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THE σ-COMPACT-OPEN TOPOLOGY AND ITS RELATIVES

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1. Introduction.

The present paper arose from an attempt to define a natural topology which would serve as the Mackey topology for the strict topology on the space $C_b(X)$ of all continuous, bounded (complex- or real-valued) functions on X. That such a topology might just conceivably exist issues on the one hand from Conway's delicate argument showing that if X is simultaneously locally compact and paracompact, then the Mackey topology is the strict topology, and on the other hand, from his example which showed that in general the strict topology is different from the Mackey topology [3].

As a locally convex topology on $C_b(X)$ which we hoped at first might be so slightly larger than the strict topology as to fill our bill we defined the "σ-compact-open topology," where convergence is simply uniform convergence on all σ-compact subsets of X. To our surprise, we found that the σ-compact-open topology is in fact the strict topology precisely when the duals are equal — whether or not it is the Mackey topology! Nevertheless, the σ-compact-open topology and the strict topology have interesting properties and inter-relationships. Along with the "bounded-compact convergence" structure — a close relative of these two topologies — these concepts are what this paper is about.

Section 2 focuses on the strict topology on $C_b(X)$, for arbitrary completely regular X. In Buck's ground-breaking papers [1], [2] he assumed that X was locally compact. Our methods differ drastically from his, since if X is no longer locally compact, there may well be a paucity of compactly supported continuous functions on X, a fact which many of his arguments could not tolerate.

Regardless, we are able to prove analogues to his results. Included among them are two we mention. First, $C_b(X)$ with the strict topology is metrizable precisely when X is compact, which is precisely when the topology is normed. The second result tells us that the dual is always

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$M(X)$, the collection of all countably additive, regular, bounded measures on X.

In Section 3 we define the bounded-compact convergence, which turns out to be stronger than convergence in the strict topology. Although we are able to talk about the linear functionals continuous with respect to this convergence, we also show that the convergence yields a topology only when X is compact.

The inauguration of the σ-compact-open topology occurs in Section 4, where we compare it with the strict topology, bounded-compact convergence, and the norm topology on $C_b(X)$. We prove that the σ-compact-open topology is metrizable only when it is normable, and prove that the space is always sequentially complete. On the other hand, it is not always complete, although if X is either locally compact or satisfies the first axiom of countability, then the space is in fact complete.

In the final section we characterize the dual of $C_b(X)$ clothed in the σ-compact-open topology. It is here that we find that the dual of $C_b(X)$ with the strict topology coincides with the dual of $C_b(X)$ with the σ-compact-open topology precisely when the two topologies coincide. We leave the question of the Mackey topology — that is, conditions under which the σ-compact-open topology is the Mackey topology — to a subsequent paper.

Before proceeding to the heart of the paper, we explain our ground rules and explain our notations. The one ground rule is: *wherever X appears, it is completely regular and Hausdorff* (though we may specify that X have additional properties). As for terminology, βX denotes the Stone–Čech compactification of X, and is the largest compactification of X. By $C_b(X)$ we mean the continuous, bounded, complex-valued functions on X, as a vector space. Since all the functions in $C_b(X)$ are bounded, there is a natural norm topology t_n, where

$$\|f\| = \sup\{|f(x)| : x \in X\}\,.$$

If $A \subseteq X$, we let

$$\|f\|_A = \sup\{|f(x)| : x \in A\}\,,$$

and we let ψ_A denote the characteristic function of A. If E is any locally convex linear topological space, we let E^* be the collection of continuous linear functionals on E. If we identify $f \in C_b(X)$ with the unique continuous extension f' on βX, then the Riesz–Kakutani Theorem says in essence that $(C_b(X), t_n)^* = M(\beta X)$, where $M(\beta X)$ consists of all countably additive, regular, bounded measures on βX. We denote the total variation measure associated with μ by $|\mu|$, and we let $\|\mu\| = |\mu|(X)$.

Finally, we let the positive integers be called N, we let Λ be an arbitrary indexing set, we let $A \setminus B$ be the collection of elements in A but not in B, and by "iff" we mean "if and only if," just as Halmos did once upon a time.

2. The strict topology.

Buck defined and studied the strict topology on $C_b(X)$, where X was assumed to be locally compact [1], [2]. Much more recently, van Rooij mentioned briefly a natural extension of the notion to arbitrary completely regular Hausdorff spaces [9]. We derive our definition from this extension.

2.1. Definition. A net $(f_\lambda)_{\lambda \in \Lambda}$ in $C_b(X)$ converges *strictly* to $f \in C_b(X)$ iff for each function φ on X which is bounded and which vanishes at ∞, $f_\lambda \varphi \to f\varphi$ uniformly on X. The topology which this convergence describes is called the *strict topology*, and is abbreviated t_s.

Surely the strict topology is locally convex and Hausdorff, and quite clearly it coincides with Buck's strict topology whenever X is locally compact. If t_c and t_n denote the compact-open and the normed topologies respectively on $C_b(X)$, then evidently $t_c \subseteqq t_s \subseteqq t_n$ and t_s has the same bounded sets as does t_n. Furthermore, a sequence converges in t_s iff it is bounded in t_s and converges in t_c. These facts are observed in Buck [1], [2]; the proofs remain valid and simple in our more general setting. Let us direct our attention to additional properties, ones which we analyze in more depth.

2.2. Proposition. *$t_s = t_c$ iff each σ-compact subset of X is relatively compact.*

Proof. The sufficiency of the condition is obvious. On the other hand, if there is a σ-compact A in X which is not relatively compact, then $A \setminus K \neq \emptyset$ for each compact K. For each compact K, let $x_K \in A \setminus K$, and for each $n \in N$, let $f_{K,n} \in C_b(X)$, such that $f_{K,n} = 0$ on K and $f_{K,n}(x_K) = n!$. If we let $(K,n) \leq (K',n')$ mean that $K \subseteqq K'$ and $n \leqq n'$ as well, then $f_{K,n} \to 0$ in t_c. To show that $f_{K,n} \nrightarrow 0$ in t_s, it suffices to show that for each (K,n), there is an $n' \geqq n$ such that $\|f_{K,n'}\varphi\| \geqq 1$, for an appropriate φ. To that end, note that A is by hypothesis σ-compact, so $A = \bigcup_{n=1}^{\infty} K_n$, with the K_n compact and increasing. Now let $\varphi = \sum_{n=1}^{\infty} \psi_{K_n}/n!$. Then for any (K,n), the chosen x_K is an element of $A \setminus K$, whence $x_K \in K_{n'}$ for some $n' \geqq n$. Then $(f_{K,n'}\varphi)(x_K) \geqq n!/n! = 1$, so that $\|f_{K,n'}\varphi\| \geqq 1$.

Even though as a rule t_s differs from t_c, they both agree on uniformly bounded subsets of $C_b(X)$, as we now prove.

2.3. PROPOSITION. *On uniformly bounded subsets of $C_b(X)$, the topologies t_s and t_c are identical.*

PROOF. We need only prove that if $(f_\lambda)_{\lambda\in\Lambda}$ is uniformly bounded and if $f_\lambda \to 0$ in t_c, then $f_\lambda \to 0$ in t_s. We may assume that $\|f_\lambda\| \leqq M$, for all λ, and let φ be bounded in norm by 1 and vanish at ∞. Our goal is to show that for any given $\varepsilon > 0$ there is a λ_n such that if $\lambda \geqq \lambda_n$, then $\|f_\lambda\varphi\|_X < \varepsilon$. To that end, let ε and φ be as just mentioned and let $1/n < \varepsilon$. Since φ vanishes at ∞ there is a compact $K_n \subseteqq X$ such that if $x \notin K_n$, then $|\varphi(x)| < 1/(nM)$. But K_n is compact and $f_\lambda \to 0$ uniformly on compact sets, so there is a λ_n such that if $\lambda \geqq \lambda_n$, then $\|f_\lambda\|_{K_n} < \varepsilon$. This means that if $\lambda \geqq \lambda_n$, then

$$|f_\lambda\varphi(x)| < \left\{\begin{matrix} \varepsilon\cdot 1, & x \in K_n \\ M(1/nM), & x \notin K_n \end{matrix}\right\} \leqq \varepsilon$$

so that $\|f_\lambda\varphi\|_X < \varepsilon$. Consequently $f_\lambda \to 0$ in t_s.

Later on, after Theorem 2.6, we will produce a criterion necessary and sufficient for t_s and t_n to agree.

Let $C_c(X)$ be the collection of functions on X which are continuous and have compact support. The upcoming proof follows naturally.

2.4. PROPOSITION. *$C_c(X)$ is dense in $(C_b(X), t_s)$ iff X is locally compact.*

PROOF. If X is locally compact, then $C_c(X)$ is dense in $(C_b(X), t_s)$ by Theorem 1 (vi) of [2]. On the other hand, if $x_0 \in X$ has no compact neighborhood, then $f(x_0) = 0$ for any $f \in C_c(X)$, which means that the constant function 1 is isolated from $C_c(X)$ in t_c, and hence is isolated from $C_c(X)$ in t_s.

When X is locally compact, one of the beautiful and profitable properties of $(C_b(X), t_s)$ is that the dual $(C_b(X), t_s)^* = M(X)$. Providentially this property remains true for non-locally compact X. The proof is very different from that in [2], since Buck's proof relies on the density of $C_c(X)$ in $(C_b(X), t_s)$, and Proposition 2.4 precludes it.

2.5. LEMMA. *If $F \in (C_b(X), t_s)^*$, then for each $\varepsilon > 0$, there is a compact set $K \subseteqq X$ such that if $f \in C_b(X)$ with $\|f\| \leqq 1$ and $f = 0$ on K, then we have $|F(f)| < \varepsilon$.*

PROOF. Otherwise, for each compact K there exists an $f_K \in C_b(X)$ such that $f_K = 0$ on K, $\|f_K\| \leqq 1$, and $|F(f_K)| \geqq \varepsilon$. But then $f_K \to 0$ in t_c and thus in t_s, whereas $F(f_K) \nrightarrow 0$. This is tantamount to F not being continuous in t_s.

2.6. THEOREM. *For any X, $(C_b(X), t_s)^* = M(X)$.*

PROOF. Let $F \in (C_b(X), t_s)^*$. Since $t_s \subseteqq t_n$ and $(C_b(X), t_n)^* = M(\beta X)$, we know that F corresponds to a measure $\mu \in M(\beta X)$. Let $\varepsilon > 0$. If $K \subseteqq X$ is compact and $|\mu|(\beta X \setminus K) > \varepsilon$, then without loss of generality we may assume that there is a compact $D \subseteqq (\beta X \setminus K)$ such that $(\mathrm{Re}\,\mu)D > \varepsilon$. Let E be the Hahn Decomposition Set for D, let C be compact such that $C \subseteqq (E \cap D)$ and $(\mathrm{Re}\,\mu)C > \varepsilon$, and finally let U be open, with $C \subseteqq U \subseteqq \beta X \setminus K$, such that $|\mu|(U \setminus C) \leqq (\mathrm{Re}\,\mu)C - \varepsilon$. Since βX is normal, we may find an $f \in C_b(X)$ such that $0 \leqq f \leqq 1$, $f = 1$ on C, and $f = 0$ on $\beta X \setminus U$ (so in particular, $f = 0$ on K). Then

$$|F(f)| = |\textstyle\int_{\beta X} f\, d\mu| = |\int_C d\mu + \int_{U \setminus C} f\, d\mu| \geqq (\mathrm{Re}\,\mu)C - |\mu|(U \setminus C) > \varepsilon\,.$$

By Lemma 2.5 this cannot hold for each compact $K \subseteqq X$. Thus for each n, there is a compact set $K_n \subseteqq X$ such that $|\mu|(\beta X \setminus K_n) < 1/n$. Consequently $|\mu|(\beta X \setminus \bigcup_{n=1}^{\infty} K_n) = 0$, so that $\mu \in M(\bigcup_{n=1}^{\infty} K_n) \subseteqq M(X)$, thereby completing the proof of half the theorem. For the converse, let $\mu \in M(X)$. Then μ has σ-compact support, so that there is an increasing sequence of compact subsets $(K_n)_{n \in N}$ of X such that $\|\mu\| = |\mu|(\bigcup_{n=1}^{\infty} K_n)$ and $|\mu|(X \setminus K_n) < \frac{1}{4}^n$. Let

$$\begin{aligned} \varphi(x) &= \tfrac{1}{2}^n, \quad x \in K_{n+1} \setminus K_n,\ n = 1, 2, 3, \ldots\,, \\ &= 0, \quad x \notin \textstyle\bigcup_{n=1}^{\infty} K_n \text{ and } x \in K_1\,. \end{aligned}$$

Then φ is bounded and vanishes at ∞, and whenever $\|f\varphi\| \leqq 1$, we have

$$|\mu(f)| \leqq \textstyle\sum_{n=1}^{\infty} 2^n/4^n = 1\,,$$

so that μ defines a continuous linear functional on $(C_b(X), t_s)$.

2.7. COROLLARY. *$t_s = t_n$ iff X is compact.*

PROOF. If X is compact, then obviously $t_s = t_n$. Conversely, if $t_s = t_n$, then Theorem 2.6 and the Riesz–Kakutani Theorem together yield

$$M(X) = (C_b(X), t_s)^* = (C_b(X), t_n)^* = (C_b(\beta X), t_n)^* = M(\beta X)\,.$$

But $M(X) = M(\beta X)$ iff X compact.

Now we turn to metrizability. The whole matter of the metrizability of $(C_b(X),t_s)$ would be easily disposed of via the open mapping theorem if we knew for certain that $(C_b(X),t_s)$ were complete. On the one hand, if X is locally compact, then $(C_b(X),t_s)$ is complete, as Buck proved, and consequently $(C_b(X),t_s)$ is usually not metrizable, as he also proved. On the other hand, if X is not locally compact, then $(C_b(X),t_s)$ need not be complete, as the following example illustrates. Let X consist of the first uncountable ordinal Ω_0, plus all the discrete ordinals less than Ω_0, clothed in the (relative) ordinal topology. We claim that $(C_b(X),t_s)$ is not complete. For any $\lambda \in X$, $\lambda \neq \Omega_0$, let $f_\lambda = \psi_{[1,\lambda]}$, so that $f_\lambda \in C_b(X)$. Then $(f_\lambda)_{\lambda \in X}$ is t_s-Cauchy, while the only possible limit would be $\psi_{[1,\Omega_0)}$, which is not continuous at Ω_0.

Nevertheless, we can show that $(C_b(X),t_s)$ is metrizable iff X is compact, irrespective of the completeness of $(C_b(X),t_s)$. If we utilize the uniqueness of the so-called "bornological topology" associated with any given locally compact topology, then it follows, as Professor Michael Powell pointed out to us, that the metrizability of $(C_b(X),t_s)$ is tantamount to the compactness of X. However, we have a more direct proof which depends on the specific characteristics of t_s, and furthermore, the proof will guide us in proving Theorem 3.6.

2.8. THEOREM. *$(C_b(X),t_s)$ is metrizable iff X is compact.*

PROOF. If X is compact, then $t_s = t_n$, so $(C_b(X),t_s)$ is metrizable, taking care of half the proof. If X is locally compact, then $(C_b(X),t_s)$ is complete. If in addition $(C_b(X),t_s)$ is metrizable, then the identity map from the Banach space $(C_b(X),t_n)$ onto $(C_b(X),t_s)$ is an open map by the open mapping theorem (p. 294 of [7]), so that $t_s = t_n$, which means X is compact by Corollary 2.7. The final case — that X is not locally compact but yet $(C_b(X),t_s)$ is metrizable — is the hardest, and we tackle it now. If $x_0 \in X$ has no compact neighborhood, let us first show that x_0 has a countable basis of neighborhoods. To that end, let $\mathscr{U}$ be the neighborhood system of x_0, and let $\mathscr{D}$ be the collection of compact subsets of X. For each $D \in \mathscr{D}$ and $U \in \mathscr{U}$, there is an $x_{D,U} \in (U \setminus D)$. Using the complete regularity of X, we find a function $f_{D,U} \in C_b(X)$ such that $f_{D,U} = 0$ on D, $f_{D,U}(x_{D,U}) = 1$, and $0 \leqq f_{D,U} \leqq 1$. If we order the (D,U) by saying that $(D,U) \leq (D',U')$ iff $D \subseteqq D'$ and $U' \subseteqq U$, then the net $\{f_{D,U} : D \in \mathscr{D},\ U \in \mathscr{U}\}$ converges uniformly on compact subsets of X to 0. Since the net is uniformly bounded and since $t_s = t_c$ on uniformly bounded sets by Proposition 2.3, this means that $f_{D,U} \to 0$ in t_s. Because of the assumption that $(C_b(X),t_s)$ is metrizable, there is a subnet of the form $(f_{D_n,U_n})_{n\in N}$ converging to 0 in t_s. This means that if U is any neighborhood of x_0, and if D is any compact subset of X, then there is an n such that $(D_n, U_n) \geq (D, U)$, which means in particular that $U_n \subseteqq U$. The upshot of this is that x_0 has a countable basis of neighborhoods. Let us continue the proof. If, in addition to x_0 not having any compact neighborhood and $(C_b(X), t_s)$ being metrizable, we also knew that $(C_b(X), t_s)$ were sequentially complete, then using the open mapping theorem we would find that X must be compact, which it is not. Thus we may as well assume that $(C_b(X), t_s)$ is not sequentially complete, which means that there is a sequence $(f_n)_{n\in N}$ which is t_s-Cauchy but which has no limit in $C_b(X)$. However, since t_s-Cauchy sequences are bounded and thus uniformly bounded, $(f_n)_{n\in N}$ converges pointwise to a bounded function f, which necessarily must be discontinuous at some point of X. Because the t_s convergence is stronger than compact convergence, that point cannot have a compact neighborhood. Call this point at which f is discontinuous by x_0. By the first part of the proof, and the fact that x_0 has no compact neighborhood, we know that x_0 has a countable basis of neighborhoods. Using the discontinuity of f at x_0, we can avail ourselves of an $\varepsilon > 0$ and a sequence $x_k \to x_0$ such that $A = \{x_k\}_{k\in N} \cup \{x_0\}$ indeed is compact and such that for each k, $|f(x_k) - f(x_0)| > \varepsilon$. Since A is compact, $(f_n)_{n\in N}$ is uniformly Cauchy on A, so that for some n_0, if $n, m \geqq n_0$, then we have $\|f_n - f_m\|_A < \frac{1}{3}\varepsilon$. Joining together this information, we have

$$|f_{n_0}(x_k) - f_{n_0}(x_0)| \geqq |f(x_k) - f(x_0)| - |f_{n_0}(x_k) - f(x_k)| - |f(x_0) - f_{n_0}(x_0)| > \tfrac{1}{3}\varepsilon \ ,$$

for all k, with the result that f_{n_0} is not continuous at x_0, contradicting the fact that $(f_n)_{n\in N} \subseteqq C_b(X)$. This latter contradiction completes the proof of the theorem.

By studying the proof of Theorem 2.8, we can produce a space X which is not locally compact but for which $(C_b(X), t_s)$ is complete. In fact, if X satisfies the first axiom of countability and is at the same time not locally compact, then $(C_b(X), t_s)$ *is* complete. (Any infinite-dimensional Banach space furnishes such an example!) What this shows us is that the defining characteristic of completeness for $(C_b(X), t_s)$ is *not* the local compactness of X.

3. Bounded-compact convergence.

It sometimes happens that you can get much more powerful results from a certain type of convergence in a locally convex space if you add the restriction that the elements involved be uniformly bounded. This has been especially true of sequential convergences. For example, a sequence of measures which converges weak* and is uniformly bounded is called a "tight" convergence sequence, and appears in works on probability (e.g., [8]). Likewise, sequences of continuous functions which converge uniformly on compacta and are uniformly bounded play a part in Herz's attack on the spectral synthesis problem [6]. We shall define a convergence criterion based on this latter example, and compare the resulting convergence with convergence in the strict topology.

3.1. DEFINITION. A net $(f_\lambda)_{\lambda \in \Lambda}$ in $C_b(X)$ converges *bounded-compactly* to a function $f \in C_b(X)$ iff it converges to f uniformly on compact subsets of X and if the f_λ are uniformly bounded. Bounded-compact convergence is denoted by γ_{bc}. If F is a linear functional on $C_b(X)$, then we say F is continuous with respect to γ_{bc} iff $f_\lambda \to f$ in γ_{bc} implies $F(f_\lambda) \to F(f)$.

To avoid any confusion, we hastily mention that "F is continuous with respect to γ_{bc}" is what van Rooij means when he says "F is tightly continuous" in [9]. We note that $f_\lambda \to f$ in γ_{bc} iff $(f_\lambda - f) \to 0$ in γ_{bc}. Consequently a linear functional on $C_b(X)$ is continuous with respect to γ_{bc} iff it is continuous with respect to γ_{bc} at 0. Now we compare γ_{bc}-convergence with t_s-convergence.

3.2. PROPOSITION. *Strict convergence is weaker than bounded-compact convergence.*

PROOF. We must prove that if $f_\lambda \to 0$ in γ_{bc}, then $f_\lambda \to 0$ in t_s. But since $(f_\lambda)_{\lambda \in \Lambda}$ must be uniformly bounded, our result is just a restatement of Proposition 2.3, couched in the language of this section.

Because a net can be convergent in γ_{bc} only when the net is uniformly bounded, it would be unreasonable to expect that γ_{bc}-convergence might coincide with t_s-convergence — unless X be compact, in which case t_s-convergence reduces to uniform convergence. In fact if $X = \Omega$, the ordinals less than the first uncountable, we can easily exhibit an example of a strictly convergent net which fails to converge bounded-compactly. Let Ω_0 be the collection of nonlimit ordinals. Let $f_\lambda = n_\lambda \psi_{\{\lambda\}}$, where λ is n_λ $(0 < n_\lambda < \infty)$ greater than a limit ordinal. Then $f_\lambda \to 0$ in t_s, but $(f_\lambda)_{\lambda \in \Omega \setminus \Omega_0}$ is not bounded, so $f_\lambda \nrightarrow 0$ in γ_{bc}.

Nevertheless, if we restrict our attention to sequences, then the two types of convergence are identical.

3.3. PROPOSITION. *A sequence converges strictly iff it converges bounded-compactly.*

PROOF. By Proposition 3.2 we need only show that if $f_n \nrightarrow 0$ in γ_{bc}, then $f_n \nrightarrow 0$ in t_s. But if $f_n \nrightarrow 0$ in γ_{bc} then either $f_n \nrightarrow 0$ in t_c or $(f_n)_{n\in N}$ is unbounded. In the former case, $f_n \nrightarrow 0$ in t_s since $t_c \subseteq t_s$. In the latter case, we assume without loss of generality that $\|f_n\| \geqq 2^n$, and pick $x_n \in X$ such that $|f_n(x_n)| \geqq 2^n$. If we let $\varphi(x_n) = 1/n$, for all n, and $\varphi(x) = 0$ for all other $x \in X$, then φ is bounded and vanishes at ∞. Also $|f_n\varphi(x_n)| \geqq 2^n/n \to \infty$, so $f_n \nrightarrow 0$ in t_s.

Now comes a more surprising result, originally proved by van Rooij in [9]. We omit the proof.

3.4. PROPOSITION. *A linear functional on $C_b(X)$ is continuous for t_s iff it is continuous for γ_{bc}.*

Let us see what the state of affairs is. On the one hand, γ_{bc} is stronger than t_s, yet they are endowed with the same continuous linear functionals. On the other hand, Conway proved in [3] that the strict topology is a Mackey topology (the largest locally-convex topology with the same continuous linear functionals), at least when X is locally compact and paracompact. So for such X — like the reals — there are only two possibilities, only two possible avenues out of a logical jam: either $\gamma_{bc} = t_s$ or γ_{bc} does not determine a locally convex topology. We have thus set the stage for the following proposition, with whose proof we acknowledge gratefully the help of Professor C. H. Cook.

3.5. PROPOSITION. *γ_{bc} determines a topology on $C_b(X)$ iff X is compact.*

PROOF. If X is compact, then γ_{bc} determines t_n, the norm topology, so $\gamma_{bc} = t_s$. Conversely, assume X is not compact. Let $\mathscr{F}$ be a filter in $C_b(X)$, and let the notation $\mathscr{F} \to 0$ mean that $\mathscr{F}$ converges to 0. Then γ_{bc} determines a topology on $C_b(X)$ iff for each compact K in X, $(\bigcap_{\mathscr{F}\to 0}\mathscr{F})K$ contains the neighborhood filter of 0 (in the complexes), and if there exists a base $(F_\lambda)_{\lambda\in\Lambda}$ for $(\bigcap_{\mathscr{F}\to 0}\mathscr{F})$ such that $\|f\| < M$, for all $f \in F_\lambda$, and all $\lambda \in \Lambda$ (see [4]). We will show that the latter stipulation cannot be met. Let $(K_\lambda)_{\lambda\in\Lambda}$ be the collection of compact subsets of X, directed by inclusion. Since X is not compact, for each $\lambda \in \Lambda$ there is an $x_\lambda \in X \setminus K_\lambda$. For each $\mu \in \Lambda$, let n_μ be a positive integer, and let every tail of the net $(n_\mu)_{\mu\in\Lambda}$ be unbounded. Since X is completely regular, there is an $f_{\lambda\mu} \in C_b(X)$ such that $f_{\lambda\mu} = 0$ on K_λ, and $f_{\lambda\mu}(x_\lambda) = n_\mu$. For any fixed $\mu \in \Lambda$, $(f_{\lambda\mu}) \to 0$ uniformly on compact sets, and $\|f_{\lambda\mu}\| \geqq n_\mu$, for all $\lambda \in \Lambda$. Let $\mathscr{F}_\mu$ be the filter generated by $[(f_{\lambda\mu})_{\lambda > \nu}]_{\nu \in \Lambda}$. Then $\mathscr{F}_\mu \to 0$ in γ_{bc}. But for each $\mu \in \Lambda$, and each set $F_\mu \in \mathscr{F}_\mu$, we know that $F_\mu \supseteqq (f_{\lambda\mu})_{\lambda > \nu}$ for some $\nu \in \Lambda$, so that there is an $f \in F_\mu$ with $\|f\| \geqq n_\mu$. Since the n_μ are not bounded, there is no uniformly bounded base for $\bigcap_{\mathscr{F} \to 0} \mathscr{F}$.

Let us say that γ_{bc} is *boundedly sequentially complete* iff every uniformly bounded sequence $(f_n)_{n \in N}$ in $C_b(X)$ which is Cauchy for γ_{bc} converges to some f in $C_b(X)$. Then by using the fact that γ_{bc}-convergence is stronger than t_c-convergence, and by altering the proof of Theorem 2.8, we can substantiate the following proposition.

3.6. PROPOSITION. *Let X be not locally compact. If sequences define γ_{bc}, then γ_{bc} must be boundedly sequentially complete.*

PROOF. If $x_0 \in X$ has no compact neighborhood, then as in the proof of Theorem 2.8 we can show that x_0 has a countable basis of neighborhoods. After doing that, we note that if γ_{bc} were not boundedly sequentially complete, then there would exist a uniformly bounded sequence $(f_n)_{n \in N}$ in $C_b(X)$ which is γ_{bc}-Cauchy but which has no limit in $C_b(X)$ with respect to γ_{bc}. Since $(f_n)_{n \in N}$ is uniformly bounded and since γ_{bc}-Cauchy is stronger than pointwise Cauchy, $(f_n)_{n \in N}$ converges pointwise to a bounded function f, and by hypothesis $f \notin C_b(X)$. Thus f must be discontinuous at some point of X. The remainder of the present proof follows precisely the same as the end of the proof to Theorem 2.8.

4. The σ-compact-open topology.

It is now time to define and discuss the σ-compact-open topology.

4.1. DEFINITION. For any X, a net $(f_\lambda)_{\lambda \in \Lambda}$ in $C_b(X)$ converges in the *σ-compact-open topology* to $f \in C_b(X)$ iff $f_\lambda \to f$ uniformly on each σ-compact subset of X. This topology is denoted by t_σ.

That this convergence yields a locally convex Hausdorff topology follows from an argument analogous to that used to show that the compact-open convergence yields a locally convex Hausdorff topology. It is clear that we have the following inclusions:

$$t_c \subseteqq t_s \subseteqq t_\sigma \subseteqq t_n .$$

With a little more effort we can determine precisely when $t_\sigma = t_n$ and when $t_\sigma = t_s$.

4.2. PROPOSITION. *$t_\sigma = t_n$ iff $X = \bar{A}$ for some σ-compact $A \subseteq X$.*

PROOF. If $X = \bar{A}$ for some σ-compact $A \subseteq X$, then $f_\lambda \to 0$ in t_σ means that $\|f_\lambda\|_A \to 0$, so that $\|f_\lambda\|_X \to 0$. Thus $f_\lambda \to 0$ in t_n. Consequently $t_\sigma = t_n$. Conversely, if $X \neq \bar{A}$, for all σ-compact subsets A, then for each A there is an $x_A \in X \setminus \bar{A}$. Using the complete regularity of X, we find an $f_A \in C_b(X)$ such that $f_A = 0$ on A, while $f_A(x_A) = 1$. If $\mathfrak{A}$ is the collection of σ-compact sets in X, ordered by inclusion, then $f_A \to 0$ in t_σ. But $\|f_A\| \geqq 1$ for all A, so that $f_A \not\to 0$ in t_n.

4.3. PROPOSITION. *$t_\sigma = t_s$ iff each σ-compact set in X is relatively compact.*

PROOF. The sufficiency is obvious. For the converse, assume that there is a σ-compact subset A of X which is not relatively compact. Then there is an $x \in \bar{A}^{\beta X} \setminus X$. Look at the linear functional F_x determined by $F_x(f) = f'(x)$, where f' is the unique continuous extension of f to βX. On the one hand, since $x \notin X$, F_x cannot be continuous with respect to t_s, by Theorem 2.6. On the other hand, if $\|f\|_A \leqq 1$, then $|F_x(f)| \leqq 1$, so that F_x is continuous with respect to the semi-norm in t_σ determined by A. Thus $t_\sigma \neq t_s$.

Spaces with the property that all σ-compact subsets are relatively compact have appeared before (e.g., Theorem 12 of [10]). Trivially, every such space is countably compact, since sequences necessarily are relatively compact, so have cluster points. The converse — that if a space is countably compact, then each σ-compact subset is relatively compact – fails, and we have an example to show it fails. In order to discuss our example, we need the definition of a p-point. An element $y \in X$ is a p-point iff every G_δ containing y is a neighborhood of p. There exist p-points in $\beta N \setminus N$ (see p. 100 of [5]). Let $X = \beta N \setminus \{x_0\}$, where x_0 is one of these p-points. Since X is open in βN and x_0 is not isolated, N is a σ-compact subset of X such that $\bar{N} = X$ is not compact. To complete our argument we need only show that X is countably compact. However, the only way that X could fail to be countably compact is for there to exist a sequence $(x_n)_{n \in N} \subseteq X$ with only the one cluster point x_0. So we assume that $\lim_n x_n = x_0$. There are two cases to consider. In the first, $(x_n)_{n \in N}$ contains a subsequence in N; so we call this subsequence $(y_n)_{n\in N}$. But then $(y_n)_{n\in N}$ is homeomorphic to N, so has an infinite number of cluster points in βN. Thus $(x_n)_{n\in N}$ must have more than one cluster point in X. In the other case we assume that $(x_n)_{n\in N}$ has no subsequence in N. Then without loss of generality we may assume that $(x_n)_{n\in N}\in \beta N\setminus N$. For each k, let U_k be a neighborhood of x_0 in βN such that $x_n\notin U_k$, $n=1,2,\ldots,k$. This is possible since βN is Hausdorff. Because x_0 is a p-point, $\bigcap_{k=1}^{\infty} U_k=U$ is a neighborhood of x_0 (see p. 63 of [5]). However this means that $(x_n)_{n\in N}\cap U=\emptyset$, so that x_0 cannot be the limit point of $(x_n)_{n\in N}$, as we advertised. This contradiction proves the assertion that X is countably compact.

We now compare σ-compact-open convergence with bounded-compact convergence.

4.4. Proposition. *t_σ-convergence is weaker than γ_{bc} convergence iff $t_\sigma=t_s$.*

Proof. The sufficiency follows from Proposition 3.2. For the reverse implication, we note that if $t_\sigma\neq t_s$, then by the proof of Proposition 4.3 there is an F_x continuous for t_σ but not continuous for t_s. However, by Proposition 3.4 this means that F_x is not continuous for γ_{bc}, so that there exists a net $(f_\lambda)_{\lambda\in\Lambda}\subseteq C_b(X)$ such that $f_\lambda\to 0$ in γ_{bc} but $F_x(f_\lambda)\not\to 0$. Since F_x is continuous for t_σ, we must have $f_\lambda\not\to 0$ in t_σ. Thus t_σ-convergence is not weaker than γ_{bc}-convergence.

The t_σ-bounded sets are just the uniformly bounded sets in $C_b(X)$, because $t_s\subseteq t_\sigma\subseteq t_n$ and because t_s and t_n have the same bounded sets. The next proposition is the analogue of Proposition 2.4.

4.5. Proposition. *$C_c(X)$ is dense in $(C_b(X),t_\sigma)$ iff X is locally compact and $t_\sigma=t_s$.*

Proof. If $C_c(X)$ is dense in $(C_b(X),t_\sigma)$, then it is dense in $(C_b(X),t_s)$, so that X is locally compact by Proposition 2.4. Furthermore, $t_\sigma=t_s$ since otherwise the constant function 1 cannot be approximated by $C_c(X)$ functions. The opposite implication also follows from Proposition 2.4.

If you remember our proof of Theorem 2.8 on metrizability of $(C_b(X),t_s)$, it was quite lengthy, with the open-mapping theorem barred from action. If we replace t_s by t_σ, then the open-mapping theorem is still disallowed (and you will see why a bit later), but a direct proof is straightforward and short.

4.6. PROPOSITION. *$(C_b(X),t_\sigma)$ is metrizable iff $t_\sigma=t_n$ (that is, iff $X=\bar{A}$ for some σ-compact $A\subseteqq X$).*

PROOF. Assume that $\bar{A}\neq X$ for all σ-compact subsets A in X. For each such A, let $x_A\notin\bar{A}$ and let $f_A\in C_b(X)$ such that $f_A=0$ on A and $f_A(x_A)=1$. If $\mathfrak{A}$ denotes the collection of σ-compact sets in X, then $(f_A)_{A\in\mathfrak{A}}$ converges in t_σ to 0. If perchance $(C_b(X),t_\sigma)$ were metrizable, then there would be a subnet $(f_{A_n})_{n\in N}$ converging to 0 in t_σ. Let $B=(x_{A_n})_{n\in N}$, a veritable σ-compact set. We then note that $\|f_{A_n}\|_B\geqq 1$, for all n, so that $(f_{A_n})_{n\in N}$ does not converge in t_σ to 0. The conclusion is that $t_\sigma\neq t_n$ implies $(C_b(X),t_\sigma)$ is not metrizable. The converse is immediate.

The last collection of properties we discuss in this section concerns various types of completeness. Remember that a space is sequentially complete iff Cauchy sequences converge, and a topological vector space is quasi-complete iff bounded Cauchy nets always converge.

4.7. LEMMA. *Let $(f_\lambda)_{\lambda\in\Lambda}\subseteqq(C_b(X),t_\sigma)$ be Cauchy with respect to t_σ. Then there is a unique function f on X such that $f_\lambda\to f$ uniformly on σ-compact sets. This f is a bounded function.*

PROOF. Since $(f_\lambda)_{\lambda\in\Lambda}$ is t_σ-Cauchy, it is Cauchy for the topology of pointwise uniform convergence, so that the net converges pointwise to a function f on X. Indeed, $f_\lambda\to f$ uniformly on all σ-compact sets and such an f is assuredly unique. Now assume that f is unbounded. Then there is a sequence $(x_n)_{n\in N}$ such that for each n, $|f(x_n)|\geqq n$. Put $A=(x_n)_{n\in N}$, a bona fide σ-compact set. Then $f_\lambda\to f$ uniformly on A because A is σ-compact. But the boundedness of each f_λ assures the boundedness of f at least on A. This contradiction yields the assertion.

4.8. PROPOSITION. *$(C_b(X),t_\sigma)$ is sequentially complete.*

PROOF. Let $(f_n)_{n\in N}$ be Cauchy in t_σ. By the previous lemma, $f_n\to f$ uniformly on σ-compact sets, and f is bounded. We need only show that f is continuous. Were it not the case, then there must be an $x_0\in X$ and $\varepsilon>0$ and $(x_\lambda)_{\lambda\in\Lambda}$ converging to x_0 such that $|f(x_\lambda)-f(x_0)|>\varepsilon$ for all $\lambda\in\Lambda$. Choose n_0 such that if $n\geqq n_0$, then $|f_n(x_0)-f(x_0)|<\frac{1}{3}\varepsilon$. Since $x_\lambda\to x_0$ and each f_n is continuous, for all $n\geqq n_0$ there is a λ_n such that whenever $\lambda\geqq\lambda_n$, we have $|f_n(x_\lambda)-f_n(x_0)|<\frac{1}{3}\varepsilon$. Then

$$\begin{aligned}|f_n(x_{\lambda_n})-f(x_{\lambda_n})| &\geqq |f(x_{\lambda_n})-f(x_0)|-|f_n(x_{\lambda_n})-f_n(x_0)|-|f_n(x_0)-f(x_0)|\\ &> \varepsilon-\tfrac{1}{3}\varepsilon-\tfrac{1}{3}\varepsilon = \tfrac{1}{3}\varepsilon\,.\end{aligned}$$

Let $A=(x_{\lambda_n})_{n\in N}$. Then $\|f_n-f\|_A>\frac{1}{3}\varepsilon$ for all $n\geqq n_0$, so that $(f_n)_{n\in N}$ does not converge on all σ-compact sets to f, as asserted. The proof is (sequentially) complete!

4.9. Proposition. *$(C_b(X),t_\sigma)$ is complete iff it is quasi-complete.*

Proof. Of course completeness implies quasi-completeness. Now assume quasi-completeness, and let $(f_\lambda)_{\lambda\in\Lambda}$ be t_σ-Cauchy. Then there is a function f on X such that $f_\lambda\to f$ uniformly on σ-compact sets, and in addition f is bounded by some $M\geqq 3$. We will be done if we show that f is continuous. To that end, let

$$\begin{aligned} f_\lambda'(x) &= f_\lambda(x), && \text{if } |f_\lambda(x)|\leqq M\ , \\ &= M f_\lambda(x)/|f_\lambda(x)|, && \text{if } |f_\lambda(x)|>M\ . \end{aligned}$$

Let us prove that $(f_\lambda')_{\lambda\in\Lambda}$ is t_σ-Cauchy. To do that, we note that $(f_\lambda)_{\lambda\in\Lambda}$ is t_σ-Cauchy and that

$$|f_\lambda'(x)-f_\mu'(x)| \leqq |f_\lambda(x)-f_\mu(x)|$$

for all x in X by the contractive property of metric projection in the complex plane. Since $(f_\lambda')_{\lambda\in\Lambda}$ is uniformly bounded, the quasi-completeness ensures the existence of a $g\in C_b(X)$ for which $f_\lambda'\to g$ in t_σ. This means that $f_\lambda'\to g$ pointwise. However, $f_\lambda'\to f$ pointwise. Consequently $f=g$ so that $f\in C_b(X)$.

4.10. Theorem. *If X is locally compact or satisfies the first countability axiom, then $(C_b(X),t_\sigma)$ is complete.*

Proof. Assume that $(C_b(X),t_\sigma)$ is not complete. Then there is a t_σ-Cauchy net $(f_\lambda)_{\lambda\in\Lambda}$ in $C_b(X)$ and a function f defined on X with the property that $f_\lambda\to f$ uniformly on σ-compact sets in X, but f is not continuous. Let f be discontinuous at $x_0\in X$, and let $\varepsilon>0$ and $x_\lambda\to x_0$ be such that $|f(x_\lambda)-f(x_0)|>\varepsilon$. If A is σ-compact, then $A\cup\{x_0\}$ is σ-compact, so that there must be a λ_0 for which $\lambda\geqq\lambda_0$ implies $\|f_\lambda-f\|_{A\cup\{x_0\}}<\frac{1}{3}\varepsilon$ Now if $y\in A\cap(x_\lambda)_{\lambda\in\Lambda}$, then

$$\begin{aligned} |f_{\lambda_0}(y)-f_{\lambda_0}(x_0)| &\geqq |f(y)-f(x_0)|-|f_{\lambda_0}(y)-f(y)|-|f(x_0)-f_{\lambda_0}(x_0)| \\ &\geqq \varepsilon-\tfrac{1}{3}\varepsilon-\tfrac{1}{3}\varepsilon = \tfrac{1}{3}\varepsilon\ . \end{aligned}$$

In particular, if $x_0\in\overline{A\cap(x_\lambda)_{\lambda\in\Lambda}}$, and $x_0\neq x_\lambda$, all λ, then f_{λ_0} could not be continuous at x_0 — which is silly. The implication of the foregoing argument is that if $(C_b(X),t_\sigma)$ is not complete, then there is an $x_0\in X$ and a net $x_\lambda\to x_0$ with $x_\lambda\neq x_0$, all λ, such that for each σ-compact $A\subseteqq X$

we have $x_0 \notin \overline{A \cap (x_\lambda)_{\lambda \in \Lambda}}$. Evidently this cannot happen if either X is locally compact or if each point in X has a countable basis for neighborhoods.

There are X's which satisfy the first countability axiom but which are not locally compact. For instance, any infinite-dimensional Banach space suffices. Consequently $(C_b(X), t_\sigma)$ is complete for a class of X's strictly including locally compact spaces. The example preceding Theorem 2.8 serves to show us that there exist X's for which $(C_b(X), t_\sigma)$ is not complete.

Incidentally, one can with some effort adapt the proof by Buck in [2] that $(C_b(X), t_s)$ is complete whenever X is locally compact, in order to show that if X is locally compact, then $(C_b(X), t_\sigma)$ is complete. But let that pass.

5. The dual.

Our final section concerns the dual of $(C_b(X), t_\sigma)$. As a consequence of the fact that $t_s \subseteqq t_\sigma \subseteqq t_n$, and as a result of the Riesz–Kakutani Theorem which identifies continuous linear functionals with measures, we know that $M(X) \subseteqq (C_b(X), t_\sigma)^* \subseteqq M(\beta X)$. However, we can be more precise.

5.1. THEOREM. *For all X, $(C_b(X), t_\sigma)^* = \bigcup M(\bar{A}^{\beta X})$, where the union is taken over all σ-compact A (in X).*

PROOF. First we show that $(C_b(X), t_\sigma)^* \subseteqq \bigcup M(\bar{A}^{\beta X})$. Let $F \in (C_b(X), t_\sigma)^*$, and let F correspond to $\mu \in M(\beta X)$ via the Riesz–Kakutani Theorem for $(C_b(X), t_n)$. Since F is t_σ-continuous, there is a σ-compact $A \subseteqq X$ such that whenever $f = 0$ on A, $F(f) = \int_{\beta X} f d\mu = 0$. The normality of X ensures that if B is compact and if $B \subseteqq \beta X \setminus \bar{A}^{\beta X}$, then $\mu(B) = 0$. Applying the Hahn Decomposition Theorem as in Theorem 2.6, we find that $|\mu| = 0$ on $\beta X \setminus \bar{A}^{\beta X}$, which means that $\mu \in M(\bar{A}^{\beta X})$. Half the theorem is proved.

In the direction of the converse inclusion, we let A be a σ-compact subset of X and assume that $\mu \in M(\bar{A}^{\beta X})$. For $f \in C_b(X)$, let f' be the unique continuous extension to βX. Then $\|f\|_A \leqq 1$ iff $\|f'\|_{\bar{A}^{\beta X}} \leqq 1$. We define F by the equation

$$F(f) = \int_{\beta X} f' d\mu, \quad \text{for all } f \in C_b(X) .$$

Note that F is linear by virtue of the additivity of $\int$. Also, F is continuous, since $\|f\|_A \leqq 1$ implies that $|F(f)| \leqq \|\mu\|$. Thus $F \in (C_b(X), t_\sigma)^*$.

5.2. Corollary. *$(C_b(X),t_\sigma)^* = M(X)$ iff $t_\sigma = t_s$.*

Proof. Theorem 2.6 provides one of the two implications. To prove the other, assume that $(C_b(X),t_\sigma)^* = M(X)$ and let A be σ-compact in X. Then

$$\{\delta_x\colon x \in \bar{A}^{\beta X}\} \subseteqq \bigcup_{A\ \sigma\text{-compact}} M(\bar{A}^{\beta X}) = (C_b(X),t_\sigma)^* = M(X)\ ,$$

which means that $\bar{A}^{\beta X} \subseteqq X$. Thus σ-compact subsets of X are relatively compact in X. Proposition 4.3 finishes the proof.

With such a reasonable result as Corollary 5.2 we might quite unwittingly conjecture that $(C_b(X),t_\sigma)^* = M(\beta X)$ iff $t_\sigma = t_n$. We cannot prove it. Nevertheless, let us see what progress we can make on the conjecture. First we need a lemma.

5.3. Lemma. *If Y is an open and closed subset of X, then βY can be homeomorphically embedded in βX — and we write $\beta Y \subseteqq \beta X$.*

Proof. Let $h\colon Y \subsetneqq X$ be the injection of Y into βX. By Stone's Theorem h has an extension $h'\colon \beta Y \to \beta X$ (Theorem 6.5 of [5]). If we can show that h' is one-to-one, then h' embeds (or rather, injects!) βY into βX. Therefore let $y,z \in \beta Y$, with $y \neq z$. Then there are disjoint open neighborhoods U and V of y and z respectively in βY. Find a continuous, bounded function f on βY such that $f=0$ on U and $f=1$ on V. Let $y_\lambda \to y$ and $z_\mu \to z$, and assume that $y_\lambda \in (U \cap Y)$, $z_\mu \in (V \cap Y)$, for all λ and μ. Then $f(y_\lambda)=0$ and $f(z_\mu)=1$, for all λ and μ. Now extend $f|_Y$ to f' on βX by first defining

$$\begin{aligned} f'(x) &= f(x), \quad && x \in Y \\ &= 0, && x \in X \setminus Y\ , \end{aligned}$$

and then taking the unique continuous extension to all of βX. The result is that $f' \in C_b(X)$ and in addition $f'(h'(y_\lambda)) = f'(y_\lambda) = 0$ and $f'(h'(z_\mu)) = f'(z_\mu) = 1$, for all λ, μ. Since f' is continuous, $(h'(y_\lambda))_{\lambda \in \Lambda}$ and $(h'(z_\mu))_{\mu \in M}$ cannot have a common cluster point in βX. This means that h' is one-to-one.

5.4. Corollary. *If X is discrete and if $Y \subseteqq X$, then $\beta Y \subseteqq \beta X$.*

5.5. Theorem. *Let X be locally compact and paracompact. Then $(C_b(X),t_\sigma)^* = M(\beta X)$ iff $t_\sigma = t_n$.*

PROOF. By hypothesis X is locally compact and paracompact, so a well-known theorem from topology tells us that X is the disjoint union of $(X_\lambda)_{\lambda\in\Lambda}$, where each X_λ is open and closed and σ-compact. Then $t_\sigma = t_n$ iff Λ is countable, so that if $t_\sigma = t_n$, then $(C_b(X), t_\sigma)^* = M(\beta X)$. Thus we need only prove that if Λ is uncountable then $(C_b(X), t_\sigma) \neq M(\beta X)$. Let us identify some subset $\Lambda_0 \subseteq \Lambda$ in a one-to-one way with the ordinals Ω less than the first uncountable Ω_0, and let $Y = \bigcup_{\lambda\in\Lambda_0} X_\lambda$. Define $F\colon Y \to \Omega$ by $F(y) = \lambda$ iff $y \in X_\lambda$. Note that F is continuous, so that Stone's Theorem says that the extension $F'\colon \beta Y \to \beta\Omega$ is continuous. But F' is onto, so that there is a $y' \in \beta Y$ such that $F'(y') = \Omega_0$. Let A be an arbitrary σ-compact subset of Y. We will show that $y' \notin \bar{A}^{\beta Y}$. In fact, $A \cap X_\lambda \neq \emptyset$ only for a countable number of λ, so $F'(A)$ is countable and in addition

$$F'(\bar{A}^{\beta Y}) \subseteq \overline{F'(A)} \subseteq \Omega ,$$

which means that $\Omega_0 \notin F'(\bar{A}^{\beta Y})$. Thus $y' \notin \bar{A}^{\beta Y}$. Since this is true for each A, Theorem 5.1 implies that $\delta_{y'} \notin (C_b(Y), t_\sigma)^*$. However, $y' \in \beta Y$, and by Lemma 5.3, $\beta Y \subseteq \beta X$ since Y is open and closed in X, so $\delta_{y'}$ can be considered as a linear functional on $C_b(X)$. Moreover, the fact that Y is open and closed in X implies that if $\delta_{y'} \notin (C_b(Y), t_\sigma)^*$, then $\delta_{y'} \notin (C_b(X), t_\sigma)^*$. On the other hand, $\delta_{y'} \in M(\beta X)$ since $y' \in \beta Y \subseteq \beta X$. Thus $(C_b(X), t_\sigma)^* \neq M(\beta X)$.

In spite of Theorem 5.5, we know of spaces X such that $t_\sigma \neq t_n$ and yet $(C_b(X), t_\sigma)^* = M(\beta X)$, although all our examples involve non-locally compact spaces. One such is our old friend: X = the ordinals less than or equal to the first uncountable, less the non-discrete countable ordinals. However, if we restrict our attention to locally compact X's, then we suspect that the statement

$$(C_b(X), t_\sigma)^* = M(\beta X) \quad \text{iff} \quad t_\sigma = t_n$$

is true. But we cannot prove it.

ADDED IN PROOF. The author has recently learned of results which overlap portions of Section 3 of this paper by R. Giles, R. Wheeler and D. Sentilles.

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