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THE CLASSIFICATION OF SIMPLY CONNECTED H-SPACES WITH THREE CELLS II

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The classification problem of H-spaces with few cells was extensively studied. The case of H-spaces with a single cell (that is spheres) was completely solved in [1]. Some further studies concerning finite dimensional H-spaces are listed in the reference which provides only a small portion of the actual line of papers and only studies directly relevant to this present paper are mentioned.

The purpose of this study is to give the complete classification of simply connected CW complexes with three (non-trivial) cells which admit H-structures. This classification is given in the following

MAIN THEOREM. *Let X be a simply connected CW complex with three non-trivial cells. If X admits an H-structure, then X is homotopy equivalent to one of the following*:

$$S^3 \times S^3, \qquad \mathrm{SU}(3), \qquad M_k^{10},\ k=0,1,3,4,5, \qquad S^7 \times S^7 ,$$

where M_k^{10} is the principal S^3 bundle over S^7 induced by

$$kw \in [S^7, BS^3] = \pi_7(BS^3) \approx \mathsf{Z}_{12} ,$$

w being a generator.

The proof of the main theorem relies heavily on some known facts. In section 1 we assemble these facts and derive some simple conclusions.

In section 2 we review the technique of „mixing" and „twisting" of homotopy types used in [8] and [10].

The actual completion of the proof of the main theorem is carried out in section 3.

We would like to note that there is an overlap between this paper and independent studies of Hilton–Roitberg [6] and of M. Curtis–G. Mislin–E. Thomas (Private communication).

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1. Some known results and their simple consequences.

1.1. DEFINITION. Let X be a finite CW complex. X is said to be of type $(i_1, i_2, \ldots, i_n)$ if

$$H^*(X, \mathbf{Q}) = \Lambda(x_{i_1}, x_{i_2}, \ldots x_{i_n}), \qquad x_{i_j} \in H^{i_j}(X, \mathbf{Q}) .$$

1.2. THEOREM. *Let X be a simply connected CW complex with three (non-trivial) cells. If X admits an H-structure, then:*

(1) *$H^*(X, \mathbf{Z})$ has no odd torsion;*

(2) *$H^*(X, \mathbf{Z})$ has no 2-torsion;*

(3) *X is of type* (3,3), (3,5), (3,7) or (7,7);

(4) *If X is of type* (3,5), *then $H^*(X, \mathbf{Z}_2) = \Lambda(x_3, Sq^2 x_3)$;*

(5) *If X is of type* (3,7), *then ${}^2\pi_6(X) \neq \mathbf{Z}_2$, where ${}^2\pi_6(X)$ is the two primary component of $\pi_6(X)$,*

(6) $S^3 \times S^3$, SU(3), M_k^{10} $(k \not\equiv 2 \bmod 4)$, *and $S^7 \times S^7$ admit H-structures, M_k^{10} being the principal S^3 bundle over S^7 induced by $kw \in [S^7, BS^3] = \pi_7(BS^3) = \mathbf{Z}_{12}$, w being a generator.*

PROOF. (1) follows from Borel's structure theorem of Hopf algebras: If $H^*(X, \mathbf{Z})$ has p-torsion (p-odd), then $QH^*(X, \mathbf{Z}_p)$ will contain at least two generators y and $\beta_r y$, where β_r is the r-order Bockstein operation. Hence there exists $z \in QH^{\text{even}}(X, \mathbf{Z}_p)$, z and $z^2 \neq 0$, and the rank of $H^*(X, \mathbf{Z}_p)$ as a vector space over $\mathbf{Z}_p$ will be greater than 3.

(2) Using Borel's structure theorem for $p = 2$ the only way for $H^*(X, \mathbf{Z})$ to have 2-torsion and $H^*(X, \mathbf{Z}_2)$ still to be of rank $\leqq 3$ as a vector space over $\mathbf{Z}_2$ is, if $H^*(X, \mathbf{Z}_2) = \Lambda(x, \beta_r x)$. By [3, corollary 4.2], $r = 1$ and x is odd dimensional. By [9, theorem 1.1], $\dim x = 2^t - 1$, $t \geqq 2$, and consequently $\dim \beta x = 2^t$. We now use the methods in [9] and observe that $H^*(B_2(X), \mathbf{Z}_2)$, where $B_2(X)$ is the projective plane of X, contains a class y with $\dim y = 2^t + 1$, such that y suspends to βx and therefore $y \in \operatorname{im} \beta$ and $y^2 \neq 0$. As $Sq^{2^t - 2} y$ vanishes when restricted to $H^*(B_1(X), \mathbf{Z}_2) = H^*(\Sigma X, \mathbf{Z}_2)$, $Sq^{2^t-2} y$ is in the image of

$$H^*(B_2(X), B_1(X); \mathbf{Z}_2) \to H^*(B_2(X), \mathbf{Z}_2) .$$

But $H^*(B_2(X), B_1(X); \mathbf{Z}_2) \approx H^*(\Sigma X, \mathbf{Z}_2) \otimes H^*(\Sigma X, \mathbf{Z}_2)$, and thus

$$H^{2^{2t}-1}(B_2(X), B_1(X); \mathbf{Z}_2) = 0 .$$

Hence $Sq^{2^t-2} y = 0$ and

$$0 = (Sq^{2,1} Sq^{2^t-2} + Sq^{2^t} Sq^1) y = Sq^{2^t+1} y = y^2 .$$

A contradiction.

(3) follows from [4, theorem 1.1].

(4) follows from [9, theorem 1.1].

(5) follows from [11, main theorem I].

The non-classical part of (6) involves only $M_k{}^{10}$, $k=3,4,5$. This follows from $M^{10}_{12-k} \approx M_k{}^{10}$, $M_0{}^{10} \approx S^3 \times S^7$ and $M_1{}^{10} \approx \mathrm{Sp}(2)$. The fact that $M_5{}^{10}$ admits an H-structure has been proved in [5], and the cases $M_3{}^{10}$, $M_4{}^{10}$ have been treated in [8, theorem 2].

2. Mixing and twisting homotopy types.

In this section we review some of the ideas introduced in [10] and used in [8]. For the sake of simplicity, we shall make a few unnecessary assumptions to suit the applications in this present study.

Let X, μ be a simply connected finite dimensional H-space and suppose $H^*(X, \mathsf{Z}) \to H^*(X, \mathsf{Z})/\text{torsion}$ splits as a morphism of Hopf algebras. There exists an H-mapping

$$\psi : X \to \textstyle\prod_j K(\mathsf{Z}, n_j) = X_0$$

yielding an isomorphism of $H^*(\cdot, \mathsf{Z})/\text{torsion}$.

2.1. If $\psi^{(1)}\colon X \to X_0{}^{(1)} = \prod_j K(\mathsf{Z}, n_j)$
$\psi^{(2)}\colon X \to X_0{}^{(2)} = \prod_i K(\mathsf{Z}, m_i)$
are H-mappings yielding isomorphisms of $H^*(\cdot, \mathsf{Z})/\text{torsion}$, then there exists $\varphi\colon X_0{}^{(1)} \overset{\approx}{\to} X_0{}^{(2)}$ with $\varphi\psi^{(1)} \approx \psi^{(2)}$.

2.2. Let P be the set of primes, $\mathsf{P}_1 \subset \mathsf{P}$. There exists an H-space $X(\mathsf{P}_1, \psi)$, unique up to homotopy type, and H-mappings ψ', ψ'' so that

$$\begin{aligned} \psi'&\colon X \to X(\mathsf{P}_1, \psi)\,, \\ \psi''&\colon X(\mathsf{P}_1, \psi) \to X_0\,, \\ \psi''\circ\psi' &= \psi\,, \end{aligned}$$

and the fibers of ψ' and ψ'' have finite homotopy groups of orders prime to P_1 and $\mathsf{P} - \mathsf{P}_1$ respectively.

2.3. Mixing homotopy types: Let X_1 and X_2 be H-spaces. Suppose $H^*(X_i, \mathsf{Z}) \to H^*(X_i, \mathsf{Z})/\text{torsion}$ splits as a morphism of Hopf algebras, $i = 1,2$. Further assume $H^*(X_1, \mathsf{Z})/\text{torsion} \approx H^*(X_2, \mathsf{Z})/\text{torsion}$ as Hopf algebras. Thus we have H-mappings ψ_i:

$$\begin{array}{ccc} & & X_2 \\ & & \big\downarrow{\scriptstyle\psi_2} \\ X_1 & \xrightarrow{\psi_1} & X_0 = \prod_j K(\mathsf{Z}, n_j)\,. \end{array}$$

Let $\mathsf{P}_1 \subset \mathsf{P}$ and put $\mathsf{P}_2 = \mathsf{P} - \mathsf{P}_1$. The mixing of X_1, P_1 and X_2, P_2 is given by the H-space W. Consider the diagram

$$\begin{array}{ccc} W & \xrightarrow{r_2} & X_2(\mathsf{P}_2, \psi_2) \\ \downarrow{\scriptstyle r_1} & & \downarrow{\scriptstyle \psi_2''} \\ X_1(\mathsf{P}_1, \psi_1) & \xrightarrow{\psi_1''} & X_0 \end{array}$$

where W is the "pull back" of ψ_1'' and ψ_2'' given by

$$W \subset X_1(\mathsf{P}_1, \psi_1) \times PX_0 \times X_2(\mathsf{P}_2, \psi_2)$$
$$W = \{x_1, \alpha, x_2 \mid \psi_1''(x_1) = \alpha(0), \psi_2''(x_2) = \alpha(1)\}\,.$$

The map r_i yields an isomorphism of rational and mod p ($p \in \mathsf{P}_i$) cohomology where $p \in \mathsf{P}_i$, $i = 1, 2$. If X_i are finite dimensional so is W.

2.4. Example (see [8]). Let $\mathsf{P}_1 = \{2\}$, $X_1 = S^3 \times S^7$, $X_2 = \mathrm{Sp}(2)$, $X_0 = K(\mathsf{Z}, 3) \times K(\mathsf{Z}, 7)$. Then $W = M_4{}^{10} \approx M_8{}^{10}$. Choosing $\mathsf{P}_1 = \{3\}$, we get $W = M_3{}^{10} \approx M_9{}^{10}$.

2.4.1. Example. Suppose $2 \in \mathsf{P}_1$, $X_1 = G_n$, $X_2 = G_{n-1} \times S^{dn-1}$, $X_0 = \prod_{m=4/d}^{n} K(\mathsf{Z}, dm-1)$, where (G_n, d) is either $(\mathrm{SU}(n), 2)$ or $(\mathrm{Sp}(n), 4)$. It follows from a theorem of Adams [2] and can be proved by a method similar to the proof of Proposition 3.1 in [10] that $S^{\text{odd}}(\mathsf{P}_2)$ is an H-space. Hence, X is an H-space. If $\pi_{dn-2}(G_{n-1}) = \mathsf{Z}_m$, it seems that X has the homotopy type of $M(n, \lambda)$, where $M(n, \lambda)$ is obtained by the induced fibration

$$\begin{array}{ccc} & G_{n-1} & \\ \swarrow & & \searrow \\ M(n, \lambda) & \longrightarrow & G_n \\ \downarrow & & \downarrow \\ S^{dn-1} & \xrightarrow{f_\lambda} & S^{dn-1}\,, \end{array}$$

where $\deg f_\lambda = \lambda$ and λ is the maximal integer satisfying $(\lambda, p) = 1$ for $p \in \mathsf{P}_1$ and $\lambda \mid m$. (See conjecture 0.1.1 in [11].)

2.5. Twisting homotopy types: Let X, X_0 and ψ be as in 2.1–2.3. Let $\varphi \colon X_0 \to X_0$ be an H-mapping yielding an isomorphism of mod p cohomology, $p \in \mathsf{P}_1 \subset \mathsf{P}$. Put $\mathsf{P}_2 = \mathsf{P} - \mathsf{P}_1$. Consider the diagram

$$\begin{array}{ccccc} & & & & X(\mathsf{P}_1,\psi) \\ & & & & \downarrow {\scriptstyle \psi_1''} \\ X(\mathsf{P}_2,\psi) & \xrightarrow{\psi_2''} & X_0 & \xrightarrow{\varphi} & X_0 \,. \end{array}$$

The φ, P_1 twisting of X is the H-space $Z = Z(\varphi, \mathsf{P}_1)$ given by

$$\begin{array}{ccc} Z & \xrightarrow{r_2} & X(\mathsf{P}_1,\psi) \\ \downarrow {\scriptstyle r_1} & & \downarrow {\scriptstyle \psi_1''} \\ X(\mathsf{P}_2,\psi) & \xrightarrow{\varphi\circ\psi_2''} & X_0 \,, \end{array}$$

Z being the „pull back" given by

$$\begin{aligned} Z &\subset X(\mathsf{P}_1,\psi) \times PX_0 \times X(\mathsf{P}_2,\psi) \\ Z &= \{x_1, \alpha, x_2 \mid \psi_1''(x_1) = \alpha(0),\ \varphi\circ\psi_2''(x_2) = \alpha(1)\}\,. \end{aligned}$$

If X is finite dimensional so is Z. Further, Z is equivalent to X if and only if the morphism

$$\varphi^*\colon H^*(X,\mathsf{Z})/\text{torsion} \to H^*(X,\mathsf{Z})/\text{torsion}$$

can be realized geometrically.

2.6. A problem (related to conjecture 0.1.2 in [10]): Let X be an H-space. Suppose

$$\begin{gathered} H^*(X,\mathsf{Z}) = \Lambda(x_{2n_1+1}, x_{2n_2+1}, \ldots, x_{2n_k+1})\,, \\ x_{2n_j+1} \in PH^{2n_j+1}(X,\mathsf{Z}), \quad n_1 < n_2 < \ldots < n_k\,. \end{gathered}$$

It follows that $\pi_{2n_j+1}(X)/\text{torsion} = \mathsf{Z}$, $j = 1, \ldots k$. Let $f_j\colon X \to K(\mathsf{Z}, 2n_j+1)$ represent x_{2n_j+1} and suppose

$$f_{j\#}\colon \pi_{2n_j+1}(X)/\text{torsion} \to \pi_{2n_j+1}(K(\mathsf{Z}, 2n_j+1))$$

is given by $f_{j\#}(g_j') = \lambda_j g_j''$, where g_j', g_j'' are generators, λ_j being an integer. Let

$$\varphi_m\colon X_0 = \prod_{j=1}^{k} K(\mathsf{Z}, 2n_j+1) \to X_0$$

be given by

$$\begin{aligned} \varphi_m^* \iota_{2n_j+1} &= \iota_{2n_j+1} && \text{if } j<k \\ &= m\iota_{2n_k+1} && \text{if } j=k\,. \end{aligned}$$

Put $\mathsf{P}_1 = \mathsf{P}_1(m) = \{p \in \mathsf{P} \mid (p,m) = 1\}$. Is $Z(\varphi_m, \mathsf{P}_1(m)) \approx Z(\varphi_n, \mathsf{P}_1(n))$ if and only if $m \equiv \pm n \bmod \lambda_k$?

In particular we have ([7, theorem 2]): Let $X = \mathrm{Sp}(2)$, $\lambda_2 = 12$, and $\mathsf{P}_1(7) = \mathsf{P} - \{7\}$. Then

$$Z(\varphi_7, \mathsf{P}_1(7)) \approx M_7{}^{10} \not\approx Z(\varphi_1, \emptyset) \approx \mathrm{Sp}(2)$$

while $Z(\varphi_5, \{5\}) \approx M_5{}^{10} \approx M_7{}^{10}$.

We would like to note that if X is a loop space, so is $Z(\varphi_m, \mathsf{P}_1(m))$.

We also have the following:

2.7. Proposition. *Let X be a simply connected CW complex. $H^*(X, \mathsf{Z}) = \Lambda(x_3, x_7)$. Then we have*:

(a) $X(\mathsf{P}_1) = (S^3 \times S^7)(\mathsf{P}_1)$ *if* $\mathsf{P}_1 = \{p \in \mathsf{P} \mid p > 3\}$.

(b) *If* $X(\{2\}) = M^{10}_{k_1}(\{2\})$, $X(\{3\}) = M^{10}_{k_2}(\{3\})$, *then* $X \approx M^{10}_{k_3}$ *for some integer* k_3.

3. The proof of the main theorem.

In view of 1.2, the 3-celled H-spaces are divided by their types. We divide the main theorem accordingly. As simply connected complexes with

$$H^*(X, \mathsf{Z}) = \Lambda(x_3, x_3'), \quad x_3, x_3' \in H^3(X, \mathsf{Z}),$$

or

$$H^*(X, \mathsf{Z}) = \Lambda(x_7, x_7'), \quad x_7, x_7' \in H^7(X, \mathsf{Z}),$$

are necessarily homotopy equivalent to products of spheres, we have the following obvious statements:

3.0. Theorem. *A simply connected CW complex with three cells of type* $(3,3)$ *or* $(7,7)$ *which admit an H-structure is homotopy equivalent to* $S^3 \times S^3$ *or* $S^7 \times S^7$ *respectively.*

Next we treat the type $(3,5)$.

3.1. Lemma. $\pi_7(\mathrm{SU}(3)) = 0$.

Proof. We have $H^*(\mathrm{SU}, \mathsf{Z}_2) = \Lambda(w_3, w_5, w_7, w_9, w_{11}, \ldots)$ and $Sq^2 w_7 = w_9$. Further $H^*(\mathrm{SU}, \mathsf{Z}) = \Lambda(\tilde{w}_3, \tilde{w}_5, \tilde{w}_7, \ldots)$. Let $f_7\colon \mathrm{SU} \to K(\mathsf{Z}, 7)$ be the map realizing $\tilde{w}_7$, then $f_7{}^*$ is a monomorphism of rational cohomology and of mod p cohomology in $\dim \leq 9$. It follows that the fiber F_7 of f_7 satisfies $H^*(F_7, G) \approx \Lambda(w_3', w_5', w_7')$ in $\dim < 9$, $G = \mathsf{Q}$ or Z_p. The inclusion $\mathrm{SU}(3) \subset \mathrm{SU}$ can be lifted to $g\colon \mathrm{SU}(3) \to F_7$ and g^* is an isomorphism of rational and mod p cohomology through dimension 8, hence the fiber of g is 7-connected, $\pi_7(\mathrm{SU}(3)) \approx \pi_7(F_7)$. We have the exact sequence

$$0 \to \pi_7(F_7) \to \pi_7(\mathrm{SU}) \approx \mathsf{Z} \xrightarrow{f_{7\#}} \pi_7(K(\mathsf{Z},7)) \approx \mathsf{Z}\,,$$

the map $f_{7\#}$ not being 0 is a monomorphism and $\pi_7(F_7)=0$.

3.2. PROPOSITION. *Let X be a simply connected CW complex, $H^*(X,\mathsf{Z})=\Lambda(\tilde{x}_3,\tilde{x}_5)$. If $H^*(X,\mathsf{Z}_2)=\Lambda(x_3,Sq^2x_3)$, then X is homotopy equivalent to* SU(3).

PROOF. Consider the Postnikov system given by

$$\begin{array}{ccccc} & & E_2 & & \\ & & \big\downarrow{\scriptstyle r_7} & & \\ K(\mathsf{Z},5) & \xrightarrow{j_6} & E_1 & \xrightarrow{h_7} & K(\mathsf{Z}_2,7)\times K(\mathsf{Z}_3,7) \\ & & \big\downarrow{\scriptstyle r_6} & & \\ & & K(\mathsf{Z},3) & \xrightarrow{h_6} & K(\mathsf{Z},6)\,, \end{array}$$

r_j being the fibration induced by h_j:

$$h_6{}^*\tilde{\iota}_6=\tilde{\iota}_3{}^2, \quad j_6{}^*h_7{}^*\iota_7{}^{(2)} = Sq^2\iota_5{}', \quad h_7{}^*\iota_7{}^{(3)} = r_6{}^*\mathscr{P}_3{}^1\iota_3{}''\,,$$

where

$$\begin{gathered} \tilde{\iota}_3 \in H^3(K(\mathsf{Z},3),\mathsf{Z}),\ \tilde{\iota}_6 \in H^6(K(\mathsf{Z},3),\mathsf{Z})\,, \\ \iota_7{}^{(k)} \in H^7(K(\mathsf{Z}_2,7)\times K(\mathsf{Z}_3,7),\mathsf{Z}_k),\ k=2,3\,, \\ \iota_5{}' \in H^5(K(\mathsf{Z},5),\mathsf{Z}_2),\ \iota_3{}'' \in H^3(K(\mathsf{Z},3),\mathsf{Z}_3) \end{gathered}$$

are the (reductions of) the fundamental classes. We have:

$$\begin{array}{ccc} & & \mathrm{SU}(3) \\ & & \big\downarrow{\scriptstyle f_2} \\ X & \xrightarrow{f_1} & E_2\,, \end{array}$$

where E_2 is the Postnikov approximation of SU(3) in $\dim \leqq 6$, $f_{2\#}\colon \pi_m(\mathrm{SU}(3)) \to \pi_m(E_2)$ is an isomorphism for $m \leqq 6$, and $\pi_m(E_2)=0$ if $m>6$. By 3.1 it follows that $\pi_7(\mathrm{SU}(3))=0$. Hence, E_2 is the Postnikov approximation in $\dim \leqq 7$:

$$f_{2\#}\colon\ \pi_7(\mathrm{SU}(3)) \overset{\approx}{\to} \pi_7(E_2)\,.$$

Hence, $f_2{}^*\colon H^*(E_2,G) \to H^*(\mathrm{SU}(3),G)$ is an isomorphism in $\dim \leqq 8$. Now, $f_1{}^*\colon H^*(E_2,G) \to H^*(X,G)$ is an isomorphism in $\dim \leqq 7$. But since $H^*(X,G) \approx H^*(\mathrm{SU}(3),G)$, $f_1{}^*$ is an isomorphism through dim 8 and both

X and SU(3) are of the homotopy type of the 8-skeleton of E_2. Combining 1.2 (4) with 3.2 we have:

3.3. THEOREM. *A simply connected CW complex with three cells that admits an H-structure and is of type* (3, 5) *is homotopy equivalent to* SU(3).

Now we turn to the type (3, 7). We first note that in this case $S^3 \to X$ is the inclusion of the 6-skeleton. Hence $\pi_6(S^3) \to \pi_6(X)$ is onto, and thus $\pi_6(X) \approx \mathsf{Z}_k$, $k \mid 12$. By 1.2 (5), $\pi_6(X) = \mathsf{Z}_k$, $k \not\equiv 2 \pmod 4$.

Throughout this section, let X be a simply connected H-space with three cells of type (3,7). Let $\mathsf{P}_1 = \{2\}$, $\mathsf{P}_2 = \{3\}$.

3.4. PROPOSITION. *If* $H^*(X, \mathsf{Z}_3) = \Lambda(z_3, \mathscr{P}_3{}^1 z_3)$, *then* $X(\mathsf{P}_2) \approx \mathrm{Sp}(2)(\mathsf{P}_2)$.

PROOF. Consider the diagram

$$\begin{array}{ccccc} & & K(\mathsf{Z}_3, 6) & & \\ & \swarrow & & \searrow & \\ & \overset{\psi_1}{\nearrow} \; E_1 & \longrightarrow & LK(\mathsf{Z}_3, 7) & \\ & \downarrow r_1 & & \downarrow & \\ X \xrightarrow{\;\psi\;} & X_0 \approx K(\mathsf{Z}, 3) \times K(\mathsf{Z}, 7) & \xrightarrow{\;h_7\;} & K(\mathsf{Z}_3, 7) , & \end{array}$$

where r_1 is the fibration induced by h_7, $h_7{}^* \iota_7 \equiv \mathscr{P}^1 \bar{\iota}_3 \otimes 1 - 1 \otimes \bar{\iota}_7$, $0 \neq \bar{\iota}_j \in H^j(K(\mathsf{Z}, j), \mathsf{Z}_3) \approx \mathsf{Z}_3$, and ψ induces isomorphism of $H^*(X_0, \mathsf{Z})/\text{torsion} \to H^*(X, \mathsf{Z})$. Now ψ may be chosen so that $h_7 \circ \psi \sim *$. Hence, ψ lifts to ψ_1. As $H^*(E_1, \mathsf{Z}_3) = \Lambda(r_1{}^*(\bar{\iota}_3 \otimes 1), r_1{}^*(1 \otimes \bar{\iota}_7)) = \Lambda(z_3{}', z_7{}')$ in $\dim \leqq 10$, E_1 is the Postnikov approximation to $\mathrm{Sp}(2)(\mathsf{P}_2)$ in $\dim \leqq 9$. Let

$$\ldots \to E_k \xrightarrow{\;r_k\;} E_{k-1} \to \ldots \to E_1$$

be the Postnikov system for $\mathrm{Sp}(2)(\mathsf{P}_2)$. As the k-invariants of E_j, $j \geqq 1$, are of $\dim > 10$ while $H^m(X, G) = 0$, $m > 10$, the map ψ_1 lifts to $\psi_\infty \colon X \to \mathrm{Sp}(2)(\mathsf{P}_2)$, and ψ_∞ is an isomorphism of rational and mod 3 cohomology factoring ψ. Therefore, we have $\mathrm{Sp}(2)(\mathsf{P}_2) \approx X(\mathsf{P}_2, \psi)$.

3.5. PROPOSITION. *If* $H^*(X, \mathsf{Z}_3) = \Lambda(z_3, z_7)$ *and* $\mathscr{P}^1 z_3 = 0$, *then*

$$X(\mathsf{P}_2) \approx (S^3 \times S^7)(\mathsf{P}_2) \approx S^3(\mathsf{P}_2) \times S^7(\mathsf{P}_2) .$$

PROOF. Consider the diagram

$$\begin{array}{ccccc}
 & \nearrow^{\tilde\psi_1} & \tilde E_1 & \longrightarrow & LK(\mathsf{Z}_3,7) \\
 & & \downarrow{\scriptstyle \tilde r_7} & & \downarrow \\
X & \xrightarrow[\tilde\psi]{} & X_0 = K(\mathsf{Z},3)\times K(\mathsf{Z},7) & \xrightarrow[\tilde h_7]{} & K(\mathsf{Z}_3,7) \\
 & & \uparrow{\scriptstyle j_2} & & \\
 & & S^7(\mathsf{P}_2)\ , & &
\end{array}$$

where $\tilde h_7{}^*\iota_7=\mathscr{P}^1\iota_3\otimes 1$, $\tilde\psi^*\colon H^*(X_0,\mathsf{Z})/\text{torsion}\to H^*(X,\mathsf{Z})$ is an isomorphism, and j_2 is the composition

$$S^7(\mathsf{P}_2)\to K(\mathsf{Z},7)\to K(\mathsf{Z},3)\times K(\mathsf{Z},7)\ .$$

The map $\tilde\psi$ lifts to $\tilde\psi_1$, and j_2 lifts to $\tilde j_2\colon S^7(\mathsf{P}_2)\to\tilde E_1$. Now, $\tilde E_1$ is the Postnikov approximation for $X(\mathsf{P}_2)$ in $\dim\leqq 6$. Hence the k-invariants leading from $\tilde E_1$ to $X(\mathsf{P}_2)$ are all of $\dim>7$. Hence, $\tilde j_2$ lifts to $j\colon S^7(\mathsf{P}_2)\to X(\mathsf{P}_2)$. Let

$$\alpha\colon S^3(\mathsf{P}_2)\times S^7(\mathsf{P}_2)\to X(\mathsf{P}_2)$$

be the mapping given by $\alpha(a,b)=\mu_2(h(a),j(b))$, where μ_2 is the multiplication in $X(\mathsf{P}_2)$ and $h\colon S^3(\mathsf{P}_2)\to X(\mathsf{P}_2)$ is induced by $S^3\subset X$. Then α is an isomorphism of $H^*(\cdot,\mathsf{Z})/\text{torsion}$ and mod 3 cohomology, hence a homotopy equivalence.

3.6. COROLLARY. $X(\mathsf{P}_2)\approx M_k{}^{10}(\mathsf{P}_2)$ *for* $k=0$ *or* 1.

3.7. PROPOSITION. *If* ${}^2\pi_6(X)\approx 0$, *then* $X(\mathsf{P}_1)\approx \mathrm{Sp}(2)(\mathsf{P}_1)$.

PROOF. Let E_5 and E_5' be the Postnikov approximation of $X(\mathsf{P}_1)$ and $\mathrm{Sp}(2)(\mathsf{P}_1)$ in $\dim\leqq 5$. Then as S^3 is the 6-skeleton of X and $\mathrm{Sp}(2)$, it follows that $E_5\approx E_5'$ as H-spaces. Since

$${}^2\pi_6(X)\ =\ \pi_6(X(\mathsf{P}_1))\ =\ 0\ =\ \pi_6(\mathrm{Sp}(2)(\mathsf{P}_1))\ ,$$

$E_6=E_5$ and $E_6'=E_5'$ are indeed the Postnikov approximations in $\dim\leqq 6$. Hence, the k-invariants from now on are of $\dim\geqq 8$. We thus have:

$$\begin{array}{ccccc} & & E_j & & \\ & & \downarrow & & \\ & & \vdots & & \\ & & E'_9 & & \\ & & \downarrow & & \\ & \xrightarrow{f'_8} & E'_8 & & \\ & & \downarrow & & \\ & \xrightarrow{f'_7} & E'_7 & & \\ & & \downarrow & & \\ X & \xrightarrow[f'_6]{} & E'_6 & \xrightarrow[h'_6]{} & K\big(\pi_7(\mathrm{Sp}(2)(\mathsf{P}_1)),8\big)\,. \end{array}$$

The map $f_6'^*$ yields an isomorphism of mod 2 cohomology and of $H^*(\cdot,\mathsf{Z})/\text{torsion}$ in $\dim \leqq 7$. Now f_6' is an H-mapping, and since $H^8(X,G)=0$, the map f_6' lifts to $f_7'\colon X \to E_7'$. The obstruction to lift the H-structure of f_6' lies in $H^7\big(X\wedge X,\, \pi_7(\mathrm{Sp}(2)(\mathsf{P}_1))\big)=0$. Hence, f_7' is an H-mapping. As $H^9(X,G)=0$, and $H^8(X\wedge X,G)=0$, the map f_7' lifts to an H-mapping $f_8'\colon X \to E_8'$.

Now the next k-invariant α_{10} is primitive and as $PH^{10}(X,G)=0$, it follows that $f_8'^*\alpha_{10}=0$, and f_8' lifts to $f_9'\colon X \to E_9'$.

From now on all k-invariants are of $\dim > 10$ and $H^k(X,G)=0$ if $k>10$. Hence, $f_6'\colon X \to E_6'$ lifts to $f=f_\infty\colon X \to \mathrm{Sp}(2)(\mathsf{P}_1)$.

f induces an isomorphism of $H^*(\cdot,\mathsf{Z})/\text{torsion}$ and of mod 2 cohomology, hence $\mathrm{Sp}(2)(\mathsf{P}_1)\approx X(\mathsf{P}_1)$.

3.8. Proposition. *If* ${}^2\pi_6(X)=\mathsf{Z}_4$, *then* $X(\mathsf{P}_1)\approx S^3(\mathsf{P}_1)\times S^7(\mathsf{P}_1)$.

Proof. The proof is similar to that of 3.5. We are seeking a lifting of f_0 as follows:

$$\begin{array}{ccc} & \xrightarrow{\quad f \quad} & X(\mathsf{P}_1) \\ & & \downarrow{\scriptstyle \varphi''} \\ S^3\times S^7 & \xrightarrow{f_0} & K(\mathsf{Z},3)\times K(\mathsf{Z},7)\,, \end{array}$$

where f_0 inducing isomorphism of $H^*(\cdot,\mathsf{Z})/\text{torsion}$. Now

$$S^3 \xrightarrow{j_1} S^3\times S^7 \xrightarrow{f_0} K(\mathsf{Z},3)\times K(\mathsf{Z},7)$$

lifts. If we can lift

$$S^7 \xrightarrow{j_2} S^3\times S^7 \xrightarrow{f_0} K(\mathsf{Z},3)\times K(\mathsf{Z},7)$$

by multiplying these two liftings we obtain a lifting f.

Now, $S^3 \subset X$ is the inclusion of the 6-skeleton. Hence, $\pi_6(S^3) \to \pi_6(X)$ is onto and so is $\pi_6(S^3(\mathsf{P}_1)) \to \pi_6(X(\mathsf{P}_1))$. But these two groups are isomorphic to Z_4. Hence

$$\pi_6(S^3(\mathsf{P}_1)) \stackrel{\approx}{\to} \pi_6(X(\mathsf{P}_1))$$

and therefore the 6-dimensional Postnikov approximations E_6'' of $S^3(\mathsf{P}_1)$ and $X(\mathsf{P}_1)$ coincide, and

$$H^*(E_6'', G) \to H^*(X(\mathsf{P}_1), G), \qquad G = \mathsf{Z} \text{ or } \mathsf{Z}_2,$$

is a monomorphism in $\dim \leqq 7$. Further $H^7(E_6'', G) = H^7(S^3(\mathsf{P}_1), G) = 0$. Consider the mapping

$$f_7'' = (f_6'' \times h_7)\Delta : X(\mathsf{P}_1) \to E_6'' \times K(\mathsf{Z}, 7) = E_7''$$

given by: $f_6'' : X(\mathsf{P}_1) \to E_6''$ is the approximation, and $h_7 : X(\mathsf{P}_1) \to K(\mathsf{Z}, 7)$ yields an isomorphism of $H^7(\cdot, \mathsf{Z})/\text{torsion}$. Now f_7'' induces isomorphism of $H^*(\cdot, \mathsf{Z})/\text{torsion}$ and of mod 2 cohomology in $\dim \leqq 7$. Hence E_7'', f_7'' is the Postnikov approximation of $X(\mathsf{P}_1)$ in $\dim \leqq 6$. It follows that the mapping $S^7 \to E_6'' \times K(\mathsf{Z}, 7)$ given by the composition

$$S^7 \xrightarrow{g_7} K(\mathsf{Z}, 7) \to E_6'' \times K(\mathsf{Z}, 7),$$

where g_7 is the inclusion of the 7-skeleton, lifts to $S^7 \to X(\mathsf{P}_1)$, and this is a lifting of

$$S^7 \xrightarrow{j_2} S^3 \times S^7 \xrightarrow{f_0} K(\mathsf{Z}, 3) \times K(\mathsf{Z}, 7).$$

3.9. Corollary. *$X(\mathsf{P}_1) \approx M_k^{10}(\mathsf{P}_1)$, $k = 0$ or 1.*

Combining 1.2 (5), 2.7 (6), 3.6 and 3.8 we obtain:

3.10. Theorem. *Let X be a simply connected CW complex with three cells and of type (3,7). If X admits an H-structure, then $X \approx M_k^{10}$, $k \not\equiv 2 \pmod 4$.*

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