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ON THE HOMOTOPY GROUPS OF COMPLEX PROJECTIVE ALGEBRAIC MANIFOLDS

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1. Introduction.

Let A be a complex algebraic manifold of dimension a everywhere imbedded in some complex projective space P_n of dimension n. In [2] it is proved that in case a and n satisfy the inequality $2a \geqq n+1$, then $\pi_1(A)=0$.

From this one can only conclude the following about the higher order homotopy groups: In case $2a=n+s$ for some $s>1$, the group $\pi_i(P_n, A)$ is finite for $1 \leqq i \leqq s+1$, and equivalently, $\pi_i(A)$ is finite for $3 \leqq i \leqq s$.

This note is concerned with the problem whether these finite groups vanish. Unfortunately the methods used require a much bigger s to get results. Actually, for $s \leqq 4$ nothing new is obtained. So suppose $s \geqq 5$. The result is stated in theorems 1 and 2.

THEOREM 1. *Let $A \subseteqq P_n$ be a complex algebraic manifold imbedded in the complex projective space P_n. Assume A has dimension a everywhere and P_n dimension n. Let $s=2a-n$ and suppose $s \geqq 5$. Then*

$$H^i(A, Z) = \begin{cases} 0 & \text{for } i \text{ odd} \\ Z & \text{for } i \text{ even} \end{cases}$$

provided $i \leqq s-2$.

COROLLARY 1. *Under the same circumstances*

$$H_i(A, Z) = \begin{cases} 0 & \text{for } i \text{ odd} \\ Z & \text{for } i \text{ even} \end{cases}$$

provided $i \leqq s-3$.

THEOREM 2. *Under the same circumstances*

$$\pi_i(A) = \begin{cases} 0 & \text{for } i=1,3,4,\ldots,s-3 \, . \\ Z & \text{for } i=2 \, . \end{cases}$$

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I wish to thank W. Barth for mentioning to me the possibility of connecting Morse theory and his work on the distance function [1].

2. Preliminaries.

First some preparations for using Morse theory. Let M be a complex n-dimensional manifold and $f\colon M \to \mathsf{R}$ a C^2-function. Then in any coordinate system $z_j = x_j + ix_{n+j}$, $j = 1,\ldots,n$, we have the quadratic Levi-form

$$L_f(p,w) = \sum_{j,k} \frac{\partial^2 f(p)}{\partial \bar{z}_k \partial z_j} w_j \bar{w}_k ,$$

for $p \in M$ and $w \in \mathsf{C}^n$. This form is known to be independent of coordinates and to be real. We can now define

$$\mathrm{Index}_{\mathsf{C}}(f,p) = \max\{\dim V \mid V \subseteqq \mathsf{C}^n \text{ and } L_f(p,w) < 0,\ \forall w \in V \setminus \{0\}\} .$$

M can be considered as a $2n$-dimensional real manifold too, and in the coordinates x_j, $j = 1,\ldots,2n$, there is the quadratic Hessian

$$H_f(p,v) = \sum_{k,j} \frac{\partial^2 f(p)}{\partial x_k \partial x_j} v_k v_j ,$$

for $p \in M$ and $v \in \mathsf{R}^{2n}$. This form is independent of coordinates in case $df(p) = 0$. We can define

$$\mathrm{Index}_{\mathsf{R}}(f,p,x) = \max\{\dim V \mid V \subseteqq \mathsf{R}^{2n} \text{ and } H_f(p,v) < 0,\ \forall v \in V \setminus \{0\}\} .$$

where x means the chosen coordinates. Define a matrix

$$\hat{E} = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}.$$

A simpel computation then shows that for $w_j = v_j + iv_{n+j}$ and $z_j = x_j + ix_{n+j}$

$$L_f(p,w) = \tfrac{1}{4}\big(H_f(p,v) + (\hat{E}^{-1} H_f \hat{E})(p,v)\big) . \tag{1}$$

One finds formula (1) in [1, Lemma 12]. Now let $q = \mathrm{Index}_{\mathsf{C}}(f,p)$ and $r = \mathrm{Index}_{\mathsf{R}}(f,p,x)$. By looking at the subspaces on which the forms are positive semi-definite formula (1) gives

$$\begin{aligned} 2n - 2q &\geqq 2(2n - r) - 2n , \\ r &\geqq q , \\ \mathrm{Index}_{\mathsf{R}}(f,p,x) &\geqq \mathrm{Index}_{\mathsf{C}}(f,p) . \end{aligned}$$

Let $G = SU(n+1)$ acting on P_n. Consider the map $\varphi\colon G \times A \to \mathsf{P}_n$ defined by $\varphi(\sigma,x) = \sigma x$. As in [2, 1(a)] there is a neighborhood B of $1 \in G$, the neutral element of the group, such that B is open and connected, $\sigma B\sigma^{-1}=B$ for all $\sigma\in G$, and there is a $b\in \mathsf{R}_+$ such that for all $x\in \mathsf{P}_n$,

$$Bx = \{y\in \mathsf{P}_n \mid \operatorname{dist}(x,y)<b\}\,.$$

Here dist is the usual Fubini–Study-metrik on P_n [1]. Hence $B\sigma A=\sigma BA$ is a tubular neighborhood of σA for all $\sigma\in G$.

Let us study $A\cap\sigma A\subseteqq A\cap B\sigma A$. Let $q\in(A\cap B\sigma A)\setminus\sigma A$. Let f_A denote the squared distance from A. According to [1, Lemma 11] there is a $\tau\in G$, such that for $g=f_{\sigma A}\,|\,U\cap\tau A$, where U is a neighborhood of $q\in\tau A$, the differential $dg(q)$ equals 0 and $H_g(q,v)<0$ for all $v\neq 0$. Hence $L_g(q,w)<0$ for all $w\neq 0$. And since $\dim A\cap\tau A\geqq s$, there is a tangent space V at q of dimension at least s, such that

$$L_{f_{\sigma A}}(q,w) = L_g(q,w) < 0 \quad \text{for all } w\in V\,.$$

Hence

$$\operatorname{Index}_{\mathsf{R}}(f_{\sigma A},q,x) \geqq \operatorname{Index}_{\mathsf{C}}(f_{\sigma A},q) \geqq s\,.$$

3. The fundamental lemma.

Let $E=B\sigma A\cap A$ and $D=\sigma A\cap A$ for some fixed $\sigma\in G$. Suppose further, that B is not maximal but somewhat smaller, say such that there exists another neighborhood B' satisfying

$$B\sigma A = \{x\in \mathsf{P}_n \mid f_{\sigma A}(x)<b\} \subset B'\sigma A = \{x\in \mathsf{P}_n \mid f_{\sigma A}(x)<b+\eta\}$$

for some $\eta>0$. Then we will show:

Lemma 1. *In the notation above we have $H^m(E,D)=0$ for $0<m<s$.*

Proof. Let $0<\varepsilon<\frac{1}{2}\eta$, assuming $\eta<b$. According to [3, Corollary 6.8, p. 37] we can find a smooth function g_ε with no degenerate critical points, so that g_ε approximates $f_{\sigma A}$ up to second derivative uniformly on

$$K(\varepsilon) = \{x\in A \mid f_{\sigma A}\leqq b+\varepsilon\}$$

with a distance from $f_{\sigma A}$ and its derivatives smaller than ε. The set

$$M(\varepsilon) = \{x\in K(\varepsilon) \mid \varepsilon\leqq g_\varepsilon(x)\leqq b+\varepsilon\}$$

has the properties that $K(\varepsilon)\supseteqq E$ and $K(\varepsilon)\setminus M(\varepsilon)\supseteqq D$, because

$$0 = \varepsilon-\varepsilon \leqq g_\varepsilon(x)-\varepsilon < f_{\sigma A}(x)\,,$$

$$\textstyle\bigcap_{\varepsilon>0}K(\varepsilon) = E, \quad \bigcap_{\varepsilon>0}K(\varepsilon)\setminus M(\varepsilon) = D\,.$$

Choosing g_ε closer to $f_{\sigma A}$, we may assume that the Hessian H_{g_ε} is so close to $H_{f_{\sigma A}}$, that we have for all $q \in M(\varepsilon)$ and all coordinates x

$$\mathrm{Index}_{\mathsf{R}}(g_\varepsilon, q, x) = \mathrm{Index}_{\mathsf{R}}(f_{\sigma A}, q, x) \geqq s .$$

Then we get from [3, Theorem 3.2, p. 14 or Theorem 3.5, p. 20] that, putting

$$M^\varepsilon = \{x \in K(\varepsilon) \mid g_\varepsilon(x) < b + \varepsilon\}, \quad M_0^\varepsilon = \{x \in M^\varepsilon \mid g_\varepsilon(x) \leqq \varepsilon\}$$

the pair $(M^\varepsilon, M_0^\varepsilon)$ is a relative CW-complex with at most such λ-cells attached for which $\lambda \geqq s$. Hence by excision (see f.ex. [3, p. 29]), we get $H_m(M^\varepsilon, M_0^\varepsilon) = 0$ for $m = 0, 1, \ldots, s-1$. Now, by duality we get

$$H^m(M^\varepsilon, M_0^\varepsilon) = 0 \quad \text{for } m = 0, 1, \ldots, s-1$$

and torsion-free for $m = s$. Now, let $\varepsilon \to 0$. Then

$$H^m(A \cap \sigma A) = \varinjlim H^m(M_0^\varepsilon) \quad \text{and} \quad H^m(A \cap B\sigma A) = \varinjlim H^m(M^\varepsilon) ,$$

and hence we have $H^m(E, D) = 0$ for $m = 0, 1, \ldots, s-1$ and torsion-free for $m = s$.

4. Some lemmata.

The group G acts on C^{n+1} and hence on S^{2n+1}, such that every $\sigma \in G$ gives a commutative diagram

$$\begin{array}{ccc} S^{2n+1} & \xrightarrow{\sigma} & S^{2n+1} \\ \downarrow & & \downarrow \\ \mathsf{P}_n & \xrightarrow{\sigma} & \mathsf{P}_n , \end{array}$$

where the vertical arrows are the Hopf fibrations with fiber S^1. Let $\hat{}$ denote the inverse image under this fibration. The idea is to go up in the diagram above, prove the theorems there, and then go down. First lemma 1 with hats on:

LEMMA 2. *If E and D are as in section 3, then $H^m(\hat{E}, \hat{D}) = 0$ for $1 \leqq m < s$.*

PROOF. The general Gysin cohomology sequence

$$\ldots \to H^m(E, D) \to H^{m+2}(E, D) \to H^{m+2}(\hat{E}, \hat{D}) \to H^{m+1}(E, D) \to \ldots$$

and lemma 1, saying $H^m(E, D) = 0$ for $m \leqq s-1$, give $H^m(\hat{E}, \hat{D}) = 0$ for $m \leqq s-1$.

Let us fix some notation. The map $\hat{\varphi}: G \times \hat{A} \to S^{2n+1}$ is defined by $\hat{\varphi}(\sigma,x) = \sigma x$. Let $p_G: G \times \hat{A} \to G$ be the projection on G, and $p'_G: \hat{\varphi}^{-1}(\hat{A}) \to G$ the restriction $p_G \mid \hat{\varphi}^{-1}(\hat{A})$. We will now compare $G \times \hat{A}$ with $\hat{\varphi}^{-1}(\hat{A})$.

Lemma 3. *There exists an isomorphism*

$$H^p(G) \otimes H^q(\hat{A}) \to H^p(G, R^q(p'_G)_* \mathsf{Z})$$

for all pairs (p,q) *where* $q \leqq s-2$.

Proof. At the point $\sigma \in G$, the stalk of the sheaf $R^q(p'_G)_* \mathsf{Z}$ is the group $H^q(\hat{D})$. There is a commutative diagram

$$\begin{array}{ccc} \hat{D} & \xleftarrow{\hat{\varphi}} & (\{\sigma\} \times \hat{A}) \cap \hat{\varphi}^{-1}(\hat{A}) \\ \downarrow & & \downarrow \\ \hat{E} & \xleftarrow{\hat{\varphi}} & (B\sigma \times \hat{A}) \cap \hat{\varphi}^{-1}(\hat{A}) \end{array},$$

which gives a commutative diagram

$$\begin{array}{ccc} H^q(\hat{E}) & \longrightarrow & H^q((B\sigma \times \hat{A}) \cap \hat{\varphi}^{-1}(\hat{A})) \\ \downarrow & & \downarrow \\ H^q(\hat{D}) & \longrightarrow & H^q((\{\sigma\} \times \hat{A}) \cap \hat{\varphi}^{-1}(\hat{A})) \end{array},$$

where by lemma 2 the left map, and obviously the lower map, are isomorphisms for $q \leqq s-2$. Hence for these q, the right map is epimorphic and the upper one is monomorphic. This gives a natural extension map

$$H^q((\{\sigma\} \times \hat{A}) \cap \hat{\varphi}^{-1}(\hat{A})) \to H^q((B\sigma \times \hat{A}) \cap \hat{\varphi}^{-1}(\hat{A}))$$

for $q \leqq s-2$. Hence the sheaf $R^q(p'_G)_* \mathsf{Z}$ is locally constant on G for $q \leqq s-2$, and since $\pi_1(G) = 0$, it is constant. That is

$$H^p(G, R^q(p'_G)_* \mathsf{Z}) \cong H^p(G) \otimes H^q(\hat{A}).$$

Proposition 1. *The restriction map* $H^i(G \times \hat{A}, \mathsf{Z}) \to H^i(\hat{\varphi}^{-1}(\hat{A}), \mathsf{Z})$ *is an isomorphism for* $i \leqq s-2$.

Proof. The spectral sequences for p_G and p'_G are for $q \leqq s-2$

$$\begin{array}{ccccccc} H^i(G \times \hat{A}, \mathsf{Z}) & \Leftarrow & E_2^{pq} & = & H^p(G, R^q(p_G)_* \mathsf{Z}) & \cong & H^p(G) \otimes H^q(\hat{A}) \\ \downarrow & & & & & & \downarrow \cong \\ H^i(\hat{\varphi}^{-1}(\hat{A}), \mathsf{Z}) & \Leftarrow & E_2^{pq} & = & H^p(G, R^q(p'_G)_* \mathsf{Z}) & \cong & H^p(G) \otimes H^q(\hat{A}) \end{array},$$

where the last isomorphism follows from lemma 3. Hence the restriction is an isomorphism for $i \leqq s-2$.

PROPOSITION 2. *$\pi_i(\hat{A})=0$ and $H_i(\hat{A})=0$ for $i=1,2,\ldots,s-3$ and $H^i(\hat{A})=0$ for $i=1,2,\ldots,s-2$.*

PROOF. The spectral sequences for the maps $\hat{\varphi}\colon G\times\hat{A}\to S^{2n+1}$ and the restriction of this map to $\hat{\varphi}^{-1}(\hat{A})$ are

$$\begin{array}{ccccc} H^i(G\times\hat{A},\mathsf{Z}) \Leftarrow E_2^{pq} & = & H^p(S^{2n+1},R^q(\hat{\varphi})_*\mathsf{Z}) & \cong & H^p(S^{2n+1})\otimes H^q(F) \\ \downarrow\cong & & & & \downarrow \\ H^i(\hat{\varphi}^{-1}(\hat{A}),\mathsf{Z}) \Leftarrow E_2^{pq} & = & H^p(A,R^q(\hat{\varphi})_*\mathsf{Z}) & \cong & H^p(\hat{A})\otimes H^q(F)\,, \end{array}$$

since both maps are fiber bundles with the same fiber F. The isomorphism to the left comes from proposition 1. For $p=0$ we have $H^q(F)$ everywhere. Suppose $H^p(\hat{A})=0$ for $1\leqq p<i<s-1$. Then there are exact sequences

$$\begin{array}{ccccccccc} 0\to H^{i-1}(F) & \to & H^i(S^{2n+1})\otimes H^0(F) & \to & H^i(G\times\hat{A}) & \to & H^i(F) & \to & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0\to H^{i-1}(F) & \to & H^i(\hat{A})\otimes H^0(F) & \to & H^i(\hat{\varphi}^{-1}(\hat{A})) & \to & H^i(F) & \to & \ldots \end{array}$$

with commutative squares. So from the five-lemma the second vertical arrow must be an isomorphism. Since $H^i(S^{2n+1})=0$, we get $H^i(\hat{A})=0$. This induction continues until $s-2$. From the Hopf fibration $\hat{A}\to A$ the sequence

$$\pi_1(S^1)\to\pi_1(\hat{A})\to\pi_1(A)$$

is exact. Therefore $\pi_1(\hat{A})$ is cyclic, because $\pi_1(S^1)=\mathsf{Z}$ and (from [2]) $\pi_1(A)=0$. Hence $\pi_1(\hat{A})\cong H_1(\hat{A})$, and by duality $H_1(\hat{A})=0$. But then it follows from $H^i(\hat{A})=0$ that $H_i(\hat{A})=0$ for $i\leqq s-3$, and, by the Hurewicz isomorphism theorem, that $\pi_i(\hat{A})=0$ for $i\leqq s-3$.

5. Proof of Theorems 1 and 2.

The Gysin cohomology sequence for the Hopf fibration gives

$$\ldots\to H^i(A)\to H^i(\hat{A})\to H^{i-1}(A)\to H^{i+1}(A)\to\ldots.$$

Proposition 2 gives $H^i(\hat{A})=0$ for $i\leqq s-2$, which implies that the maps $H^i(A)\to H^{i+2}(A)$ are isomorphisms for $0\leqq i\leqq s-4$. Using $H^0(A)=\mathsf{Z}$ and $H^1(A)=0$, it follows from [2] that these groups are computable:

$$H^i(A) = \begin{cases} \mathsf{Z} & \text{for } i \text{ even} \\ 0 & \text{for } i \text{ odd} \end{cases}$$

provided $i \leqq s-2$.

The proof of the corollary goes by duality.

PROOF OF THEOREM 2. The Hopf fibration gives the exact sequence

$$\ldots \to \pi_i(\hat{A}) \to \pi_i(A) \to \pi_{i-1}(S^1) \to \pi_{i-1}(\hat{A}) \to \ldots .$$

Since $\pi_i(\hat{A})=0$ for $i \leqq s-3$, we have isomorphisms $\pi_i(A) \to \pi_{i-1}(S^1)$ for $i \leqq s-3$. But $\pi_1(S^1)=\mathsf{Z}$ and $\pi_i(S^1)=0$ for $i \geqq 2$, so $\pi_2(A)=\mathsf{Z}$ and $\pi_i(A)=0$ for $3 \leqq i \leqq s-3$.

ADDED IN PROOF. Meanwhile the author has been able to prove that $\pi_i(\mathsf{P}_n, A)=0$ for $1 \leqq i \leqq s+1$. See M. E. Larsen, *On the topology of complex projective manifolds*, Københavns Universitet, Matematisk Institut, Preprint Series 1972 No. 12.

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