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A MAXIMAL ALGEBRA

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1. Introduction.

1.1. Let (R, S) be a flow. By this we mean that S is a locally compact Hausdorff space on which the real line R acts as a topological transformation group. We will denote by T the function from $\mathsf{R} \times S$ to S that defines the action of R on S. Then by definition T is continuous, $T(0,x)=x$ for all x in S, and $T(s+t,x)=T(s,T(t,x))$ for all (s,t,x) in $\mathsf{R} \times \mathsf{R} \times S$. We recall that if f is a function on $\mathsf{R} \times S$, $t \in \mathsf{R}$, and $x \in S$, then f_t and f^x are the functions on S and R respectively defined by $f_t(y)=f(t,y)$ for all y in S and $f^x(s)=f(s,x)$ for all s in R. (Thus T_t is a homeomorphism of S for every t in R.) If X is a locally compact Hausdorff space, then we will denote by $C_0(X)$ the uniform algebra of all continuous complex functions on X that vanish at infinity. (Thus if $f \in C_0(S)$ and $x \in S$, then $f \circ T^x$ is a uniformly continuous bounded complex function on R.) The class of all functions f in $C_0(S)$ such that $f \circ T^x \in H^\infty(\mathsf{R})$ for every x in S is a uniformly closed subalgebra of the algebra $C_0(S)$ which we will denote by A (see Section 2.1 for the definition of $H^\infty(\mathsf{R})$). If S is the unit circle in the complex plane and T_t is the rotation of S through an angle of t radians for every t in R, then A is the familiar disc algebra. Wermer showed that the disc algebra is a maximal closed subalgebra of the algebra of all continuous functions on the circle [8, Theorem 1]. The purpose of this paper is to state a generalization and to give its proof. If $x \in S$, then by the orbit of x we mean $T^x(\mathsf{R})$, which is a subset of S. The flow (R, S) is called minimal if for every x in S the orbit of x is dense in S. The rotation flow on the circle is of course minimal since there is just one orbit. There is the following generalization of the Wermer maximality theorem.

1.2. THEOREM. *If the flow* (R, S) *is minimal, then* A *is a maximal closed subalgebra of* $C_0(S)$.

The proof of Theorem 1.2 is in Section 2.7. Sections 2.1–2.6 are preparatory. With regard to the proof we refer to [5].

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1.3. We remark that Theorem 1.2 is a theorem of Hoffman and Singer if (R, S) is the flow that is associated with a dense subgroup of R [6, Theorem 4.7].

1.4. A subalgebra B of $C_0(X)$ is called pervasive if for every proper closed subset E of X the restriction of B to E is uniformly dense in $C_0(E)$ [7].

1.5. THEOREM. *If the flow* (R, S) *is minimal, then* A *is a pervasive subalgebra of* $C_0(S)$.

The proof of Theorem 1.5 is in Section 3. We remark that Theorem 1.5 is a theorem of Hoffman and Singer if (R, S) is the flow that is associated with a dense subgroup of R [7]. Furthermore, with regard to Theorem 1.5 we refer to [3, Theorem 4.2, 1°]. For some examples of minimal flows we refer to [1].

2. The proof of Theorem 1.2.

2.1. We recall that $H^\infty(\mathsf{R})$ (which serves to define A) is the class of all functions F in $L^\infty(\mathsf{R})$ such that

$$\int \operatorname{Im}(1/(t-z))F(t)\,dt \tag{2.1}$$

is holomorphic on

$$\{z:\ \operatorname{Im}(z) > 0\}\,. \tag{2.2}$$

An equivalent definition of $H^\infty(\mathsf{R})$ is that it is the class of all functions F in $L^\infty(\mathsf{R})$ such that the spectrum of F is contained in $[0, \infty)$. Furthermore we recall that $H^1(\mathsf{R})$ is the class of all functions F in $L^1(\mathsf{R})$ such that (2.1) is holomorphic on (2.2). An equivalent definition of $H^1(\mathsf{R})$ is that it is the class of all functions F in $L^1(\mathsf{R})$ such that $\hat{F} = 0$ on $(-\infty, 0)$ where $\hat{F}$ is the Fourier transform of F,

$$\hat{F}(s) = \int e^{-ist}\,F(t)\,dt\,.$$

The following lemma expresses the well-known relationship between $H^\infty(\mathsf{R})$ and $H^1(\mathsf{R})$.

2.2 LEMMA. *If* $F \in L^\infty(\mathsf{R})$, *then* $F \in H^\infty(\mathsf{R})$ *if and only if*

$$\int F(t)G(t)\,dt = 0$$

for every G *in* $H^1(\mathsf{R})$.

Lemma 2.2 will be used at the end of Section 2.7.

2.3. The following lemma is a particular case of [2, Lemma 2, 2].

2.4. LEMMA. *If* $f \in C_0(S)$, $F \in H^1(\mathsf{R})$, *and*

$$g = \int f \circ T_t F(-t)\, dt ,$$

then $g \in A$.

The proof is easy and is in [2]. We remark that $g \in C_0(S)$ because $f \in C_0(S)$ and T is continuous.

2.5. With regard to measure theory we will follow Halmos [4]. If X is a locally compact Hausdorff space, then we will denote by $M(X)$ the space of all complex Baire measures on X. We remark that every complex measure is bounded. If β is a complex measure, then we will denote by $|\beta|$ the total variation measure of β. (Thus if $\beta \in M(X)$, then $|\beta| \in M(X)$.) A measure β in $M(S)$ is called quasi-invariant if whenever a Baire set E is of $|\beta|$ measure 0, then for every t in R the Baire set $T_t(E)$ is of $|\beta|$ measure 0. We define transformations U and V of $\mathsf{R} \times S$ by $U(t,x)=(t,T(t,x))$ and $V(t,x)=(t,T(-t,x))$ for all (t,x) in $\mathsf{R} \times S$. We have $U \circ V = V \circ U =$ the identity transformation of $\mathsf{R} \times S$, and thus U and V are homeomorphisms of $\mathsf{R} \times S$. We will need the following lemma.

2.6. LEMMA. *Let* λ *and* μ *be nonnegative measures in* $M(\mathsf{R})$ *and* $M(S)$ *respectively and consider the product measure* $\lambda \times \mu$, *which is in* $M(\mathsf{R} \times S)$. *If* μ *is quasi-invariant, then there is a finite nonnegative Baire measurable function* φ *on* $\mathsf{R} \times S$ *such that*

$$\int F\, d(\lambda \times \mu) = \int F \circ U \varphi\, d(\lambda \times \mu) \tag{2.3}$$

for all F *in* $L^1(\lambda \times \mu)$.

PROOF. We define a nonnegative measure γ in $M(\mathsf{R} \times S)$ by

$$\gamma(E) = (\lambda \times \mu)(U(E)) \tag{2.4}$$

for every Baire subset E of $\mathsf{R} \times S$.

We assert that γ is absolutely continuous with respect to the product measure $\lambda \times \mu$. We recall that if E is a subset of $\mathsf{R} \times S$ and $t \in \mathsf{R}$, then E_t is the subset of S consisting of all x in S such that $(t,x) \in E$. For the purpose of proving the assertion let E be any Baire subset of $\mathsf{R} \times S$ such that

$$(\lambda \times \mu)(E) = 0 . \tag{2.5}$$

Then (the Fubini theorem)

$$\mu(E_t) = 0 \tag{2.6}$$

for λ almost all t. If t in R is such that (2.6) holds, then because μ is quasi-invariant we have $\mu(T_t(E_t))=0$. Consequently since $(U(E))_t = T_t(E_t)$ we have $\mu\big((U(E))_t\big)=0$ for λ almost all t, and therefore by (2.4)

$$\gamma(E) = 0 . \tag{2.7}$$

Thus we see that (2.5) implies (2.7), i.e. $\gamma \ll \lambda \times \mu$.

It now follows (the Radon–Nikodym theorem) that there is a finite nonnegative Baire measurable function φ on $R \times S$ such that

$$\int G\, d\gamma = \int G\varphi\, d(\lambda \times \mu) \tag{2.8}$$

for all G in $L^1(\gamma)$. Since $V=U^{-1}$ we have by (2.4) $\gamma\big(V(E)\big)=(\lambda\times\mu)(E)$ for every Baire subset E of $\mathsf{R}\times S$. It follows from this that

$$\int F\circ U\, d\gamma = \int F\, d(\lambda\times\mu) \tag{2.9}$$

for every nonnegative Baire measurable function F on $\mathsf{R}\times S$, and hence for all F in $L^1(\lambda\times\mu)$. The desired (2.3) follows from (2.9) and (2.8) (with $G=F\circ U$).

2.7. We will now prove Theorem 1.2. Let B be any subalgebra of $C_0(S)$ that contains A. It is to be shown that either $B=A$ or B is uniformly dense in $C_0(S)$. Suppose then that B is not uniformly dense in $C_0(S)$. Then (following Wermer [8]) there is a nonzero measure β in $M(S)$ that annihilates B: $\int f d\beta \neq 0$ for some f in $C_0(S)$ and $\int g\, d\beta=0$ for every g in B. By [2, Theorem 3] the measure β is quasi-invariant, and thus $|\beta|$ satisfies the hypothesis of Lemma 2.6. Fix a nonzero function G in $H^1(\mathsf{R})$, let $d\lambda=|G(t)|\,dt$, let $\mu=|\beta|$, and let φ be a finite nonnegative Baire measurable function on $\mathsf{R}\times S$ such that (2.3) holds. Furthermore let X and χ be bounded complex Baire measurable functions on R and S respectively such that $G=X|G|$ and $\beta=\chi\mu$. We will denote by Z_+ the class of all nonnegative integers, and by Q_+ the class of all nonnegative rational numbers.

Let $g\in B$. We will use (2.3) to prove that $g\in A$. In (2.3) let

$$F(t,x) = e^{irt}X(t)\chi(x)g(x)^k f\big(T(-t,x)\big)$$

where $r\in\mathsf{Q}_+$, $k\in\mathsf{Z}_+$, and $f\in C_0(S)$. Then the right side of (2.3) is equal to

$$\int \Big(\int e^{irt}G(t)\,\chi\big(T(t,x)\big)\,\varphi(t,x)\,g\big(T(t,x)\big)^k\,dt\Big)\,f(x)\,d\mu(x) \tag{2.10}$$

and the left side of (2.3) is equal to

$$\int \Big(\int f\circ T_{-t}\,e^{irt}G(t)\,dt\Big)\,g^k\,d\beta . \tag{2.11}$$

By Lemma 2.4 the inner integral in the expression (2.11) is a member of A, and therefore because of the conditions on B and β the expression (2.11) vanishes. Consequently the expression (2.10) vanishes for all f in $C_0(S)$, and therefore the inner integral in the expression (2.10) vanishes for μ almost all x. Although the Baire set of μ measure 0 where the inner integral in the expression (2.10) does not vanish depends on r in Q_+ and k in Z_+, since Q_+ and Z_+ are countable there is a single Baire set N of μ measure 0 such that if $x \in N'$, then

$$\int e^{irt} G(t)\,\chi(T(t,x))\,\varphi(t,x)\,g(T(t,x))^k\,dt = 0 \tag{2.12}$$

for all r in Q_+ and all k in Z_+. We remark that if $x \in N'$, then

$$\int |G(t)\,\chi(T(t,x))|\,\varphi(t,x)\,dt < \infty\,.$$

By (2.3) (with $F(t,x) = |\chi(x)|$) we have

$$\int d(\lambda \times \mu) = \int |\chi \circ T| \varphi\, d(\lambda \times \mu)\,,$$

and therefore there is an x in N' such that

$$\int |G(t)\,\chi(T(t,x))|\,\varphi(t,x)\,dt > 0\,. \tag{2.13}$$

Fix such an x. Then for this x (2.12) and (2.13) state the following:

There is a nonzero function F in $L^1(\mathsf{R})$ such that

$$F(g \circ T^x)^k \in H^1(\mathsf{R}) \text{ for every } k \text{ in } \mathsf{Z}_+. \tag{2.14}$$

It is well-known nowadays that (2.14) implies that

$$g \circ T^x \in H^\infty(\mathsf{R})\,. \tag{2.15}$$

This fact of the theory of $H^\infty(\mathsf{R})$ can be obtained from the theory of simply invariant subspaces. Wermer [8] stated it in terms of the disc algebra and the Hardy class H^1 on the circle, and proved it by means of the theory of functions, thereby completing the proof of his maximality theorem.

From (2.15) it follows that if $y \in T^x(\mathsf{R})$ (the orbit of x), then $g \circ T^y \in H^\infty(\mathsf{R})$, for $H^\infty(\mathsf{R})$ is a translation invariant space and

$$g(T(t, T(s,x))) = g(T(t+s,x))\,.$$

We will complete the proof of Theorem 1.2 by showing that

$$g \circ T^y \in H^\infty(\mathsf{R}) \quad \text{for every } y \text{ in } S\,.$$

Let $G \in H^1(\mathsf{R})$ and consider the function in $C_0(S)$ defined by the integral

$$\int g \circ T_t\, G(t)\,dt\,.$$

By Lemma 2.2 this function vanishes on the orbit of x, and therefore because it is continuous it vanishes on the closure of the orbit of x. This is true for every G in $H^1(\mathsf{R})$, and therefore by Lemma 2.2 we have $g \circ T^y \in H^\infty(\mathsf{R})$ for all y in the closure of the orbit of x. Since the flow (R, S) is minimal the closure of this orbit is S.

3. The proof of Theorem 1.5.

3.1. We will denote by Q the class of all rational numbers.

3.2. LEMMA. *If the flow* (R, S) *is minimal, then*

$$S = \bigcup_{t \in \mathsf{Q}} T_t(G) \tag{3.1}$$

for every nonempty open set G.

PROOF. We will denote by H the right side of (3.1). If $t \in \mathsf{Q}$, then $T_t(H) = H$. Therefore because H is open we have $T_t(H) = H$ for all t in R, and hence $T_t(H') = H'$ for all t in R. It now follows since (R, S) is minimal that the closed set H' is empty.

3.3. We will now prove Theorem 1.5. Let X be a proper closed subset of S. If f is a function on S, then we will denote by $f^\#$ the restriction of f to X. Let $A^\#$ be the subalgebra of $C_0(X)$ consisting of all functions on X of the form $f^\#$ where f is any member of A. It is to be shown that $A^\#$ is uniformly dense in $C_0(X)$. For this purpose let α be any measure in $M(X)$ that annihilates $A^\#$. We will show that $\alpha = 0$. The desired density of course follows from this. The measure α is defined on the class of all Baire subsets of X. It is easily seen that this class coincides with the class of all sets of the form $E \cap X$ where E is any Baire subset of S. We define a measure β in $M(S)$ by

$$\beta(E) = \alpha(E \cap X) \tag{3.2}$$

for every Baire subset E of S. It follows that

$$\int f \, d\beta = \int f^\# \, d\alpha$$

for every bounded complex Baire measurable function f on S. Consequently β annihilates A, and therefore by [2, Theorem 3] the measure β is quasi-invariant. Let $\mu = |\beta|$ and let G be a nonempty open Baire subset of S that is disjoint from X. If E is any Baire subset of S that is disjoint from X, then by (3.2) we have $\beta(E) = 0$. Consequently $\mu(G) = 0$, and therefore by (3.1) and the fact that μ is quasi-invariant we have $\mu(S) = 0$. Hence by (3.2) we have $\alpha(E \cap X) = 0$ for every Baire subset E of S. Thus $\alpha = 0$.

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