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RINGS IN WHICH PURE IDEALS ARE GENERATED BY IDEMPOTENTS

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1. Introduction.

In this note all rings considered are associative with an identity element 1, ring homomorphisms preserve 1 and subrings have the same 1.

A left ideal I is called pure if the ring modulo the ideal is a flat left module. Rings in which pure ideals are generated by a single idempotent are the class of rings for which cyclic flat left modules are projective.

Commutative rings with cyclic flat modules projective have been studied by several authors, for instance S. Endo [5], K. H. Mount [12], D. Lazard [11] and W. V. Vasconcelos [16] and [17]. Not necessarily commutative rings with cyclic flat left modules projective are treated in the following papers: I. I. Sahaev [14] and [15] and the author [7], [8], [9].

The aim of this paper is to study rings in which any pure left ideal is generated, not necessarily by a single idempotent, but by a whole family of idempotents. This class of rings will be denoted by $\mathscr{C}$.

In the first part of this paper we prove that if any projective ideal is a direct sum of finitely generated ideals, then the ring is in $\mathscr{C}$. As a corollary we get that any left or right semihereditary ring is in $\mathscr{C}$. In the rest of the paper we only consider commutative rings in $\mathscr{C}$.

For a commutative ring in $\mathscr{C}$ we can prove that any projective ideal is a direct sum of finitely generated ideals.

In the second part of the paper we prove that if R (commutative) is in $\mathscr{C}$, then the polynomial ring, $R[X]$, and the ring of power series, $R[[X]]$, are again in $\mathscr{C}$. Furthermore we can characterize the rings in $\mathscr{C}$ by topological properties of the prime spectrum of the rings.

We conclude the paper by results concerning products of rings in $\mathscr{C}$, in particular we are interested in conditions on the rings, R_i, which ensures that the complete direct product is in $\mathscr{C}$.

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2. Generel results.

This section includes all the results we are able to prove for not necessarily commutative rings in $\mathscr{C}$.

LEMMA 2.1. *Let A be any rings. The following conditions are equivalent:*

(1) $A \in \mathscr{C}$.
(2) *All countably generated pure left ideals are generated by idempotents.*
(3) *All countably generated projective pure left ideals are generated by idempotents.*
(4) *Given any sequence $(a_i)_{i\in\mathsf{N}}$ of elements in A satisfying $a_i = a_i a_{i+1}$ for all i, then the left ideal generated by the a_i's is generated by idempotents.*

PROOF. Obviously (1) implies (2) and (2) implies (3).

To prove that (3) implies (4) we note that the ideal generated by the a_i's is pure [4, proposition 2.2]. Any countably generated pure ideal is projective by Jensen's theorem [6, lemma 2], and the proof of the implication is completed.

It remains to prove that (4) implies (1). Assume that I is a pure ideal which can't be generated by idempotents and let $a_1 \in I$ be an element not contained in any left ideal generated by idempotents from I.

We want to construct a sequence of elements $a_1, a_2, \ldots$, where all the a_i's are in I and $a_i a_{i+1} = a_i$ for all i. If we can construct such a sequence, then we are done since the ideal I_0 generated by the a_i's can't be generated by idempotents from I and then, of course, I_0 is not generated by idempotents.

Since I is pure there exists a homomorphism $u \in \mathrm{Hom}_A(I, A)$ with $u(a_1) = a_1$. Put $a_2 = u(1)$, then we have $a_1 = a_1 a_2$. If we repeat the argument with a_1 replaced by a_2, then we get an element a_3 satisfying $a_2 a_3 = a_2$. Continuing the process we get the required sequence.

Let us state some corollaries of the lemma.

COROLLARY 2.2. *A ring in which any projective ideal is a direct sum of finitely generated ideals is in $\mathscr{C}$.*

PROOF. It suffices to prove that any projective pure ideal I can be generated by idempotents. By assumption, I is a direct sum of finitely generated ideals $I_j, j \in J$. Each I_j is clearly pure and hence I_j is generated by a single idempotent. (Since A/I_j is a flat and finitely related left A-module, A/I_j is projective.) It is now obvious that I can be generated by idempotents.

Corollary 2.3. *Any left or right semihereditary ring is in* $\mathscr{C}$.

Proof. If A is left semihereditary, then $A \in \mathscr{C}$ by F. Albrecht [1], and if A is right semihereditary, then a theorem by H. Bass [2, theorem 3] implies that $A \in \mathscr{C}$.

Proposition 2.4. *Any left* p.p. *ring is in* $\mathscr{C}$.

Proof. R is a left p.p. ring means that all principal left ideals of R are projective or equivalently any left annihilator of a principal left ideal is generated by a single idempotent.

Given an ideal $I = \sum_j Aa_j$, where $a_j = a_j a_{j+1}$, we have to prove that I is generated by idempotents.

It is easily seen that the left annihilator of $(1-a_j)$ is contained in the left annihilator of $(1-a_{j+1})$. By assumption there exists an idempotent e_j such that Ae_j is equal to the left annihilator of $(1-a_j)$ for any j, so e_j is in the left annihilator of $(1-a_{j+k})$ for all k, hence

$$e_j = e_j a_{j+k} \quad \text{for all } j \text{ and } k\,. \tag{1}$$

It follows from (1) that e_j is in I for all j. The element a_j is clearly contained in the left annihilator of $(1-a_{j+1})$ and consequently $a_j = x_j e_{j+1}$ for some x_j. We have now proved that the left ideal generated by the e_j's is equal to I.

I don't know whether any projective left ideal over a left p.p. ring is a direct sum of finitely generated ideals. But one can prove the following result:

Theorem 2.5. *Let R be a commutative ring in $\mathscr{C}$. Any projective ideal is a direct sum of finitely generated ideals.*

Proof. It suffices to prove that any countably generated projective ideal is a direct sum of finitely generated ideals (I. Kaplansky [10]).

By $t(I)$ we denote the trace ideal of I. Since I is projective and countably generated it follows that $t(I)$ is countably generated.

For all prime ideals P, I_P is free, hence $t(I)_P = t(I_P)$ is $(0)_P$ or R_P, this gives us that $R/t(I)$ is flat and by hypothesis $t(I)$ is generated by idempotents, $f_j, j \in N$, say. The idempotents

$$\bar{f}_1 = f_1, \quad \bar{f}_2 = \bar{f}_1 + f_2 - \bar{f}_1 f_2, \quad \bar{f}_3 = \bar{f}_2 + f_3 - \bar{f}_2 f_3, \quad \ldots$$

do also generate $t(I)$ and $\bar{f}_i \bar{f}_{i+1} = \bar{f}_i$. Put

$$e_1 = \bar{f}_1, \quad e_2 = \bar{f}_2 - \bar{f}_1, \quad e_3 = \bar{f}_3 - \bar{f}_2, \quad \ldots ,$$

then $t(I)$ is generated by the e_i's, and it is easily seen that each e_i is idempotent and $e_i e_j = 0$, $i \neq j$. We claim that

$$I = \textstyle\sum_i \oplus e_i I . \tag{2}$$

It is obvious that the sum of the ideals $e_i I$ is direct and contained in I. To prove equality in (2) it suffices to prove that for any prime ideal P,

$$I_P = \left(\textstyle\sum_i \oplus e_i I\right)_P .$$

If $I_P = 0_P$, then there is nothing to prove. So let us assume that $I_P \cong R_P$. In this case $t(I)_P = t(I_P) = R_P$, thus there exists one and only one element e_δ such that $e_\delta \notin P$. By the complete additivity of the functor $- \otimes_R R_P$, we get

$$\left(\textstyle\sum_i \oplus e_i I\right)_P = \textstyle\sum_i \oplus (e_i I)_P = (e_\delta I)_P = I_P ,$$

since e_δ is a unit in R_P.

The proof of theorem 2.5 is completed if we can prove that $e_i I$ is finitely generated. Let us remark that $t(e_i I) = e_i R$. Since $e_i I$ is $e_i R$ projective, it suffices to prove that a projective ideal J in a ring S with $t(J) = S$ is finitely generated. We can write

$$1 = \textstyle\sum_p f_p(a_p) ,$$

where $f_p \in \operatorname{Hom}_S(J, S)$ and $a_p \in J$ for all p. For any $a \in J$ we have

$$a = \textstyle\sum_p a f_p(a_p) = \sum_p a_p f_p(a) ,$$

hence the a_p's generate J.

The last part of the proof can be replaced by a reference to [17, lemma 1.2].

Theorem 2.5 is due to Vasconcelos (unpublished), but with a different proof.

3. Stability.

In this section all rings considered are commutative rings.

Proposition 3.1. *If $R \in \mathscr{C}$, then the polynomial ring $R[x] \in \mathscr{C}$, too.*

Proof. Using the notation from [13, section 4] we have that for any pure ideal I in $R[x]$, $\varphi(I)$ is pure in R and hence generated by idempotents. Further $I = \varphi(I)R[x]$ by [13, lemma 4.2], so I is generated by idempotents.

PROPOSITION 3.2. *If* $R \in \mathscr{C}$, *then* $R[[x]] \in \mathscr{C}$. *Here* $R[[x]]$ *denotes the ring of power series over* R.

PROOF. For $a \in R[[x]]$ we denote by $\varphi(a)$ the constant term of the power series a. Let I be any pure ideal in $R[[x]]$. Clearly $I_0 = \varphi(a)$, $a \in I$, is a pure ideal in R. Hence there exist elements $(a_s)_{s \in S}$, where $a_s \in I$ for all s, such that $\varphi(a_s) = e_s$ is an idempotent in R for all $s \in S$ and the e_s's generate I_0.

We claim that $e_s \in I$ for all s. Since $\varphi(a_s e_s) = e_s$, we have $e_s a_s = e_s(1 + b_s x) \in I$. The element x is in the Jacobson-radical of R, hence $e_s \in I$.

To prove proposition 3.2 it suffices to prove that the ideal in $R[[x]]$ generated by the e_s's is equal to I.

If we can prove that for any maximal ideal P, where $e_s \in P$ for all $s \in S$, $I_P = (0)_P$, then we are done.

Since $(R[[x]]/I)_P$ is a cyclic flat module over a local ring, it is free of rank zero or one. Thus if $(I)_P$ is nonzero, then $I_P = R_P$. Now assume $I_P = R_P$. We can find an $a \in I, a \notin P$, say $a = a_0 + a_1 x + \dots$. Since $x \in P$, we get $a_0 \notin P$. Hence there exists an idempotent $e \in I$ such that $a_0 e = a_0$. This implies that $e \notin P$, and the proof of proposition 3.2 is completed.

REMARK 1. The ring R of continuous real-valued functions on $[0,1]$ shows that a subring of a ring in $\mathscr{C}$ is not necessarily again in $\mathscr{C}$. (The classical ring of quotients of R is von Neumann regular and the ideal consisting of all functions vanishing on some neighbourhood of zero is projective, indecomposable and not finitely generated.)

REMARK 2. If R is a subring of S, S faithfully flat and $S \in \mathscr{C}$, then it is readily checked that $R \in \mathscr{C}$.

Let us recall that a subset E of Spec(R) is called D-closed, if E is closed and for each $x \in \text{Spec}(R)$ with $\{\bar{x}\} \cap E \neq \emptyset$, $x \in E$ (cf. D. Lazard [11]).

THEOREM 3.3. *The following conditions are equivalent for the ring* R, *with* Spec$(R) = X$.

(1) $R \in \mathscr{C}$.

(2) *Any D-closed subset of X is an intersection of open-closed subsets of X.*

PROOF. (1) implies (2). By [11, proposition 5.4] any D-closed set E is of the form Supp(R/I), where R/I is R-flat. Since I is pure, it is readily checked that $\mathrm{Supp}(R/I) = V(I)$. By hypothesis $I = \sum_j Re_j$, where e_j is idempotent for all j. Further, $V(I) = V(\sum_j Re_j) = \bigcap_j V(Re_j)$ and we are done by [3, chap. II, § 4, n° 3, proposition 15].

(2) implies (1). Let I be any pure ideal in R. Then $V(I) = \mathrm{Supp}(R/I)$ is D-closed and consequently $V(I) = \bigcap_j V(Re_j)$ for a suitable set of idempotents in R. From $V(I) = V(\sum_i Re_j)$, it follows that

$$\mathrm{rad}(I) = \mathrm{rad}\left(\textstyle\sum_j Re_j\right).$$

Obviously $I \supseteqq \sum_j Re_j$. To prove equality it suffices to prove that

$$I_P = \left(\textstyle\sum_j Re_j\right)_P \quad \text{for all } P \in X .$$

If I_P is non-zero, then we can find an element a, $a \in I$ and $a \notin P$. Thus we can find an m such that a^m is in the ideal generated by the e_j's. Hence there exists an i such that e_i does not belong to P, that is, $(\sum_j Re_j)_P = R_P$.

COROLLARY 3.4. *Let I be a nil ideal. Then $R \in \mathscr{C}$ if and only if $R/I \in \mathscr{C}$.*

4. The pure length of a ring.

R denotes still a commutative ring. In [14] I. I. Sahaev has proved that any cyclic flat left module over a ring R is projective if and only if any ascending chain of principal left ideals

$$(1) \qquad (a_1) \subseteqq \ldots \subseteqq (a_m) \subseteqq \ldots, \quad \text{where } a_m = a_m a_{m+1},$$

terminates. We define the pure length of a ring R to be the supremum of all n for which there exists a strictly ascending chain

$$(2) \qquad (a_1) \subseteqq \ldots \subseteqq (a_n), \quad \text{where } a_j = a_j a_{j+1},\ j \in \{1,\ldots,n-1\}.$$

The pure length may be infinite even if all finitely generated flat modules are projective, as the following example shows:

EXAMPLE 1. Let F be a commutative field. We take R to be the F-algebra on the generators x_{ij}, $i \leqq j$, $j \in \mathrm{N}$ and defining relations

$$(3) \qquad x_{ij} x_{i_0 j_0} = 0 \quad \text{for } i = i_0 .$$

$$(4) \qquad x_{i-1j} x_{ij} = x_{i-1j} \quad \text{for all } i \text{ and } j .$$

The relations in (4) show that the pure length of R is infinite. By means of the theorem of I. I. Sahaev, mentioned above, it is not hard to verify that any cyclic flat R-module is projective and hence by Mount's theorem [12] any finitely generated flat R-module is projective.

Proposition 4.1. *Let* $(R_i, \gamma_{ij})_{i\in I}$ *be a direct system of rings. If there exists an* n *such that* R_i *has pure length less than or equal to* n *for all* i, *then* $\varinjlim_I R_i$ *has finite pure length.*

Proof. We have the following diagram

$$\begin{array}{ccc} R_i & \xrightarrow{\gamma_{ij}} & R_j \\ & {\scriptstyle \gamma_i}\searrow \quad \swarrow{\scriptstyle \gamma_j} & \\ & \varinjlim_I R_i & \end{array}$$

If $(a_1)\subseteqq \ldots \subseteqq (a_{n+1})$ is an ascending chain of elements from $\varinjlim_I R_i$ and $a_i a_{i+1} = a_i$, then there exists an $i_0 \in I$ and elements $b_1, \ldots, b_{n+1} \in R_{i_0}$ satisfying $\gamma_{i_0}(b_j) = a_j$ for all j. Since $\gamma_{i_0}(b_j - b_j b_{j+1}) = 0$, there exists an i_1 such that $i_0 \leqq i_1$ and $\gamma_{i_0 i_1}(b_j) = \gamma_{i_0 i_1}(b_j)\gamma_{i_0 i_1}(b_{j+1})$. Hence by assumption $R_{i_1}\gamma_{i_0 i_1}(b_j) = R_{i_1}\gamma_{i_0 i_1}(b_{j+1})$ for some $j \in \{1, \ldots, n\}$ and consequently $(\varinjlim_I R_i)(a_j) = (\varinjlim_I R_i)(a_{j+1})$.

The proof of the following lemma is straightforward.

Lemma 4.2. *A finite direct sum of rings* R_i, $1 \leqq i \leqq n$, *has finite pure length, if and only if each ring* R_i *has finite pure length.*

Lemma 4.3. (Cf. Sahaev [15] or Jøndrup [8]). *If the ring* R *is a subring of* S *and* S *has finite pure length, then* R *has finite pure length.*

Proof. R has no infinite set of orthogonal idempotents by S. Endo [5] and since R is commutative we can assume that R is indecomposable.

If $(a_1) \subseteqq \ldots \subseteqq (a_n)$ is a finite sequence of the form (2) and if n is strictly greater than the pure length of S, then there exist an $s \in S$ and an $i \in (2, \ldots, n\}$ such that $sa_{i-1} = a_i$. If we multiply this equation by a_i, we get that a_i is idempotent, and since a_i is nonzero, we conclude that $a_i = 1$.

Remark. In general the pure length of R may be arbitrarily larger than the pure length of S. If $R = \mathsf{Z}\oplus \ldots \oplus \mathsf{Z}$ (m copies of the ring of integers) and $S = \mathsf{Q}\oplus \ldots \oplus \mathsf{Q}$ (m copies of the ring of rational numbers), then it is easily seen that R has pure length equal to $2m+1$ and S has pure length equal to $m+1$.

Lemma 4.4. *Let* I *be a nil ideal. If the pure length of* R/I *is finite, then* R *has finite pure length.*

Proof. Since I is nil, there is a 1–1 correspondence between idempotents in R and R/I, so we can assume that both R and R/I are indecomposable rings.

For $a \in R$, $\bar{a}$ denotes the homomorphic image of a under the canonical map from R to R/I.

Let $(a_1) \subseteqq \ldots \subseteqq (a_n)$ be a chain in R of the form (2). Then $(\bar{a}_1) \subseteqq \ldots \subseteqq (\bar{a}_n)$ is a chain of the form (2) in R/I, hence if n is greater than the pure length of R/I plus one, there exists an element $\bar{r} \in R/I$ and an index $j > 2$ such that $\bar{r}\bar{a}_{j-1} = \bar{a}_j$. If we multiply this equation by $\bar{a}_j$, we get $\bar{a}_j = \bar{a}_j^2$. Since R/I is indecomposable, we conclude that $a_j \in I$ or that a_j is a unit modulo I. If $a_j \in I$, then the equation $a_{j-1}a_j = a_{j-1}$ shows that $a_{j-1} = 0$, a contradiction. If a_j is a unit modulo I, then it is well-known that a_j is a unit itself.

Proposition 4.5. *Let R be a noetherian ring. Then R has finite pure length.*

Proof. The proof of theorem 4.6 in [8] can be repeated verbatim having lemma 4.2, lemma 4.3 and lemma 4.4 in mind.

Proposition 4.6. *Let $(R_i)_{i \in I}$ be a family of rings and suppose there exists an integer n such that all rings R_i have pure length less than n. Then $\prod_i R_I \in \mathscr{C}$.*

Proof. Let (a_τ) be a sequence in $\prod_i R_i$ with $a_\tau = a_\tau a_{\tau+1}$. For each τ let a_τ be equal to $(a_{\tau i})_{i \in I}$. If $\tau > n$, then for all i there exists a $\tau_i \leqq \tau$ such that $a_{\tau_i i} = e_i$ is idempotent. It is readily checked that $(e_i)(a_{\tau+1\, i}) = (e_i)$ and $(e_i)a_1 = a_1$. The first equation shows that $(e_i)_{e \in I}$ is in the ideal generated by the a_τ's. The second equation shows that a_1 is contained in the ideal generated by the elements $(e_i)_{i \in I}$.

Remark. The assumption that the pure lengths of the R_i's are limited is essential.

Let K be a commutative field. We take R_i, $i \in \mathsf{N}$, to be the commutative K-algebra on the generators x_{ji}, $j \leqq i$, and defining relations

$$x_{j-1\,i}x_{ji} = x_{j-1\,i} \quad \text{for all } i \text{ and } j\,. \tag{5}$$

It is easily seen that R_i is a noetherian ring. Let $a_{\tau i}$ be equal to 1 if $\tau > i$ and $x_{\tau i}$ if $\tau \leqq i$ and put $(a_\tau) = ((a_{\tau i})_{i \in \mathsf{N}})$. Then it is readily checked that the ideal generated by the a_τ's can't be generated by idempotents.

Corollary 4.7. *The direct product of any family of copies of a noetherian ring R is in $\mathscr{C}$.*

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