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THE CLASSIFICATION OF SIMPLY CONNECTED H-SPACES WITH THREE CELLS I

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0. Introduction.

Recently the list of known H-spaces with few cells was considerably enriched see [2], [4], [7] and [8]. Most of the non-classical H-spaces newly discovered were principal G-bundles over spheres where G is a classical Lie group. In order to regain control of this new flow of H-spaces it seems desirable to obtain some necessary or sufficient conditions for G-bundles over spheres to be H-spaces. The following seems highly probable:

0.1. CONJECTURE. *Let* $(G_n,d)=(SU(n),2)$ *or* $(Sp(n),4)$. *Then it is well-known that*

$$\pi_{dn-2}(G_{n-1}) = \mathsf{Z}_m, \quad m = 2^r(2k+1),\ r>0\ .$$

Let $M(n,\lambda)$ *be the total space of the following induced principal fibration:*

$$\begin{array}{ccc} & G_{n-1} & \\ \swarrow & & \searrow \\ M(n,\lambda) & \longrightarrow & G_n \\ \downarrow & & \downarrow \\ S^{dn-1} & \xrightarrow{f_\lambda} & S^{dn-1}\ , \end{array}$$

$\deg f_\lambda=\lambda$. (*In* [7] *and in the sequel,* $M(2,\lambda)$ *is denoted by* M_λ^{10} *for* $d=4$.)

0.1.1. *If* $(\lambda,m)=1$, *then* $M(n,\lambda)$ *is a loop space.* $M(n,\lambda)\approx M(n,\lambda')$ *if and only if* $\lambda\equiv\pm\lambda'$ $(\mathrm{mod}\, m)$. *For* $d=4$ *and* $n=2$ *one has* $m=12$, $M(2,1)=Sp(2)$, *and* $M_5^{10}=M(2,5)\approx M(2,7)$ *is the Hilton-Roitberg manifold proved to be a loop space by Stasheff* [7].

0.1.2. *If* λ *is odd and satisfies:* $p\,|\,\lambda$ *and* $p^r\,|\,m$ *implies* $p^r\,|\,\lambda$ *for every prime* p, *then* $M(n,\lambda)$ *is an* H*-space. If* $dn-1=7$, *the restriction that* λ *is odd is not necessary.*

0.1.3. *If* $dn-1>7$ *and* $M(n,\lambda)$ *admits an* H*-structure, then* λ *is odd.*

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This paper is first in a sequence of two referred to as CSCH3 I and CSCH3 II. In CSCH3 II, conjectures 0.1.1 and 0.1.2 are given some consideration and an outline of a proof is given. In the case $dn-1<7$, conjecture 0.1.3 is very simple. As a whole 0.1 is verified in [2] for the case $d=2$, $dn-1=7$. The main object of CSCH3 I is to settle the case $d=4$, $dn-1=7$. It turns out that the validity of 0.1.3 for $d=4$, $n=2$ was the last obstruction to the classification of simply connected H-spaces with three cells. This was also realized independently by Hilton–Roitberg [5] and by M. Curtis–Mislin–E. Thomas (Private communications). Thus, the ultimate goal of the present papers is to prove

0.2. THE CLASSIFICATION THEOREM. *Let X be a simply connected CW complex with three cells. If X admits an H-structure, then X is homotopy equivalent to one of the following eight complexes*:

$$S^3\times S^3, \quad SU(3), \quad M_k^{10},\ k=0,1,3,4,5, \quad S^7\times S^7 .$$

This theorem is proved in CSCH3 II. The main theorem in CSCH3 I is the following:

0.3. MAIN THEOREM I. *Let X be a simply connected CW complex satisfying*:

(1) $\qquad H^*(X,\mathsf{Z}_2) = \Lambda(x_3,x_7)$ *in* $\dim\leqq 13,\ x_i\in H^i(X,\mathsf{Z}_2)$,

(2) $\qquad {}^2\pi_6(X) = \mathsf{Z}_2$.

If X admits an H-structure, then $H^{14}(X,\mathsf{Z}_2)\neq 0$. In particular, the complexes M_k^{10}, $k\equiv 2 \bmod 4$, do not admit H-structures.

The proof of 0.3 is based on calculations involving high order cohomology operations carried out in chapter 2. The proof is quite complex as the obstruction for M_k^{10} to be an H-space can be essentially detected by a cohomology operation of order 5. Fortunately this operation can be decomposed in such a way that the calculations only involve operations of order at most three.

More precisely: Two operations φ and $\bar\varphi$ are studied. Both are operations defined on $H^*(\cdot,\mathsf{Z}_4)$ classes with values being cosets of $H^*(\cdot,\mathsf{Z}_2)$ (referred to as Z_4-Z_2 operations). The operation φ is a third-order operation of degree 8, while $\bar\varphi$ is a secondary operation of degree 4. The following type of relation between them is established (Proposition 2.3):

$$\varphi 2 \supset Sq^4\bar\varphi + Sq^8\varrho_2 ,$$

where ϱ_2 is the reduction $H^*(\cdot,\mathsf{Z}_4)\to H^*(\cdot,\mathsf{Z}_2)$. The evaluations of φ and $\bar\varphi$ on the projective plane $B_2(\hat X)$ of $\hat X$ (where $\hat X$ is essentially X made 4-connected) imply the condition $H^{14}(X,\mathsf{Z}_2)\neq 0$.

1. Some definitions and notations.

Let $I=(n_1,n_2,\ldots,n_k)$ be a finite sequence of natural numbers. Put $k=l(I)$ and write $K(\mathsf{Z}_p,I)$ for the product $\prod_{j=1}^{k}K(\mathsf{Z}_p,n_j)$. The vector in $H^*(K(\mathsf{Z}_p,I),\mathsf{Z}_p)$ consisting of the images of the fundamental classes of $H^{n_j}(K(\mathsf{Z}_p,n_j),\mathsf{Z}_p)$ will be referred to as the fundamental vector.

If I_1 and I_2 are two sequences, an H-mapping h (and hence ∞-loop map) between $K(\mathsf{Z}_p,I_1)$ and $K(\mathsf{Z}_p,I_2)$ can be given by an $l(I_2)\times l(I_1)$ matrix B with entries in the Steenrod algebra $\mathcal{A}(p)$: $h^*\iota_2=B\iota_1$, where the ι_k are the fundamental vectors.

A (stable) generalized k-stage Postnikov system (mod p) is an ∞-deloopable diagram $(\mathscr{G})$ of the form

$$\begin{array}{ccccccc}
\Omega K_{k-1} & \xrightarrow{j_{k-1}} & E_k & & & & \\
 & & \downarrow{\scriptstyle r_{k-1}} & & & & \\
\Omega K_{k-2} & \xrightarrow{j_{k-2}} & E_{k-1} & \xrightarrow{h_{k-1}} & K_{k-1} & = & K(\mathsf{Z}_p,I_{k-1}) \\
 & & \downarrow{\scriptstyle r_{k-2}} & & & & \\
 & & \vdots & & & & \\
 & & \downarrow{\scriptstyle r_2} & & & & \\
\Omega K_1 & \xrightarrow{j_1} & E_2 & \xrightarrow{h_2} & K_2 & = & K(\mathsf{Z}_p,I_2) \\
 & & \downarrow{\scriptstyle r_1} & & & & \\
K(\mathsf{Z}_p,I_0) = & & E_1 & \xrightarrow{h_1} & K_1 & = & K(\mathsf{Z}_p,I_1)\ ,
\end{array}$$

where

$$\Omega K_{i-1} \xrightarrow{j_{i-1}} E_i \xrightarrow{r_{i-1}} E_{i-1}$$

is the principal ΩK_{i-1} fibration induced by h_{i-1}. Let B_i be the $l(I_i)\times l(I_{i-1})$ matrix with entries in $\mathcal{A}(p)$ corresponding to h_1 if $i=1$ and to $h_i\circ j_{i-1}$ if $k>i>1$. We refer to $(\mathscr{G})$ as to the geometric realization of $B_1,\ldots,B_{k-1}$. Conversely, given $B_1,\ldots,B_{k-1}$, B_i an $m_i\times m_{i-1}$ matrix with entries in $\mathcal{A}(p)$, we write $0\in\langle B_1,\ldots,B_{k-1}\rangle$ if $B_1,\ldots,B_{k-1}$ admit a geometric realization. A necessary condition for $0\in\langle B_1,\ldots,B_{k-1}\rangle$ is that $B_{i+1}B_i=0$. If $k=3$ this condition is sufficient.

Let $0\in\langle B_1,\ldots,B_k\rangle$. A stable k-order cohomology operation φ associated with $B_1,\ldots,B_k$ is an operation on $m_0=l(I_0)$ variables with $m_k=l(I_k)$ values given by the universal example (in the sense of [1]) $\langle x,E_k,y\rangle$, where E_k is obtained from a $k+1$ stage Postnikov system $(\mathscr{G})$ realizing $B_1,\ldots,B_k$ geometrically, and where

$$x = r_{k-1}^* r_{k-2}^* \dots r_1^* \iota_0, \qquad y = h_k^* \iota_k \,.$$

If X is a CW complex, the domain $D(\varphi)$ of φ consists of all vectors z of length $l(I_0)$ of cohomology classes in $H^*(X, Z_p)$ with the property that

$$f_z^{(1)}\colon X \to E_1, \quad f_z^{(1)*}\iota_0 = z\,,$$

can be lifted to $f_z^{(k)}\colon X \to E_k$. The operation $\varphi(z)$ is then the set $\{f_z^{(k)*}(y)\}$, $f_z^{(k)}$ running over all such liftings.

A similar situation occurs when E_1 is not $K(Z_p, I)$ but $K(Z_{p^r}, I)$. The operation φ is then referred to as being a Z_{p^r}-Z_p operation as $D(\varphi) \subset H^*(\cdot, Z_{p^r})$ while $\varphi \subset H^*(\cdot, Z_p)$. In this study, we restrict ourselves to Z_4-Z_2 operations defined on a single class (that is, $l(I_0)=1$) with a single value $\big(l(I_k)=1\big)$ and $k \leqq 3$. One only has to note that in this case, the condition $B_2 B_1 = 0$ should read $B_2 B_1 \equiv 0 \bmod\big(\mathcal{A}(2)\, Sq^1\big)$.

Throughout this paper we shall consider only Z_2 Postnikov approximations, that is: we consider only the $Z_{2m} - k$ invariants or the integral k-invariants of order 2^m.

2. Some relations among high-order operations.

We consider here two high-order Z_4-Z_2 operations: A third-order operation φ and a secondary operation $\bar{\varphi}$.

Let

$$B_1 = \begin{pmatrix} Sq^2 \\ Sq^6 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} Sq^2 & 0 \\ 0 & Sq^2 \\ Sq^7 + Sq^{4,2,1} & Sq^{2,1} \end{pmatrix},$$

$$B_3 = (Sq^{4,2} \;\; Sq^2 \;\; Sq^1)\,.$$

2.1 Lemma. $0 \in \langle B_1, B_2, B_3 \rangle$.

Proof. In order to prove this lemma one should construct a three-stage Postnikov system:

$$\begin{array}{ccccc}
\Omega K_2 & \xrightarrow{j_2} & E_3 & \xrightarrow{h_3} & K(Z_2, n+8) = K_3 \\
 & & \downarrow{\scriptstyle r_2} & & \\
\Omega K_1 & \xrightarrow{j_1} & E_2 & \xrightarrow{h_2} & K(Z_2; n+3, n+7, n+8) = K_2 \\
 & & \downarrow{\scriptstyle r_1} & & \\
 & & E_1 = K(Z_4, n) & \xrightarrow{h_1} & K(Z_2, n+2, n+6) = K_1
\end{array} \qquad (\mathscr{G})$$

$$h_1{}^* = B_1\varrho_2\,,$$

(where ϱ_2: $H^*(\cdot, \mathsf{Z}_4) \to H^*(\cdot, \mathsf{Z}_2)$ is the reduction),

$$j_{i-1}^* h_i{}^* = B_i, \quad i = 2, 3\,.$$

Instead, one seeks a Z_2-Z_2 geometric realization for $\tilde{B}_1$, $\tilde{B}_2$, and $\tilde{B}_3$, where

$$\tilde{B}_1 = \begin{pmatrix} Sq^1 \\ Sq^2 \\ Sq^6 \end{pmatrix},$$

$$\tilde{B}_2 = \begin{pmatrix} Sq^3 & Sq^2 & 0 \\ Sq^7 + Sq^{4,2,1} & 0 & Sq^2 \\ Sq^{6,2} & Sq^7 + Sq^{4,2,1} & Sq^{2,1} \end{pmatrix},$$

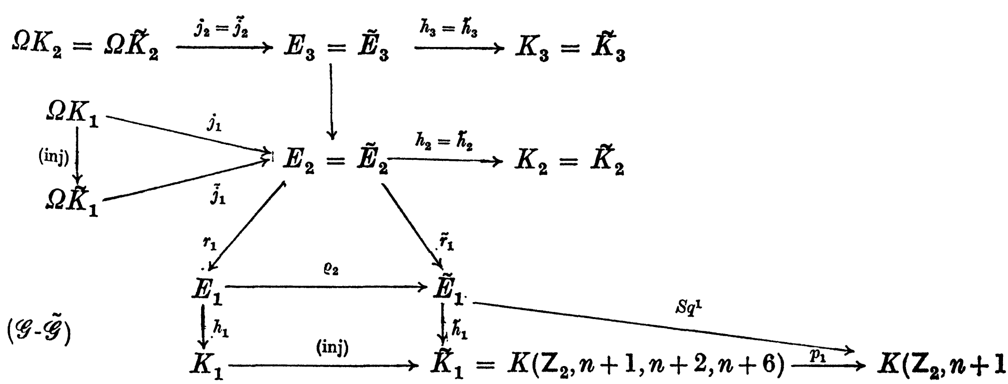
$$\tilde{B}_3 = B_3 = (Sq^{4,2}\ \ Sq^2\ \ Sq^1)\,.$$

That is, one seeks a three-stage Postnikov system:

$$(\tilde{\mathscr{G}})\qquad \begin{array}{ccccl} \Omega\tilde{K}_2 & \xrightarrow{\tilde{j}_2} & \tilde{E}_3 & \xrightarrow{\tilde{h}_3} & K(\mathsf{Z}_2, n+8) = \tilde{K}_3 = K_3 \\ & & \downarrow{\scriptstyle \tilde{r}_2} & & \\ \Omega\tilde{K}_1 & \xrightarrow{\tilde{j}_1} & \tilde{E}_2 & \xrightarrow{\tilde{h}_2} & K(\mathsf{Z}_2, n+3, n+7, n+8) = \tilde{K}_2 = K_2 \\ & & \downarrow{\scriptstyle \tilde{r}_1} & & \\ & & \tilde{E}_1 = K(\mathsf{Z}_2, n) & \xrightarrow{\tilde{h}_1} & K(\mathsf{Z}_2, n+1, n+2, n+6) = \tilde{K}_1 \end{array}$$

$$\tilde{h}_1{}^* = \tilde{B}_1 \quad \text{and} \quad \tilde{j}_{i-1}{}^*\tilde{h}_i{}^* = \tilde{B}_i, \quad i = 2, 3\,.$$

Once such a realization is established one gets the following comparison:



$(\mathscr{G}$-$\tilde{\mathscr{G}})$

where p_1 is the projection and E_i, K_i, r_i, h_i, j_i are the desired realization as

$$(\mathrm{inj})^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

To show $0 \in \langle \tilde B_1, \tilde B_2, \tilde B_3 \rangle$ (or equivalently the existence of $(\tilde{\mathscr{G}})$) one realizes that $\tilde B_2 \tilde B_1 = 0$. Hence $0 \in \langle \tilde B_1, \tilde B_2 \rangle$ and there exists a partial realization

$$(\tilde{\mathscr{G}}\text{-partial}) \qquad \begin{array}{ccccc} \Omega\tilde K_2 & \xrightarrow{\tilde j_2} & \tilde E_3 & & \\ & & \downarrow{\scriptstyle \tilde r_2} & & \\ \Omega\tilde K_1 & \xrightarrow{\tilde j_1} & \tilde E_2 & \xrightarrow{\tilde h_2} & \tilde K_2 \\ & & \downarrow{\scriptstyle \tilde r_1} & & \\ & & \tilde E_1 & \xrightarrow{\tilde h_1} & \tilde K_1 . \end{array}$$

As $\tilde B_3 \tilde B_2 = 0$ if $\sigma^* \tilde\iota_1$ and $\tilde\iota_2$ are the fundamental vectors of $H^*(\Omega\tilde K_1, \mathbf{Z}_2)$ and $H^*(\tilde K_2, \mathbf{Z}_2)$ respectively,

$$\tilde j_1^* \tilde h_2^* \tilde B_3 \tilde\iota_2 = \tilde B_3 \tilde B_2 \sigma^* \tilde\iota_1 = 0 .$$

Hence $\tilde h_2^* \tilde B_3 \tilde\iota_2 \in \ker \tilde j_1^*$, therefore,

$$\tilde h_2^* \tilde B_3 \tilde\iota_2 \in \tilde r_1^* PH^{(n+9)}(\tilde E_1, \mathbf{Z}_2) \quad \text{and} \quad \tilde h_2^* \tilde B_3 \tilde\iota_3 = \tilde r_1^* \alpha \tilde\iota_1 ,$$

$\alpha \in \mathcal{a}(2)$, $\deg\alpha = 9$. Now, as

$$Sq^9 = (Sq^8 + Sq^{6,2})Sq^1 + (Sq^7 + Sq^{4,2,1})Sq^2, \quad Sq^{8,1} = Sq^8 Sq^1 ,$$
$$Sq^{7,2} = Sq^7 Sq^2, \quad Sq^{6,3} = Sq^{6,1} Sq^2, \quad Sq^{6,2,1} = Sq^{6,2} Sq^1$$

form a basis for $\mathcal{a}(2)$ in dim 9,

$$\alpha = \alpha_1 Sq^1 + \alpha_2 Sq^2 \quad \text{and} \quad \alpha\tilde\iota_1 \in \ker \tilde r_1^* .$$

It follows that

$$\tilde h_2^* \tilde B_3 \tilde\iota_2 = 0, \quad \tilde B_3 \tilde\iota_2 \in \ker \tilde h_2^* ,$$

and consequently

$$\tilde B_3 \sigma^* \tilde\iota_2 \in \operatorname{im} \tilde j_2^*$$

where $\sigma^* \tilde\iota_2$ is the fundamental vector in $H^*(\Omega\tilde K_2, \mathbf{Z}_2)$, and $(\tilde{\mathscr{G}}$-partial) can be completed to $(\tilde{\mathscr{G}})$ by adding $\tilde h_3 \colon \tilde E_3 \to \tilde K_3$, $\tilde j_2^* \tilde h_3^* \tilde\iota_3 = \tilde B_3 \sigma^* \iota_2$.

2.1.1. Remark. Note that during the proof of 2.1 the arbitrary original choice of $h_2^* \iota_2$ satisfying

$$\tilde{j}_1^*\tilde{h}_2^*\iota_2 = \tilde{j}_1^* h_2^*\iota_2 = \tilde{B}_2\sigma^*\iota_1$$

was not altered and hence $h_2^*\iota_2$ can be freely changed by any element in $\operatorname{im}\tilde{r}_1^*$.

Let
$$\bar{B}_1 = \begin{pmatrix} Sq^2 \\ Sq^4 \end{pmatrix}, \quad \bar{B}_2 = (Sq^{2,1}, Sq^1)\,.$$

As $\bar{B}_2\bar{B}_1 \equiv 0 \bmod \mathcal{A}(2)Sq^1$, one gets a geometric realization of $0 \in \langle \bar{B}_1, \bar{B}_2\rangle$ as follows:

$$(\bar{\mathcal{G}}_1) \qquad \begin{array}{ccccc} \Omega\bar{K}_1 & \xrightarrow{\bar{j}_1} & \bar{E}_2 & \xrightarrow{\bar{h}_2} & \bar{K}_2 = K(\mathbf{Z}_2, n+4) \\ & & \downarrow{\scriptstyle \bar{r}_2} & & \\ & & \bar{E}_1 = K(\mathbf{Z}_4, n) & \xrightarrow{\bar{h}_1} & \bar{K}_1 = K(\mathbf{Z}_2, n+2, n+4)\,. \end{array}$$

$$\bar{h}_1^* = \bar{B}_1, \quad \bar{j}_1^*\bar{h}_2^* = \bar{B}_2\,.$$

Our main concern in this section is to choose h_2 and h_3 in $(\mathcal{G})$ in such a way that the third-order operation φ defined by $(\mathcal{G})$ and the secondary operation $\bar{\varphi}$ defined by $(\bar{\mathcal{G}})$ will satisfy the conditions described in the following propositions 2.2 and 2.3.

Let $BSp^{(k)}$ be the Postnikov approximation of BSp in $\dim \leqq k$:

$$\pi_m(BSp) \xrightarrow{\approx} \pi_m(BSp^{(k)}) \quad \text{for } m \leqq k$$

and $\pi_m(BSp^{(k)}) = 0$ for $m > k$. Denote $B = BSp^{(14)}$, let $j : B \to BSp^{(13)}$ and let $\theta : \hat{B} \to B$ be the $K(\mathbf{Z}, 3)$ principal fibration induced by $\tilde{g}_1 : B \to K(\mathbf{Z}, 4)$, where $\tilde{g}_1^*\iota_4$ is a generator.

2.2. Proposition. *For every choice of a non-decomposable generator z in $H^8(BSp^{(13)}, \mathbf{Z}_4)$, h_2 and h_3 in $(\mathcal{G})$ can be so chosen that one gets a commutative diagram*

$$(\mathcal{G}_1) \qquad \begin{array}{ccc} \hat{B} & \xrightarrow{\sim *} & K_3 \\ \downarrow{\scriptstyle \theta} & \searrow{\scriptstyle g_3} & \uparrow{\scriptstyle h_3} \\ B & & E_3 \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle r_2} \\ BSp^{(13)} & \xrightarrow{g_2} & E_2 \\ & \searrow{\scriptstyle g_1} & \downarrow{\scriptstyle r_1} \\ & & E_1 = K(\mathbf{Z}_4, 8)\,, \end{array}$$

with $g_1^*\iota_1 = z$, and E_i, h_i and r_i from $(\mathcal{G})$ for $n = 8$.

2.3. PROPOSITION. *With respect to the choices of h_i in $(\mathscr{G})$ made in 2.2, $\bar{h}_2$ in $(\bar{\mathscr{G}})$ can be chosen so that the following commutative diagram is obtained:*

$$(\mathscr{G}_2)\qquad \begin{array}{ccccccc} K(\mathsf{Z}_2,n+4) = \bar{K}_2 & \xleftarrow{\bar{h}_2} & \bar{E}_2 & \xrightarrow{\varphi_2} & E_3 & \xrightarrow{h_3} & K_3 = K(\mathsf{Z}_2,n+8) \\ & & \downarrow{\scriptstyle \bar{r}_1} & & \downarrow{\scriptstyle r_2} & & \\ & & \bar{E}_1 = K(\mathsf{Z}_4,n) & \xrightarrow{\varphi_1} & E_2 & \xrightarrow{r_1} & E_1 = K(\mathsf{Z}_4,n) \end{array}$$

with $\varphi_1{}^* r_1{}^* \iota_1 = 2\bar{\iota}_1$ and $\varphi_2{}^* h_3{}^* \iota_3 = Sq^4 \bar{h}_2{}^* \bar{\iota}_2 + Sq^8 \varrho_2 \bar{r}_1{}^* \bar{\iota}_1$, where ι_1, $\bar{\iota}_1$, $\bar{\iota}_2$, and ι_3 are the fundamental vectors.

PROOF OF 2.2. One can obviously obtain the following part of $(\mathscr{G}_1)$:

$$(\mathscr{G}_1)\text{part}\qquad \begin{array}{ccc} \hat{B} & & \\ \downarrow{\scriptstyle \theta} & & \\ B & & \\ \downarrow{\scriptstyle j} & & \\ BSp^{(13)} & \xrightarrow{g_2} & E_2 \\ & & \downarrow{\scriptstyle r_1} \\ & \xrightarrow{g_1} & E_1 \end{array}$$

Note that $H^k(BSp, \mathsf{Z}_2) \approx H^k(BSp^{(13)}, \mathsf{Z}_2)$ for $k \leqq 14$ and hence $Sq^2 z = 0$, $Sq^6 z = 0$.

If μ_B and Δ_B denote the multiplication $B \times B \to B$ and diagonal $B \to B \times B$, put $\lambda_2 = \mu_B \Delta_B$. As $\pi_{15}(BSp) = 0$ and as the k-invariant of BSp in dim 17 is integral of order divisible by 4,

$$H^*(B, \mathsf{Z}_4) = H^*(BSp^{(15)}, \mathsf{Z}_4) = H^*(BSp, \mathsf{Z}_4) = \mathsf{Z}_4[w_4, w_8, w_{12}, w_{16}]$$

in dim $\leqq 16$ and

$$\mu_B{}^* w_{4k} = \sum_{j=0}^{k} w_{4j} \otimes w_{4(k-j)}, \qquad w_0 = 1 ,$$

it follows that

$$\lambda_2{}^* w_4 = 2w_4, \quad \lambda_2{}^* w_8 = 2w_8 + w_4{}^2 ,$$
$$\lambda_2{}^* w_{12} = 2w_{12} + 2w_8 w_4 ,$$
$$\lambda_2{}^* w_{16} = 2w_{16} + 2w_4 w_{12} + w_8{}^2 .$$

If $j^* z = \pm w_8 + k w_4{}^2$, $k \in \mathsf{Z}_4$, then $\lambda_2{}^* j^* z = 2w_8 \pm w_4{}^2$ is primitive. Consequently $g_1 \circ j \circ \lambda_2$ is an H-mapping, and as $H^k(B \wedge B, \mathsf{Z}_2) = 0$ for $k = 9$, 13, it follows that $[B \wedge B, \Omega K_1] = 0$ and $g_2 \circ j \circ \lambda_2$ is an H-mapping too. As $PH^k(B, \mathbf{Z}_2) = 0$, $k = 11, 15$, and $PH^{16}(B, \mathbf{Z}_2)$ is generated by $\varrho_2 w_4{}^4$, it follows that

$$(g_2 \circ j \circ \lambda_2)^* h_2{}^* \iota_2 = (0, 0, \varepsilon \varrho_2 w_4{}^4), \quad \varepsilon \in \mathbf{Z}_2 .$$

Changing h_2 in $(\mathscr{G})$ if necessary so that $h_2{}^* \iota_2$ is altered by $Sq^8 \tilde{r}_1{}^* \tilde{\iota}_1 = Sq^8 \varrho_2 r_1{}^* \iota_1$ (and see remark 2.1.1) one may assume that $(g_2 \circ j \circ \lambda_2)^* h_2{}^* \iota_2 = 0$.

Let $r_2' : E_3' \to E_2$ be the fibration induced by

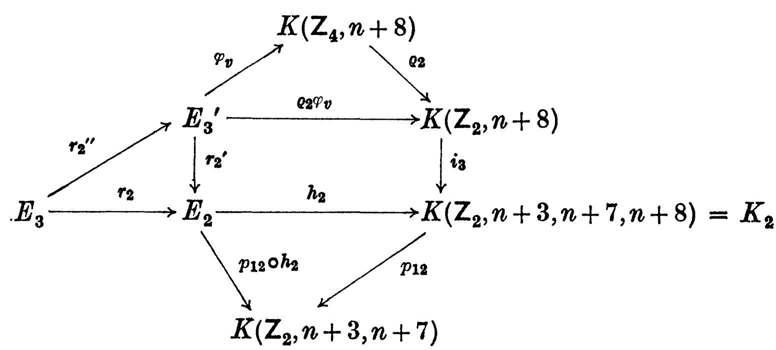
$$p_{12} \circ h_2 : E_2 \to K(\mathbf{Z}_2, n+3, n+7) ,$$

where $p_{12} : K(\mathbf{Z}_2; n+3, n+7, n+8) = K_2 \to K(\mathbf{Z}_2, n+3, n+7)$ is the projection. Now, if $h_2{}^* \iota_2 = (v_1, v_2, v_3)$, then $(p_{12} \circ h_2)^* = (v_1, v_2)$, and hence $v_1, v_2 \in \ker r_2'^*$. Further, as

$$0 = B_3 h_2{}^* \iota_2 = Sq^{4,2} v_1 + Sq^2 v_2 + Sq^1 v_3 ,$$

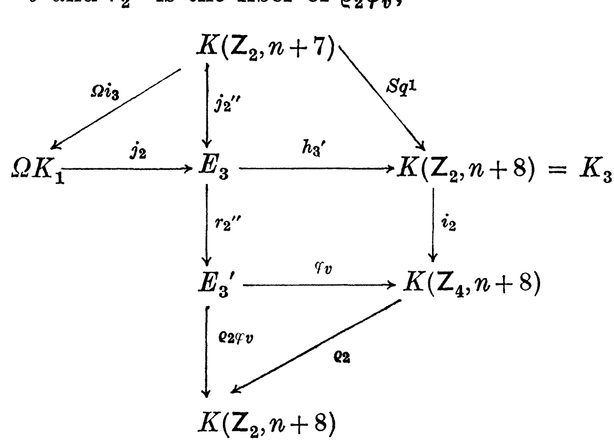
$Sq^1 v_3 \in \ker r_2'^*$. It follows that $r_2'^* h_2{}^* \iota_2 = (0, 0, \varrho_2 v)$ for some class $v \in H^{n+8}(E_3', \mathbf{Z}_4)$.

One gets the following diagrams:



where $\varphi_v{}^* \iota_{n+8} = v$ and r_2'' is the fiber of $\varrho_2 \varphi_v$,

$(\mathscr{G}_3)$



As $h_3 j_2'' \sim h_3 \circ j_2 \circ \Omega i_3 \sim h_3' j_2''$ if $\iota_3 \in H^*(K_3, Z_2)$ is the fundamental class, then

$$(h_3'^* - h_3^*)\iota_3 \in \ker j_2''^*, \qquad (h_3'^* - h_3^*)\iota_3 = r_2''^* v_1'$$

where $v_1' \in H^{n+8}(E_3', Z_2)$. Altering φ_v (and hence v) by $i_2\varphi_{v_1'}$, where $\varphi_{v_1'}: E_3' \to K_3$ and $\varphi_{v_1'}^* \iota_3 = v_1'$, $\varrho_2 v$ is not altered and one may assume $h_3' = h_3$. Moreover, any further alteration of v by an element in $r_2'^*(\ker \varrho_2)$ will correspond to a change of $h_3 = h_3'$ in $(\mathscr{G})$ without changing the relation $j_2^* h_3^* = B_3$.

Return now to the case $n=8$ and diagram $(\mathscr{G}_1)_{\text{part}}$. As $H^k(B, Z_2) = 0$, $k = 11, 15$, it follows that $g_2 \circ j$ lifts to

$$(\mathscr{G}_4) \qquad \begin{array}{ccc} & & E_3' \\ & {}^{g_3'}\nearrow & \downarrow{\scriptstyle r_2'} \\ B & \xrightarrow{g_2 \circ j} & E_2 \end{array} .$$

As $[B \wedge B, K(Z_2, 10, 14)] = 0$, $g_3' \circ \lambda_2$ is an H-mapping. As v is primitive, $\lambda_2^* g_3'^* v = 2aw_4^4$ ($a = 0$ or 1 in Z_4).

If $\iota_0 \in H^n(E_1, Z_4) = H^n(K(Z_4, n), Z_4)$ is the fundamental class $Sq^9 \varrho_2 r_1^* \iota_0 = 0$ for all n, then

$$Sq^8 \varrho_2 r_1^* \iota_0 = \varrho_2 \bar{v} \quad \text{for some} \quad \bar{v} \in H^{n+8}(E_2, Z_4) ,$$

and for $n = 8$,

$$Sq^8 \varrho_2 r_1^* \iota_0 = \varrho_2 (r_1^* \iota_0)^2 = \varrho_2 \bar{v} .$$

It follows that $2(r_1^* \iota_0)^2 = 2\bar{v}$. Now,

$$\lambda_2^* g_3'^* r_2'^* (r_1^* \iota_0)^2 = \lambda_2^* j^* g_1^* \iota_0^2 = \lambda_2^* (\pm w_8 + k w_4^2)^2 = w_4^4$$

and

$$\lambda_2^* g_3'^* r_2'^* (2a\bar{v}) = \lambda_2^* g_3'^* r_2'^* 2a (r_1^* \iota_0)^2 = 2aw_4^4 .$$

Replacing v by $v + 2ar_2'^* \bar{v}$ if necessary, we may assume

$$\text{(a)} \qquad \lambda_2^* g_3'^* v = 0 .$$

As $g_3'^* v \in \ker \lambda_2^*$, $\dim v = 16$, it follows that $g_3^* v$ must be in the ideal generated by w_4 and hence, $g_3'^* v \in \ker \theta^*$. Using this fact, diagrams $(\mathscr{G}_3)$ and $(\mathscr{G}_4)$ for $n = 8$ yield

$$\begin{array}{ccccc} & & K(Z_2, 15) & \xrightarrow{\quad Sq^1 \quad} & \\ & & \downarrow{\scriptstyle j_2''} & & \downarrow \\ \hat{B} & \xrightarrow{g_3} & E_3 & \xrightarrow{h_3} & K(Z_2, 16) \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle r_2''} & & \downarrow \\ B & \xrightarrow{g_3'} & E_3' & \xrightarrow{\varphi_v} & K(Z_4, 16) \end{array} ,$$

with $\varphi_v \circ g_3' \circ \theta \sim *$. As $h_3 \circ g_3$ can be lifted to $D\colon \hat{B} \to K(\mathsf{Z}_2, 15)$, a change in g_3 by $j_2'' \circ D$ yields $h_3 \circ g_3 \sim *$, and 2.2 follows.

As a side result of this proof one gets (a) and therefore one has

$$\begin{array}{ccc} & & E_3 \\ & \overset{\tilde{g}_3}{\nearrow} & \big\downarrow {\scriptstyle r_1 \circ r_2} \\ B & \xrightarrow[g_1 \circ j \circ \lambda_2]{} & K(\mathsf{Z}_4, 8) = E_1 . \end{array}$$

As it was done for g_3, the map $\tilde{g}_3$ can be chosen so that $h_3 \circ \tilde{g}_3 \sim *$. If $i\colon BS^3 \to B$ is induced by

$$BS^3 = BSp(1) \subset BSp \to B$$

one gets $i^* \lambda_2^* \lambda_2^* \tilde{g}_3^* h_3^* \iota_3 = 0$. As

$$i^* \lambda_2^* \lambda_2^* j^* g_1^* \iota_8 = 2w^2$$

where $w \in H^4(BS^3, \mathsf{Z}_4)$ is a generator, one has:

2.4. Corollary. *If φ is the third order operation induced by ($\mathscr{G}$) (with the* 2.2 *choices of h_i), then $0 = \varphi(2w^2)$ (with* 0 *indeterminancy).*

Proof of 2.3. First consider the following commutative ladder:

$$\begin{array}{ccccccc} \Omega\tilde{K}_1 = K(\mathsf{Z}_2, n, n+1, n+5) & \xrightarrow{\tilde{j}_1} & E_2 & \xrightarrow{\tilde{r}_1} & K(\mathsf{Z}_2, n) & \xrightarrow{\tilde{h}_1} & K(\mathsf{Z}_2, n+1, n+2, n+6) = \tilde{K}_1 \\ {\scriptstyle i_1} \uparrow \downarrow {\scriptstyle p_1} & & \downarrow {\scriptstyle r_1} & & \downarrow {\scriptstyle 1} & & \downarrow {\scriptstyle p_1} \\ K(\mathsf{Z}_2, n) & \xrightarrow{i_2} & E_1 = K(\mathsf{Z}_4, n) & \xrightarrow{\varrho_2} & K(\mathsf{Z}_2, n) & \xrightarrow{Sq^1} & K(\mathsf{Z}_2, n+1) , \end{array}$$

where ϱ_2, $\tilde{h}_1$, $E_2 = \tilde{E}_2$, $\tilde{r}_1$ and $\tilde{j}_1$ are the same as in diagram ($\tilde{\mathscr{G}}$) of 2.1. Further $\tilde{j}_1^* h_2^* \iota_2 = \tilde{B}_2 \bar{\iota}_2$ with $\tilde{B}_2$ as in the proof of 2.1. If $i_1\colon K(\mathsf{Z}_2, n) \to \Omega\tilde{K}_1$ is the injection, then $\tilde{\varphi}_1 = \tilde{j}_1 \circ i_1$ is a lifting of i_2, and

$$\tilde{\varphi}_1^* h_2^* \iota_2 = i_1^* \tilde{j}_1^* h_2^* \iota_2 = i_1^* \tilde{B}_2 \iota_2 = \begin{pmatrix} Sq^3 \\ Sq^7 + Sq^{4,2,1} \\ Sq^{6,2} \end{pmatrix} \bar{\iota}_0 .$$

Put $\varphi_1 = \tilde{\varphi}_1 \circ \varrho_2$ and note that

$$\varphi_1^* r_1^* \iota_1 = \varrho_2^* \tilde{\varphi}_1^* r_1^* \iota_1 = \varrho_2^* i_2^* \iota_1 = 2\bar{\iota}_1 .$$

Now consider the following commutative diagram:

$$\begin{array}{ccccc}
\Omega\bar{K}_1 & \xrightarrow{\text{(inj)}} & \Omega\tilde{\bar{K}}_1 & \xrightarrow{\Omega\tilde{\varphi}_3} & \Omega K_2 \\
\downarrow{\scriptstyle \bar{j}_1} & & \downarrow{\scriptstyle \tilde{\bar{j}}_1} & & \downarrow{\scriptstyle j_2} \\
\bar{E}_2 & \xrightarrow{1} & \tilde{\bar{E}}_2 & \xrightarrow{\varphi_2} & E_3 \\
\downarrow{\scriptstyle \bar{r}_1} & & \downarrow{\scriptstyle \tilde{\bar{r}}_1} & & \downarrow{\scriptstyle r_2} \\
\bar{E}_1 = K(\mathsf{Z}_4,n) & \xrightarrow{\varrho_2} & K(\mathsf{Z}_2,n) = \tilde{\bar{E}}_1 & \xrightarrow{\tilde{\varphi}_1} & E_2 \\
\downarrow{\scriptstyle \bar{h}_1} & & \downarrow{\scriptstyle \tilde{\bar{h}}_1} & & \downarrow{\scriptstyle h_2} \\
K(\mathsf{Z}_2,n+2,n+4) & \xrightarrow{\text{(inj)}} & K(\mathsf{Z}_2,n+1,n+2,n+4) & \xrightarrow{\tilde{\varphi}_3} & K(\mathsf{Z}_2,n+3,n+7,n+8) = K_2
\end{array}$$

$$\tilde{\varphi}_3{}^* = \tilde{B} = \begin{pmatrix} 0 & Sq^1 & 0 \\ Sq^{4,2} & 0 & Sq^3 \\ Sq^7+Sq^{6,1} & 0 & Sq^4 \end{pmatrix}, \quad \tilde{\bar{h}}_1{}^* = \begin{pmatrix} Sq^1 \\ Sq^2 \\ Sq^4 \end{pmatrix}.$$

Put $\tilde{\varphi}_3\circ(\text{inj}) = \varphi_3$. There exists

$$z \in H^{n+4}(\bar{E}_2, \mathsf{Z}_2), \quad \tilde{\bar{j}}_1{}^* z = (Sq^4, Sq^{2,1}, Sq^1)\sigma^* \tilde{\bar{\iota}}_1 ,$$

and one can choose $\bar{h}_2$ so that $z = \bar{h}_2{}^* \bar{\iota}_2$ and hence

$$\tilde{\bar{j}}_1{}^* \bar{h}_2{}^* = (Sq^4, Sq^{2,1}, Sq^1) ,$$

since

$$\begin{aligned} \bar{j}_1{}^* \varphi_2{}^* h_3{}^* &= (\Omega\varphi_3)^* j_2{}^* h_3{}^* \iota_3 = (\text{inj})^* B_3 \tilde{B} \sigma^* \iota_1 \\ &= Sq^4(Sq^{2,1} Sq^1)\sigma^* \bar{\iota}_1 = Sq^4 \bar{j}_1{}^* h_2{}^* \bar{\iota}_2 . \end{aligned}$$

Hence it follows that

$$\varphi_2{}^* h_3{}^* \iota_3 + Sq^4 h_2{}^* \iota_2 \in \bar{r}_1{}^* PH^{n+8}(\bar{E}_1, \mathsf{Z}_2)$$

and, consequently,

$$\varphi_2{}^* h_3{}^* \iota_3 = Sq^4 h_2{}^* \bar{\iota}_2 + \varepsilon Sq^8 \varrho_2 \bar{r}_1{}^* \tilde{\bar{\iota}}_0, \quad \varepsilon \in \mathsf{Z}_2, \ \tilde{\bar{\iota}}_0 \in H^*(\bar{E}_1, \mathsf{Z}_2) .$$

If one shows that $\bar{\varphi}(w^2) = \varrho_2 w^3$, where $w \in H^4(BS^3, \mathsf{Z}_4)$ is a generator, then by 2.4, one obtains

$$0 = \varphi(2w^2) = Sq^4 \bar{\varphi}(w^2) + \varepsilon Sq^8 \varrho_2 w_4{}^2 = (1+\varepsilon)\varrho_2 w_4{}^4 .$$

Hence $\varepsilon = 1$, and 2.3 follows.

Now, the relation $\bar{\varphi}(w^2) = \varrho_2 w^3$ follows from general calculations carried out independently by Kristensen [6], Gitler and Milgram [3] and others, but as the adaptation of their calculations to this simple case is not simpler than a direct proof, we give here an outline of the proof of the following:

2.5. LEMMA. *Let $\langle x, Y, u\rangle$ be the universal example for the (unique) Z-Z_2 secondary operation induced by the (non-stable) relation*

$$e[(Sq^{2,1}Sq^2)\circ\varrho_2] > 4 .$$

Then $\bar\varrho_4 x^2 \in D(\bar\varphi)$ and $\bar\varrho_2 x^3 \in \bar\varphi(x^2)$, where $\varrho_i\colon H^*(\ ,\mathsf{Z}) \to H^*(\ ,\mathsf{Z}_i)$ is the reduction.

PROOF. One has the (once deloopable) commutative diagram:

$$\begin{array}{ccc}
K(\mathsf{Z}_2,5) & \xrightarrow{(Sq^{2,1},\,0,\,Sq^6)} & K(\mathsf{Z}_2,8,9,11) \\
\downarrow{\scriptstyle j_0} & & \downarrow{\scriptstyle \tilde{\bar j}_1} \\
Y & \xrightarrow{\qquad \bar f \qquad} & \bar E_2 \\
\downarrow{\scriptstyle r_0} & & \downarrow{\scriptstyle \tilde{\bar r}_1} \\
K(\mathsf{Z},4) & \xrightarrow{\quad Sq^4\bar\varrho_2 \quad} & K(\mathsf{Z}_2,8) = \tilde{\bar{\bar E}}_1 \\
\downarrow{\scriptstyle Sq^2\bar\varrho_2} & & \downarrow{\scriptstyle \tilde{\bar h}_1} \\
K(\mathsf{Z}_2,6) & \xrightarrow{(Sq^{2,1},\,0,\,Sq^6)} & K(\mathsf{Z}_2,9,10,12)
\end{array}$$

$$\begin{pmatrix} Sq^1 \\ Sq^2 \\ Sq^4 \end{pmatrix} = \tilde{\bar h}_1{}^*$$

$$j_0{}^*\bar f^* h_2{}^*\iota_2 = Sq^4 Sq^{2,1}\iota_5 = j_0{}^* Sq^4 u ,$$

where $j_0{}^*u = Sq^{2,1}\iota_5$. Now, if μ_Y is the multiplication in Y, one has

$$\mu_Y{}^*u = u\otimes 1 + 1\otimes u + \bar\varrho_2 x\otimes\bar\varrho_2 x, \qquad x = r_0{}^*\iota_4 .$$

It follows that

$$Sq^4u + \bar\varrho_2 x^3 + \bar f^* h_2{}^*\iota_2 \in PH^{12}(Y,\mathsf{Z}_2) \cap \ker j_0{}^* = r_0{}^*PH^{12}\big(K(\mathsf{Z},4),\mathsf{Z}_2\big) = 0.$$

Changing $\bar f$ so that $\bar f^* h_2{}^*$ is altered by Sq^4u, that is, altering $\bar f$ by $\alpha\colon Y \to K(\mathsf{Z}_2,8,9,11)$, $\alpha^*\iota_8 = u$, $\alpha^*\iota_j = 0$, $j = 9, 11$, one gets $\bar\varrho_2 x^3 \in \bar\varphi(x^2)$.

3. The proof of the main theorem.

Let X, μ be an H-space satisfying (1) and (2).

Suppose $PH^{14}(X,\mathsf{Z}_2) = 0$. Let $\hat\theta\colon \hat X \to X$ be the $K(\mathsf{Z},2)$-principal fibration induced by $\hat{g}_0 : X \to K(\mathbf{Z},3)$, $\hat{g}_0$ inducing isomorphism of $H^3(\cdot,\mathbf{Z})/$ torsion.

3.1 Lemma. *$H^*(\hat{X},\mathbf{Z}_2) = [H^*(X,\mathbf{Z}_2)/(x_3)] \otimes \mathbf{Z}_2(\hat{x}_4) \otimes \Lambda(Sq^1\hat{x}_4)$ as a Hopf algebra over the Steenrod algebra.*

Proof. If $\iota_3 \in H^3(K(\mathbf{Z},3),\mathbf{Z})$ is the fundamental class, then $\ker \hat{g}_0{}^*$ is the $\mathfrak{a}(2)$ ideal generated by $Sq^2\bar{\varrho}_2\iota_3$. Hence, $H^*(\hat{X},\mathbf{Z}_2)//\operatorname{im}\hat{\theta}^*$ is generated by the algebraic suspension of

$$Sq^2\bar{\varrho}_2\iota_3, \quad Sq^{4,2}\bar{\varrho}_2\iota_3, \ldots, Sq^{2^n,2^{n-1},\ldots 4,2}\bar{\varrho}_2\iota_3, \ldots, \quad Sq^3\bar{\varrho}_2\iota_3 .$$

Consequently

$$H^*(\hat{X},Z_2)//\operatorname{im}\hat{\theta}^* = \mathbf{Z}_2(\hat{x}_4)\otimes\Lambda(Sq^1\hat{x}_4) ,$$

as $Sq^2Sq^2\bar{\varrho}_2 = 0$, $Sq^2\hat{x}_4 = 0$ and, hence, $Sq^{4,1}\hat{x}_4 = Sq^5\hat{x}_4 = 0$. It follows that

$$H^*(\hat{X},\mathbf{Z}_2) \approx H^*(X,\mathbf{Z}_2)//(x_3)\otimes\mathbf{Z}_2[\hat{x}_4]\otimes\Lambda(Sq^1\hat{x}_4)$$

as algebras. Since $\hat{x}_4$ is obviously primitive, the above is actually an isomorphism of Hopf algebras. The only possible non-trivial extension over $\mathfrak{a}(2)$ of the above splitting is given by $Sq^{2,1}\hat{x}_4 = \varepsilon\hat{\theta}^*x_7$.

Now, X can be mapped into $Sp^{(5)} \approx Sp^{(6)}$. Moreover,

$$X^{(5)} \approx Sp^{(5)} \approx (S^3)^{(5)}$$

where $Y^{(k)}$ is the Postnikov approximation of Y. Since

$$\mathbf{Z}_2 \approx {}^2\pi_6(X) \neq {}^2\pi_6(Sp^{(6)}) = 0 ,$$

one has a fibration

$$X^{(6)} \xrightarrow{f^{(6)}} Sp^{(6)} \xrightarrow{h^{(6)}} K(\mathbf{Z}_2,7) .$$

where $h^{(6)*}\iota_7$ restricts to $\varrho_2\sigma^*w_8 \in H^7(Sp^{(6)},\mathbf{Z}_2)$. Lifting $f^{(6)}$ to $\hat{f}^{(6)} : \hat{X} \to \hat{S}p^{(6)} = Sp^{(6)}$ made 3-connected, one has

$$H^*(\widehat{Sp}^{(6)},\mathbf{Z}_2) = \mathbf{Z}_2[h^{(6)*}\iota_7]\otimes\mathbf{Z}_2[\hat{w}_4]\otimes\Lambda(Sq^1\hat{w}_4)$$

in $\dim \leqq 7$,

$$Sq^{2,1}\hat{w}_4 = h^{(6)*}\iota_7 \quad \text{and} \quad \hat{f}^{(6)*}\hat{w}_4 = \hat{x}_4 .$$

Hence $Sq^{2,1}\hat{x}_4 = 0$, and 3.1 follows.

3.2 Corollary. *$QH^*(X,\mathbf{Z}_2)/PH^*(X,\mathbf{Z}_2) \approx QH^*(\hat{X},\mathbf{Z}_2)/PH^*(\hat{X},\mathbf{Z}_2)$, where QA stands for the module of indecomposables.*

3.3. LEMMA. *Let $B_2(X)$ and $B_2(\hat{X})$ be the projective planes of X and $\hat{X}$, respectively. There exists a commutative diagram*

$$\begin{array}{ccc} B_2(\hat{X}) & \xrightarrow{\hat{f}} & \hat{B} \\ \Big\downarrow{\scriptstyle B_2(\theta)} & & \Big\downarrow{\scriptstyle j\circ\theta} \\ B_2(X) & \xrightarrow{f} & BSp^{(13)} \\ \Big\downarrow{\scriptstyle \hat{g}_1} & & \Big\downarrow{\scriptstyle g_2} \\ K(\mathsf{Z}_4, 8) & \xrightarrow{\varphi_1} & E_2 , \end{array}$$

where g_2 and φ_1 are given in 2.2 and 2.3, respectively, and

$$\varrho_2 \sigma^* \hat{g}_1{}^* \iota_0 = x_7$$

where $\iota_0 \in H^8(K(\mathsf{Z}_4, 8), \mathsf{Z}_4)$ is the fundamental class.

PROOF. As in the proof of 3.1 one has a fibration

$$[B_2(X)]^{(7)} \xrightarrow{f^{(7)}} BSp^{(7)} \to K(\mathsf{Z}_2, 8)$$

and hence, a diagram

$$\begin{array}{ccccc} [B_2(X)]^{(7)} & \xrightarrow{f^{(7)}} & BSp^{(7)} & & \\ \Big\downarrow{\scriptstyle \hat{g}_1'} & & \Big\downarrow & \searrow & \\ K(\mathsf{Z}_2, 8) & \longrightarrow & K(\mathsf{Z}_4, 8) & \xrightarrow{\varrho_2} & K(\mathsf{Z}_2, 8) . \end{array}$$

Since $\hat{g}_1{}'^* \iota_8$ is a reduction of a Z_4 class in $H^*(B_2(X), \mathsf{Z}_2)$, one gets

$$\begin{array}{ccc} B_2(X) & \xrightarrow{\tilde{f}^{(7)}} & BSp^{(7)} \\ \Big\downarrow{\scriptstyle \hat{g}_1} & & \Big\downarrow{\scriptstyle g_1{}^{(7)}} \\ K(\mathsf{Z}_4, 8) & \xrightarrow{\times 2} & K(\mathsf{Z}_4, 8) , \end{array}$$

where $g_1{}^{(7)}$ corresponds to g_1 in 2.2. Now, consider the mappings

$$B_1(X) = \Sigma X \xrightarrow{i} B_2(X) \xrightarrow{k} B_2(X), B_1(X)$$

and the induced sequence of Z_2 cohomology

$$H^*(B_2(X), B_1(X)) \approx \bar{H}^*(\Sigma X) \otimes \bar{H}^*(\Sigma X) \xrightarrow{k^*} H^*(B_2(X)) \xrightarrow{i} H^*(B_1(X)) .$$

It follows that $H^*(B_2(X), \mathsf{Z}_2)$ is generated (as a Z_2 module) in $\dim < 16$ by

$$u_4, u_8,\ u_4\cdot u_8,\ k^*(i^*u_4\otimes\bar{u}_{11}),\ k^*(\bar{u}_{11}\otimes i^*u_4) \quad \text{and} \quad (\Sigma^*\bar{u}_{11}=x_3x_7)\,.$$

Note that

$$\Sigma^* i^* u_{i+1} = x_i \in H^i(X, \mathsf{Z}_2),\ i = 3,7, \quad \text{and} \quad \hat{g}_1{}^*\varrho_2\iota_8 = u_8\,.$$

All are reductions of integral classes. Hence,

$$H^{15}\big(B_2(X), \mathsf{Z}_2\big) \subset \ker B_2(\hat\theta)^*\,,$$

and as $\pi_{11}(BSp)=0$, all k-invariants of BSp in $\dim \leqq 14$ vanish on $H^*\big(B_2(X), \pi_*(BSp)\big)$. It follows that $f^{(7)}$ can be lifted to $f\colon B_2(X)\to BSp^{(13)}$ and as

$$H^{15}\big(B_2(X), \mathsf{Z}_2\big) \subset \ker B_2(\hat\theta)^*$$

one gets $f'\colon B_2(\hat X) \to BSp^{(14)}=B$ to obtain the commutative diagram:

$$\begin{array}{ccc} & \overset{\hat f}{\nearrow} & \hat{B} \\ & & \downarrow{\scriptstyle \hat\theta} \\ B_2(\hat X) & \xrightarrow{f'} & B = BSp^{(14)} \\ \downarrow{\scriptstyle B_2(\hat\theta)} & & \downarrow{\scriptstyle j} \\ B_2(X) & \xrightarrow{f} & BSp^{(13)} \\ \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_1} \\ K(\mathsf{Z}_4, 8) & \xrightarrow{\times 2} & K(\mathsf{Z}_4, 8)\,. \end{array}$$

Finally

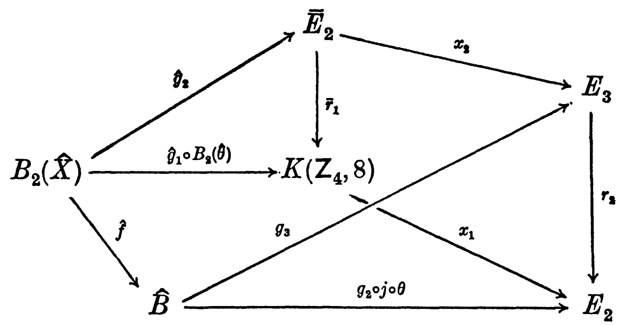
$$\begin{array}{ccc} B_2(X) & \xrightarrow{f} & BSp^{(13)} \\ \downarrow{\scriptstyle \hat g_1} & & \downarrow{\scriptstyle g_2} \\ K(\mathsf{Z}_4; 8) & \xrightarrow{\varphi_1} & E_2 \\ & \underset{\times 2\cdot}{\searrow} & \downarrow{\scriptstyle \bar r_1} \\ & & K(\mathsf{Z}_4, 8) \end{array}$$

is commutative as

$$[B_2(X), \Omega K_1] = H^9\big(B_2(X), \mathsf{Z}_2\big) \oplus H^{13}\big(B_2(X), \mathsf{Z}_2\big) = 0\,,$$

and 3.3 follows.

To complete the proof of the main theorem, we consider the diagram



The bottom face of this diagram is commutative by 3.3, the front and right hand faces commutativity follows from 2.2 and 2.3, respectively, and $g_3{}^*h_3{}^*\iota_3=0$.

As

$$\hat{g}_1{}^*\bar{h}_1{}^*\bar{\iota}_1 = \bar{B}_1 u_8 = (Sq^2, Sq^4)u_8 = (0, \varepsilon u_8 u_4), \quad \varepsilon\in \mathsf{Z}_2 ,$$

$B_2(\theta)^*\hat{g}_1{}^*\bar{h}_1{}^*\iota_1=0$; and $\hat{g}_2$ exists, and the back face of the above diagram is commutative.

Now the difference between $x_2\circ\hat{g}_2$ and $g_3\circ\hat{f}$ can be measured by an element α in

$$[B_2(\hat{X}),\Omega K_2] = H^{10}(B_2(\hat{X}),\mathsf{Z}_2)\oplus H^{14}(B_2(\hat{X}),\mathsf{Z}_2)\oplus H^{15}(B_2(\hat{X}),\mathsf{Z}_2) .$$

Further, $\hat{k}^*\colon H^m(B_2(\hat{X}),B_1(\hat{X});\,\mathsf{Z}_2) \to H^m(B_2(\hat{X}),\mathsf{Z}_2)$ is onto for $10\leqq m<16$, and consequently

$$\hat{g}_2{}^*x_2{}^*h_3{}^*\iota_3+\hat{f}^*g_3{}^*h_3{}^*\iota_3 = B_3\alpha = B_3\hat{k}^*\bar{\alpha} .$$

As $g_3{}^*h_3{}^*=0$ by 2.2 and

$$x_2{}^*h_3{}^*\iota_3 = Sq^4\bar{h}_2{}^*\iota_2+Sq^8\varrho_2\bar{r}_1{}^*\bar{\iota}_1$$

by 2.3, and since $Sq^4H^{12}(B_2(\hat{X},\mathsf{Z}_2))=0$, one has

$$B_3\hat{k}^*\bar{\alpha} = \hat{g}_2{}^*Sq^8\varrho_2\bar{r}_1{}^*\bar{\iota}_1 = B_2(\theta)^*u_8{}^2 = \hat{u}_8{}^2$$

or as $\hat{u}_8{}^2=\hat{k}^*(\hat{\imath}^*\hat{u}_8\otimes\hat{\imath}^*\hat{u}_8)$, where $\hat{\imath}\colon B_2(\hat{X}) \to B_1(\hat{X})$,

$$\hat{k}^*[\,\hat{\imath}^*\hat{u}_8\otimes\hat{\imath}^*\hat{u}_8)+B_3\bar{\alpha}] = 0 .$$

As $\hat{\imath}^*\hat{u}_8 \notin \overline{\mathcal{A}(2)}H^*(B_1(\hat{X}),\mathsf{Z}_2)$,

$$w = \hat{\imath}^*\hat{u}_8\otimes\hat{\imath}^*\hat{u}_8+B_3\bar{\alpha} \neq 0 ,$$

and there exists $\hat{u}_{15} \in H^{15}(B_1(\hat{X}), Z_2)$, $\delta\hat{u}_{15} = w$ with

$$\delta\colon H^*(B_1(\hat{X}), Z_2) \to H^*(B_2(\hat{X}), B_1(\hat{X}), Z_2).$$

Observe that

$$0 \neq \Sigma^* u_{15} \in QH^{14}(\hat{X}, Z_2)/PH^{14}(\hat{X}, Z_2) = QH^{14}(X, Z_2)/PH^{14}(X, Z_2)$$

by 3.2, and the main theorem follows.

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