\mathsf{R}^n} \exp(2\pi i\langle x, y\rangle) f(x)\, dx .$$

The Fourier transform is defined for matrices and vector valued tempered distributions by applying the transform elementwise. If $K \subseteqq S'$, FK denotes the corresponding set of Fourier transforms.

By L_p^N we mean the set of functions $v = (v_1, \ldots, v_N)$ with $v_j \in L_p$, $j = 1, \ldots, N$. For $p < +\infty$ we let

$$\|v\|_p = \left(\int_{\mathsf{R}^n} |v(x)|^p\, dx\right)^{1/p},$$

and for $p = \infty$,

$$\|v\|_\infty = \operatorname{ess\,sup}\{|v(x)|;\ x \in \mathsf{R}^n\} .$$

We shall assume that $1 \leqq p \leqq \infty$.

We say that an $N \times N$-matrix μ with elements in S' is a multiplier on FL_p^N, $\mu \in M_p^{N,N}$, if

$$M_p^{N,N}(\mu) = \sup\{\|\hat{\mu} * f\|_p;\ f \in S^N, \|f\|_p \leqq 1\} < +\infty .$$

We use the convention that $M_p^{N,N}(\mu) = \infty$ if $\mu \notin M_p^{N,N}$. One can prove that (cf. [1], [2]) $M_p^{N,N} = M_{p'}^{N,N}$ for $1/p + 1/p' = 1$; that $M_1^{N,N} \subseteqq M_p^{N,N} \subseteqq M_2^{N,N}$ and that $M_2^{N,N}$ is the set of $N \times N$-matrices with elements that are L_∞-functions. Further, $M_1^{1,1}$ can be identified with the set of Fourier–Stieltjes transforms of bounded measures on R^n. In general $M_p^{N,N}$ is a Banach algebra of matrix-valued functions with norm $M_p^{N,N}(\cdot)$. We notice that $FL_1 \subseteqq M_1^{1,1}$ and that the w^*-closure of the unit ball in L_p is the unit ball in L_p for $1 < p \leqq \infty$, and is the unit ball in $FM_1^{1,1}$ for $p=1$. Let us denote the union of w^*-closures of the compact balls in L_p by W_p and the set of corresponding N-vectors by W_p^N. For later reference we state the following well-known result.

LEMMA 1. *The unit ball in W_p^N is w^*-compact.*

By the above it will cause no confusion if we use the same notation for the norms in L_p^N and in W_p^N.

Finally we will give a short discussion of pseudo-differential operators, mainly following [3], [4]. The pseudo-differential operator $P(x,D)$ is defined by

$$P(x,D)u(x) = \int \exp(-2\pi i\langle x,y\rangle)\, P(x,y)\, \hat{u}(y)\, dy, \qquad u \in S^N,$$

where $P(x,y)$ is an $N\times N$-matrix, the symbol of $P=P(x,D)$. We assume that for some $d>0$ and some sequence $\{P_{d-j}\}_{j=0}^{\infty}$ of $N\times N$-matrix functions which are homogeneous of degree $d-j$, respectively, in y, the following relations hold for any integer K:

$$D_x^{\alpha} D_y^{\beta}\Big(P(x,y) - \sum_{j=0}^{K} P_{d-j}(x,y)\Big) = O(|y|^{d-|\beta|-K}) \tag{5}$$

for $|\alpha| \leqq \lambda$, $|\beta| \leqq \nu$, uniformly for x in compact subsets of R^n as $|y| \to \infty$. Here we have assumed that $P(x,y) \in \mathscr{C}^{\lambda}$ in x (y fixed), and belongs to $\mathscr{C}^{\nu}$ for x fixed, and correspondingly for P_{d-j}. We will below assume $\lambda=\infty$, but will specify ν if we assume more than $\nu \geqq 0$. From (5) it follows that $P_d(x,y)$, the principal part of $P(x,y)$, is given by

$$P_d(x,y) = \lim_{\lambda\to\infty} \lambda^{-d} P(x,\lambda y)$$

uniformly for (x,y) in compact subsets of $\mathsf{R}^n \times (\mathsf{R}^n \setminus \{0\})$. We say that P has *exact* order d if $P_d(x,y)$ does not vanish identically. We will below assume that $P(x,D)$ has exact order $d>0$.

3. Proof of Theorem 1.

Let $s>0$, $x_0 \in \mathsf{R}^n$. For $u \in L_p^N$ we define $U_s u$ by

$$U_s u(x) = s^{-(n/p)} u(s^{-1}(x-x_0)). \tag{6}$$

Then

$$\|U_s u\|_p = \|u\|_p, \tag{7}$$

and U_s has an inverse U_s^{-1} such that

$$U_s^{-1}u(x) = s^{n/p} u(sx+x_0) \tag{8}$$

$$\|U_s^{-1}u\|_p = \|u\|_p. \tag{9}$$

Let $s>0$ and $t=\tau s^d$, where d is the order of P. We assume that (3) holds. By (7), (9) and (3) we have

$$\|U_s^{-1}E(\tau s^d)U_s u\|_p \leqq C(T)\|u\|_p . \tag{10}$$

By Lemma 1 there exists a $\varphi_\tau(u) \in W_p^N$ and a sequence $s_j \to 0$, such that with $t_j=\tau s_j^d$,

$$U_{s_j}^{-1}E(t_j)U_{s_j}u \xrightarrow{w^*} \varphi_\tau(u), \quad s_j \to 0 . \tag{11}$$

By (10) this implies, again by Lemma 1, that

$$\|\varphi_\tau(u)\|_p \leqq C(T)\|u\|_p . \tag{12}$$

Let $\tilde{P}_d(y)=P_d(x_0,y)$. We want to prove that $(\varphi_\tau(u))\hat{}=\exp(\tau\tilde{P}_d)\hat{u}$. From (12) it follows, approximating $\hat{u}$ by functions with compact support, that $\varphi_\tau(u) \in L_p^N$ also for $p=1$.

We first make a simple computation

Lemma 2. *Let $P_d(s;D)=s^dU_s^{-1}PU_s$. Then for $u \in S^N$,*

$$P_d(s;D)u(x) = \int \exp(-2\pi i\langle x,y\rangle)\, s^dP(sx+x_0,s^{-1}y)\,\hat{u}(y)\,dy . \tag{13}$$

Proof. We have

$$(U_su)\hat{}(y) = s^{n(1-1/p)}\exp(2\pi i\langle x_0,y\rangle)\,\hat{u}(s,y) ,$$

and so

$$\begin{aligned} PU_su(x) &= s^{n(1-1/p)}\int \exp(-2\pi i\langle x-x_0,y\rangle)\,P(x,y)\,\hat{u}(sy)\,dy \\ &= s^{-n/p}\int \exp(-2\pi i\langle s^{-1}(x-x_0),y\rangle)\,P(x,s^{-1}y)\,\hat{u}(y)\,dy . \end{aligned}$$

This proves (13).

For a shorter notation, let $E_s(\tau)=U_s^{-1}E(t)U_s$. Then notice that $P_d(s;D)$ and $E_s(\tau)$ commute since P and $E(t)$ do. Hence

$$P_d(s;D)E_s(\tau)u=E_s(\tau)P_d(s;D)u$$

is well defined, and is in L_p^N for $u \in S^N$ by (3) and (3)′.

We want to prove that for any $g \in C_0^\infty$,

$$\int g(x)P_d(s;D)E_s(\tau)u(x)\,dx \to \int \hat{g}(y)\tilde{P}_d(y)(\varphi_\tau(u))\hat{}\,(y)\,dy, \quad s=s_j \to 0 . \tag{14}$$

We will need the following lemma (cf. the proof of Theorem 3.6 in Hörmander [3]).

Lemma 3. *Let $g \in C_0^\infty$, and let*

$$g^*P_d(s;D)(y) = \int \exp(2\pi i\langle x,y\rangle)\,g(x)s^dP(x_0+sx,s^{-1}y)\,dx .$$

Assume that for fixed y, $P(x,y) \in \mathscr{C}^{\infty}$. *Then for any integer R,*

$$|g_* P_d(s;D)(y)| \leqq C(1+|y|)^{-R}, \tag{15}$$

uniformly in $s \to 0$. *Further,* $g_* P_d(s;D)(y) \to \hat{g}(y)\tilde{P}_d(y)$ *in* L_p.

PROOF. Since $g_* P_d(s;D) \to \hat{g}\tilde{P}_d$ uniformly on compact sets as $s \to 0$, it is sufficient to prove (15), and then use dominated convergence to complete the proof of the lemma. Let $|\alpha| = R+d$, α a multi-index. Then

$$y^{\alpha} \int \exp(2\pi i \langle x,y \rangle)\, g(x) s^d P(x_0+sx, s^{-1}y)\, dx$$
$$= \int \exp(2\pi i \langle x,y \rangle)\, D_x^{\alpha}(g(x) s^d P(x_0+sx, s^{-1}y))\, dy .$$

By Leibnitz' formula the right hand side is $O(|y|^d)$, uniformly for $s \to 0$, since $g \in C_0^{\infty}$ by assumption. This proves (15).

We assume for the moment that $1 \leqq p \leqq 2$. Let us then complete the proof of (14). The case $p \geqq 2$ will be proved afterwards by duality.

Writing out the definition of $P_d(s;D)$ and changing the order of integration we get

$$\int g(x) P_d(s;D) E_s(\tau) u(x)\, dx = \int g_* P_d(s;D)(y) (E_s(\tau)u)\hat{}\,(y)\, dy . \tag{16}$$

By Lemma 3, $g_* P_d(s;D) \to \hat{g}\tilde{P}_d$ in L_p, and by the Hausdorff–Young inequality $(E_s(\tau)u)\hat{} \to (\varphi_\tau(u))\hat{}$ weakly in L_q^N since $1 \leqq p \leqq 2$. Hence the right hand side of (16) converges to

$$\int g(y) P_d(y) (\varphi_\tau(u))\hat{}\,(y)\, dy ,$$

which is the right hand side of (14). This completes the proof of (14).

Since (1) holds we also have

$$\frac{\partial}{\partial \tau}(U_s^{-1} E(\tau s^d) U_s u) = s^d U_s^{-1} \left(\frac{\partial}{\partial t} E(t)\right)_{t=\tau s^d} U_s u$$
$$= (s^d U_s^{-1} P(x,D) U_s)(U_s^{-1} E(\tau s^d) U_s u) = P_d(s;D) E_s(\tau) u .$$

This, together with (14), proves that with convergence in D' (i.e. in the distribution sense),

$$\frac{\partial}{\partial \tau}(\varphi_\tau(u))\hat{}\,(y) = \tilde{P}_d(y)(\varphi_\tau(u))\hat{}\,(y) . \tag{17}$$

But from (17) we get that $(\varphi_\tau(u))\hat{} = \exp(\tau \tilde{P}_d)\hat{u}$, since $\varphi_0(u) = E(0)u = u$. By (12) we finally have

$$M_p^{N,N}(\exp(\tilde{P}_d)) \leqq C_0 = \inf\{C(T);\, T>0\}\,, \tag{18}$$

and so Theorem 1 is proved for $1 \leqq p \leqq 2$.

For $p \geqq 2$ we notice that if we could define P^* so that

$$\int \langle P^*u,w\rangle\, dx = \int \langle u,Pw\rangle\, dx, \qquad u,w \in \mathscr{C}_0^\infty\,,$$

then the dual problem corresponding to (1) for P^* and $E(t)^*$, the adjoint of $E(t)$, would by (3) and (3)′ be well posed in L_q, $1/p+1/q=1$. If further P^* has a symbol with principal part $P_d(x,y)^*$, the proof above implies that

$$M_p^{N,N}(\exp(\tilde{P}_d^*)) \leqq C_0\,.$$

This is equivalent to (18), and so Theorem 1 would follow also for $2 \leqq p \leqq \infty$.

It remains to verify the above assertions about P^*. Using Parseval's formula it is easy to see that P^* exists and is uniquely determined by the symbol.

$$Q_v(x,y) = \mathscr{F}_\eta\big(\textstyle\int P(\xi,\eta+y)^* v(\xi)\, e^{-2\pi i\langle \xi,\eta\rangle}\, d\xi\big)(x)$$

where $\mathscr{F}_\eta$ denotes the Fourier transform with respect to η and where $C_0^\infty \ni v=1$ in a neighborhood of x (for the computations, see [3], [4]). Rewriting this as

$$Q_v(x,y) = P(x,y)^* + R_v(x,y)$$

the Fourier inversion formula gives

$$R_v(x,y) = \mathscr{F}_\eta\big(\textstyle\int (P(\xi,\eta+y)^* - P(\xi,y)^*)\, v(\xi)\, e^{-2\pi i\langle \xi,\eta\rangle}\, d\xi\big)(x)\,.$$

We can then, as above, use Lemma 2 to prove that $s^d U_s^{-1} R_v U_s(x,y) = o(1)$ as $s \to 0$. Thus R_v contains only lower order terms, and the proof of Theorem 1 is complete.

REMARK. If $P(x,D)$ were a differential operator, then $g_* P_d(s;D) \to \hat{g}\tilde{P}_d$ in S. Since $(E_s(\tau)u)\hat{} \to (\varphi_\tau(u))\hat{}$ in S' we have (14) at once, and so (17) in the distribution sense. The non-regularity of P and P_d force us to use the slightly more complicated argument above.

4. A generalization.

Let $\alpha \geqq 0$ and $\omega_\alpha^*(y) = |y|^\alpha$, and define for $u \in S^N$ the semi-norm

$$\|u\|_{p,\alpha}^* = \|F^{-1}(\omega_\alpha^* \hat{u})\|_p\,,$$

where F^{-1} denotes the inverse Fourier transform. We say that (1) is strictly well posed in $L_{p,\alpha}$ if instead of (3) the following inequality holds: (Notice that the degree of $P(x,D)$ is $d>0$.),

$$(19) \qquad \|E(t)u_0\|_p \leqq C(T)(\|u_0\|_p + t^{\alpha/d}\|u_0\|^*_{p,\alpha}), \quad 0 \leqq t \leqq T,\ u_0 \in S^N .$$

One can show that if $P(x,D)$ is a homogeneous partial differential operator with constant coefficients then (1) is strictly well posed in $L_{p,\alpha}$ if and only if

$$\|E(t)u_0\|_p \leqq C(T)(\|u_0\|_p + \|u_0\|^*_{p,\alpha}), \quad 0 \leqq t \leqq T,\ u_0 \in S^N ,$$

that is, if and only if (1) is well posed in $L_{p,\alpha}$ (cf. [1]).

Using (19) instead of (3), it is not hard to prove the following result, modifying the proof above slightly.

THEOREM 2. *Let* $\omega_\alpha(y)=(1+|y|)^\alpha$. *If* (1) *is strictly well posed in* $L_{p,\alpha}$ (*i.e. if* (19) *holds*), *then*

$$(20) \qquad \sup_{x_0\in \mathbf{R}^n} M_p{}^{N,N}\big(\omega_\alpha{}^{-1}\exp\big(P_d(x_0,\cdot)\big)\big) < +\infty .$$

Define the rank $r(x)$ of $P_d(x,y)$ as the largest integer $r(x)$ such that there exist some imaginary eigenvalue $\alpha(x,y)$ of $P_d(x,y)$ and some ball $B \subseteqq \mathbf{R}^n$ on which $\alpha(x\cdot) \in C^2(B)$, and such that the rank of the hessian

$$\left(\frac{\partial^2\alpha(x,y)}{\partial y_k \partial y_l}\right)_{k,l}$$

is at least $r(x)$ for $y \in B$. From Theorem 2 above and Theorem 5.4 in [1], we then get the following result.

THEOREM 3. *Assume that* $P(x,y) \in \mathscr{C}^\infty(\mathbf{R}^n \times \mathbf{R}^n)$. *Let* $r(x)$ *be the rank of* $P_d(x,y)$ *and let* $r=\sup r(x)$. *Then the Cauchy problem* (1) *is not strictly well posed in* $L_{p,\alpha}$ *for* $0 \leqq \alpha < rd|\frac{1}{2}-p^{-1}|$.

We end this note by giving an example of an application of Theorem 3. Let $(a_{kl}(x))$ be a real symmetric $\mathscr{C}^\infty$-matrix, and let $b_j(x)$ and $c(x)$ be C^∞-functions. Assume also that

$$\sum_{k,l=1}^{n} a_{kl}(x)\, y_k y_l \geqq 0, \quad y_k, y_l \in \mathbf{R},\ x \in \mathbf{R}^n .$$

Then consider the Cauchy problem for the hyperbolic system

$$(21) \qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \sum a_{kl}(x)\dfrac{\partial^2 u}{\partial x_k \partial x_l} + \sum_{j=1}^{n} b_j(x)\dfrac{\partial u}{\partial x_j} + c(x)u \\ u(x,0) = u_{01}(x) \\ \dfrac{\partial}{\partial t}u(x,0) = u_{02}(x) . \end{cases}$$

Let $|y|_\alpha = (\sum_{k,l=1}^n a_{kl}(x) y_k y_l)^{\frac{1}{2}}$. As in section 5 in [1], it is easy to see that we can transform (21) to a Cauchy problem for a system of pseudo-differential operators, where the eigenvalues of the principal part P_1 are $\pm 2\pi i |y|_\alpha$. A simple computation shows then that the rank of P_1 is $r(x)-1$, where $r(x) = \operatorname{rank}(a_{kl}(x))$. Hence we have the following corollary of Theorem 3.

COROLLARY 3. With the above notations and assumptions, let $r = \sup_x(r(x))$. Then the Cauchy problem (21) is not strictly well posed in $L_{p,\alpha}$ for $0 \leqq \alpha < (r-1)|\frac{1}{2} - p^{-1}|$.

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