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THE ATTAINMENT SET OF THE φ-ENVELOPE AND GENERICITY PROPERTIES

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Abstract

The attainment set of the φ-envelope of a function at a given point is investigated. The inclusion of the attainment set of the φ-envelope of the closed convex hull of a function into the attainment set of the function is preserved in sufficiently general settings to encompass the case φ being a norm in a power not less than 1. The non-emptiness of the attainment set is guaranteed on generic subsets of a given space, in several fundamental cases.

1. Introduction

Given a normed space $(X, \|\cdot\|)$, two reals $\lambda > 0$ and $p \geq 1$, and an extended real-valued function $f\colon X \to \mathbb{R} \cup \{+\infty\}$, the *Klee envelope* of f with index λ and power p is defined by

$$\kappa_{\lambda,p} f(x) := \sup_{y\in X}\left(\frac{1}{p\lambda}\|x-y\|^p - f(y)\right) \quad \text{for all } x \in X,$$

and the *attainment set* is

$$Q_{\lambda,p} f(x) := \left\{ y \in X : \frac{1}{p\lambda}\|x-y\|^p - f(y) = \kappa_{\lambda,p} f(x)\right\}.$$

The Klee envelope is extensively studied. There are a lot of results concerning its properties and its attainment set, see, e.g., [3], [7], [17] and references therein. The Klee envelope is a suitable extension of the farthest distance function widely developed, e.g., in [1], [9], [10], [13], [14]. Recently, various new properties of the Klee envelope of a function have been obtained through the notion of *norm subdifferential local uniform convexity* (NSLUC, for short), see [12]. Among them, assuming that f is a proper lower semicontinuous function whose effective domain $\operatorname{dom} f := \{x \in X : f(x) < +\infty\}$ is bounded, the inclusions

$$Q_{\lambda,p} f(x) \subset Q_{\lambda,p}(\overline{\operatorname{co}}\, f)(x) \subset \operatorname{co} Q_{\lambda,p} f(x) \tag{1}$$

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are established, see [12, Theorem 2]. In the case $p = 1$, it is also shown (for any real $\alpha > 0$) in [12, Theorem 3] that, whenever $\kappa_{\lambda,1} f$ is finite at some point, then

$$\kappa_{\lambda,1} f(x) + \frac{1}{\lambda} \operatorname{dist}(x, W_\alpha) = m_\alpha, \tag{2}$$

where $m_\alpha := \alpha + \inf_{y \in X} \kappa_{\lambda,1} f(y)$; this allowed in [12] to reduce the important problem of singleton property of sets with unique farthest point to the other important problem of convexity of Chebyshev sets.

The Klee envelope is an important particular case of the Moreau supremal convolution. Thus, it is natural to look for counterparts of those results (for the Klee envelope) in more general cases. The aim of the present paper is to investigate attainment sets of the supremal convolution, whenever a general convex function $\varphi\colon X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is considered in place of the norm to the power p above. This is motivated by recent results in [6], where the concept of φ-envelope of a function is considered in depth. Given an extended real-valued function $\varphi\colon X \to \mathbb{R} \cup \{-\infty, +\infty\}$, the φ*-envelope* of the function $f\colon X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is defined by

$$f^\varphi(x) := \sup_{y \in X} \bigl(\varphi(x - y) \dot{+} (-f(y))\bigr) \quad \text{for all } x \in X,$$

where following Moreau [15]

$$\forall r, s \in \mathbb{R}, \quad r \dot{+} s := r + s, \quad r \dot{+} \infty := +\infty, \quad r \dot{+} (-\infty) := -\infty,$$
$$\text{and} \quad (+\infty) \dot{+} (-\infty) = (-\infty) \dot{+} (+\infty) = -\infty.$$

Of course, as said in [6], the function f^φ corresponds to the Moreau c-conjugate f^c of f with the coupling $c\colon X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ given by $c(x, y) := \varphi(x - y)$. The structure of the transform $f \mapsto f^\varphi$ is examined in great detail in [6] and various properties of φ-envelopes are provided.

The paper is organized as follows. In the next section we present preliminary results used throughout the development. In Section 3 we recall the notion of NSLUC sets and give many examples from [12] of NSLUC sets, relating in this way that concept to the ones of locally uniformly rotund property (LUR, for short), Kadec-Klee property, etc. We also prove some new results for NSLUC sets which are crucial in the following sections, see, e.g., Proposition 3.7. Section 4 is devoted to generalize (1) to the general fundamental case when the φ-envelope of f is involved instead of the Klee envelope. This is achieved in Theorem 4.1. The genericity of non-emptiness of the attainment set is considered in Section 5 and shown in Theorem 5.1 therein. Finally, in Section 6 the identity given in (2) is obtained for the φ-envelope of f, see Theorem 6.1.

2. Preliminaries

Throughout the paper, given a normed space $(X, \|\cdot\|)$ we will denote by $B[x, r]$ (resp. $B(x, r)$) the closed (resp. open) ball centered at $x \in X$ and with radius $r > 0$. It will be convenient to denote by $\mathbb{B}_X$ and $\mathbb{S}_X$ the closed unit ball and closed unit sphere of X, that is $\mathbb{B}_X := B[0, 1]$ and $\mathbb{S}_X := \{x \in X : \|x\| = 1\}$. As usual, we will also denote by $\mathbb{N}$ the set of all positive integers and by $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ the extended real line. The indicator function of a set $S \subset X$ is defined by

$$\delta_S(x) = \begin{cases} 0, & \text{if } x \in S, \\ +\infty, & \text{if } x \notin S. \end{cases}$$

The closure, the convex hull and the closed convex hull of $S \subset X$ are denoted by $\operatorname{cl} S$, $\operatorname{co} S$ and $\overline{\operatorname{co}}\, S$ respectively. For a function $f: X \to \overline{\mathbb{R}}$, we denote its effective domain by $\operatorname{dom} f$, that is,

$$\operatorname{dom} f := \{x \in X : f(x) < +\infty\}.$$

We call f a proper function if $f(x) < +\infty$ for at least one $x \in X$, and $f(x) > -\infty$ for all $x \in X$, or in other words, if $\operatorname{dom} f$ is a non-empty set on which f is finite. The function which is constantly equal to $+\infty$ (resp. $-\infty$) on X is denoted by ω_X (resp. $-\omega_X$). A function $f: X \to \overline{\mathbb{R}}$ is said to be *Fréchet differentiable* at a point x where it is finite whenever as usual there is some $x^* \in X^*$ (the topological dual space of X) such that

$$\frac{f(x') - f(x) - \langle x^*, x' - x\rangle}{\|x' - x\|} \to 0 \quad \text{as } x' \to x;$$

in such a case f is necessarily finite near x. The continuous linear functional x^* is called the *Fréchet derivative* of f at x and it is denoted by $D_F f(x)$. Similarly, when for any $h \in X$,

$$t^{-1}\big(f(x + th) - f(x) - t\langle x^*, h\rangle\big) \to 0 \quad \text{as } t \to 0,$$

one says that f is *Gâteaux differentiable* and such an element x^* is generally denoted by $D_G f(x)$.

If f is convex, its *Moreau-Rockafellar subdifferential* is defined as $\partial f(x) = \emptyset$ if f is not finite at x and if f is finite at x

$$\partial f(x) := \{x^* \in X^* : \langle x^*, x' - x\rangle \le f(x') - f(x), \ \forall x' \in X\}.$$

More generally, given a real $\delta \ge 0$ and a point $x \in X$ with $|f(x)| < +\infty$ the set

$$\partial_\delta f(x) := \{x^* \in X^* : \langle x^*, x' - x\rangle \le \delta + f(x') - f(x), \ \forall x' \in X\}$$

is the δ-subdifferential of f at x, and as above one puts $\partial_\delta f(x) = \emptyset$ when $f(x)$ is not finite. Of course, $\partial_\delta f(x)$ coincides with $\partial f(x)$ whenever $\delta = 0$. The domain of the set-valued mapping $\partial_\delta f$ is

$$\operatorname{dom} \partial_\delta f := \{x \in X : \partial_\delta f(x) \neq \emptyset\}.$$

The *normal cone* to a convex set $S \subset X$ at $x \in \operatorname{cl} S$ is defined by

$$N(S, x) := \{x^* \in X^* : \langle x^*, u - x\rangle \leq 0 \,\forall u \in S\}.$$

Consider now the concepts of convex hull and closed convex hull of an extended real-valued function. For a function $f\colon X \to \mathbb{R} \cup \{+\infty\}$, one defines its *convex hull* $\operatorname{co} f\colon X \to \overline{\mathbb{R}}$ by

$$\operatorname{co} f(x) := \inf\{r \in \mathbb{R} : (x, r) \in \operatorname{co}(\operatorname{epi} f)\},$$

where $\operatorname{epi} f$ denotes the epigraph of f, that is,

$$\operatorname{epi} f := \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}.$$

Obviously, it is the greatest convex function majorized by f and

$$\operatorname{co} f(x) = \inf\left\{\sum_{i=1}^{m} t_i f(y_i) : y_i \in X,\ t_i > 0,\ \sum_{i=1}^{m} t_i y_i = x,\ \sum_{i=1}^{m} t_i = 1\right\}. \quad (3)$$

Similarly, the *lower semicontinuous convex hull* (or *closed convex hull*) $\overline{\operatorname{co}}\, f\colon X \to \mathbb{R} \cup \{-\infty, +\infty\}$ of f is defined by

$$\overline{\operatorname{co}}\, f(x) := \inf\{r \in \mathbb{R} : (x, r) \in \overline{\operatorname{co}}(\operatorname{epi} f)\}.$$

From the above formulas one sees that $\overline{\operatorname{co}}\, f$ is convex and lower semicontinuous and it is the greatest lower semicontinuous convex function less or equal to f. It also satisfies the following properties

$$\overline{\operatorname{co}}(\operatorname{epi} f) = \operatorname{epi}(\overline{\operatorname{co}}\, f), \quad \operatorname{co}(\operatorname{dom} f) \subset \operatorname{dom} \overline{\operatorname{co}}\, f.$$

This allows us to express this closed convex hull function in the following manner in the case when f is lower semicontinuous: for all $x \in X$ and $n \in \mathbb{N}$ there exist $m_n \in \mathbb{N}$, $t_1^n, \ldots, t_{m_n}^n$ in $]0, 1]$ with $\sum_{i=1}^{m_n} t_i^n = 1$, and $y_1^n, \ldots, y_{m_n}^n$ in $\operatorname{dom} f$, such that

$$\lim_{n\to\infty} \sum_{i=1}^{m_n} t_i^n y_i^n = x \quad \text{and} \quad \overline{\operatorname{co}}\, f(x) = \lim_{n\to\infty} \sum_{i=1}^{m_n} t_i^n f(y_i^n).$$

Let us first state a result from [6] on the relationship between the *φ-envelope* of f and the *φ-envelope* of $\overline{\text{co}}\, f$; for the previous case when φ is a norm to a power not less than one see, for example, [12]. For the sake of completeness, we give a detailed proof.

PROPOSITION 2.1. *Let $(X, \|\cdot\|)$ be a normed space, $f: X \to \mathbb{R} \cup \{+\infty\}$ be a proper function and $\varphi: X \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. Let $x \in X$ be such that φ is lower semicontinuous on $x - \text{dom}\,\overline{\text{co}}\, f$. Then*

$$f^{\varphi}(x) = g^{\varphi}(x) = (\text{co}\, f)^{\varphi}(x) = (\overline{\text{co}}\, f)^{\varphi}(x), \tag{4}$$

where g is any function satisfying $\overline{\text{co}}\, f \le g \le f$.

PROOF. Since $\overline{\text{co}}\, f \le g \le f$, we see that $f^{\varphi} \le g^{\varphi} \le (\overline{\text{co}}\, f)^{\varphi}$. Thus (4) holds true, whenever $f^{\varphi}(x) = +\infty$. The case $f^{\varphi}(x) = -\infty$ is excluded since f is proper. Let us consider the case $f^{\varphi}(x) \in \mathbb{R}$. Let us take any $y \in X$, $m \in \mathbb{N}$, and $t_i > 0$, and $y_i \in X$ for every $i = 1, \ldots, m$, such that $\sum_{i=1}^{m} t_i = 1$ and $\sum_{i=1}^{m} t_i y_i = y$. We have

$$\varphi(x - y_i) \le f^{\varphi}(x) + f(y_i),$$

thus

$$\sum_{i=1}^{m} t_i \varphi(x - y_i) \le f^{\varphi}(x) + \sum_{i=1}^{m} t_i f(y_i),$$

so by the convexity of φ

$$\varphi(x - y) \le f^{\varphi}(x) + \sum_{i=1}^{m} t_i f(y_i).$$

Taking the infimum over all admissible convex combinations, that is, the infimum over the set in (3) with y playing the role of x, we obtain

$$\varphi(x - y) \le f^{\varphi}(x) + \inf\left\{\sum_{i=1}^{m} t_i f(y_i) : \sum_{i=1}^{m} t_i y_i = y,\ t_i > 0, \sum_{i=1}^{m} t_i = 1\right\},$$

or equivalently $\varphi(x - y) \le f^{\varphi}(x) + \text{co}\, f(y)$. This, combined with the lower semicontinuity of φ on $x - \text{dom}(\overline{\text{co}}\, f)$, yields $\varphi(x - y) \le f^{\varphi}(x) + \overline{\text{co}}\, f(y)$ for all $y \in \text{dom}(\overline{\text{co}}\, f)$. From the latter inequality we obtain that $(\overline{\text{co}}\, f)^{\varphi}(x) \le f^{\varphi}(x)$, hence the desired equalities

$$f^{\varphi}(x) = g^{\varphi}(x) = (\overline{\text{co}}\, f)^{\varphi}(x)$$

are true. Finally, taking $g = \text{co}\, f$ in the latter equalities gives the equality with $(\text{co}\, f)^{\varphi}(x)$.

Given functions $f, \varphi: X \to \overline{\mathbb{R}}$, and a real $\varepsilon \geq 0$, consider $x, y \in X$ such that $\varphi(x-y) \dotplus (-f(y)) \geq f^{\varphi}(x) - \varepsilon$ and note, for such x, y, that both $\varphi(x-y)$ and $f(y)$ are finite whenever $f^{\varphi}(x)$ is finite. Then, for every $x \in X$ the ε-*approximate attainment set* $M_{\varepsilon,\varphi}(x)$ is defined as follows: if $|f^{\varphi}(x)| = +\infty$ then $M_{\varepsilon,\varphi} f(x) := \emptyset$, and if $|f^{\varphi}(x)| < +\infty$ then

$$M_{\varepsilon,\varphi} f(x) := \{y \in X : \varphi(x-y) \dotplus (-f(y)) \geq f^{\varphi}(x) - \varepsilon\}$$
$$= \{y \in X : \varphi(x-y) - f(y) \geq f^{\varphi}(x) - \varepsilon\}, \tag{5}$$

where the second equality is due to the above comment. For $\varepsilon = 0$ the set $M_{\varepsilon,\varphi} f(x)$, called the *attainment set*, will be denoted by $M_{\varphi} f(x)$, so

$$M_{\varphi} f(x) := \{y \in X : \varphi(x-y) \dotplus (-f(y)) = f^{\varphi}(x)\}$$
$$= \{y \in X : \varphi(x-y) - f(y) = f^{\varphi}(x)\}$$

if $|f^{\varphi}(x)| < +\infty$, and $M_{\varphi} f(x) = \emptyset$ if $|f^{\varphi}(x)| = +\infty$. Our objective is to study the domain of the set-valued mapping $x \mapsto M_{\varphi} f(x)$, that is, the set of $x \in \operatorname{dom} f^{\varphi}$ for which the attainment set $M_{\varphi} f(x)$ is non-empty. Under suitable conditions, we will study, in particular, properties of sequences of approximating points, connections between $M_{\varphi}(\overline{\text{co}}\, f)(x)$ and $M_{\varphi} f(x)$, and the generic property of attainment points.

The next lemma, providing relations between the Fréchet subdifferentiability of $-f^{\varphi}$ and the differentiability of φ, will be used in Section 4. Recall that the Fréchet subdifferential of a function $g: X \to \overline{\mathbb{R}}$ at a point x where g is finite is given by

$$\partial_F g(x) = \left\{x^* \in X^* : \liminf_{x' \to x} \frac{g(x') - g(x) - \langle x^*, x' - x\rangle}{\|x' - x\|} \geq 0\right\}.$$

We adopt the convention $\partial_F g(x) = \emptyset$ if $g(x)$ is not finite.

LEMMA 2.1. *Let $(X, \|\cdot\|)$ be a normed space, and let $\varphi: X \to \mathbb{R} \cup \{+\infty\}$ and $f: X \to \overline{\mathbb{R}}$ be extended real-valued functions. Assume that f^{φ} is finite at x.*

(1) *If f^{φ} is continuous at x and φ is proper and convex, then for every $y \in \operatorname{dom} \overline{\text{co}}\, f$, the function φ is finite and continuous at $x - y$.*

(2) *If $\overline{y} \in M_{\varphi} f(x)$, then the following hold:*
  (i) *$\partial_F \varphi(x - \overline{y}) \subset \partial_F f^{\varphi}(x)$;*
  (ii) *$\partial_F(-f^{\varphi})(x) \subset \partial_F(-\varphi)(x - \overline{y})$;*
  (iii) *if $\partial_F(-f^{\varphi})(x) \neq \emptyset$ and if $\partial_F \varphi(x - \overline{y}) \neq \emptyset$, then f^{φ} is Fréchet differentiable at x and φ is Fréchet differentiable at $x - \overline{y}$ with $D_F f^{\varphi}(x) = D_F \varphi(x - \overline{y})$;*

(iv) *if f^φ is Fréchet differentiable at x and φ is proper and convex, then φ is Fréchet differentiable at $x - \overline{y}$ with $D_F\varphi(x - \overline{y}) = D_F f^\varphi(x)$.*

PROOF. (1) Following the proof of Proposition 2.1, we have

$$\forall x' \in \operatorname{dom} f^\varphi,\ \forall y \in X, \quad \varphi(x' - y) \le f^\varphi(x') + \operatorname{co} f(y).$$

Fix $\varepsilon > 0$ and, by the continuity of f^φ at x, take $\delta > 0$ such that

$$f^\varphi(x') \le f^\varphi(x) + \varepsilon, \quad \forall x' \in B(x, \delta),$$

which implies that for every $y \in \operatorname{dom} \operatorname{co} f$

$$\varphi(x' - y) \le \operatorname{co} f(y) + f^\varphi(x) + \varepsilon, \quad \forall x' \in B(x, \delta). \tag{6}$$

Let $y \in \operatorname{dom} \overline{\operatorname{co}} f$. There exists a sequence $\{z_k\}_{k\in\mathbb{N}}$ converging to y such that $\lim_{k\to+\infty} \operatorname{co} f(z_k) = \overline{\operatorname{co}} f(y)$. We may choose an integer $k \in \mathbb{N}$ such that $\operatorname{co} f(z_k) < \overline{\operatorname{co}} f(y) + \varepsilon$ and $\|z_k - y\| < \delta/2$. Thus relation (6) ensures that

$$\varphi(x' - z_k) \le \overline{\operatorname{co}} f(y) + f^\varphi(x) + 2\varepsilon, \quad \forall x' \in B(x, \delta),$$

and this implies that φ is bounded from above on $B(x - y, \delta/2)$. This guarantees, because of the convexity of φ, that φ is continuous at $x - y$.

(2)(i) The function f^φ is the supremal convolution of the functions φ and $-f$, hence the result can be deduced from [12, Lemma 1].

(ii) Let $x^* \in \partial_F(-f^\varphi)(x)$. There is some function $\varepsilon(x') \to 0$ as $x' \to x$ such that, for all x' near x,

$$\begin{aligned}\langle -x^*, x' - x\rangle &\ge f^\varphi(x') - f^\varphi(x) - \varepsilon(x')\|x' - x\| \\ &\ge \varphi(x' - \overline{y}) - f(\overline{y}) - \big(\varphi(x - \overline{y}) - f(\overline{y})\big) - \varepsilon(x')\|x' - x\| \\ &= \varphi(x' - \overline{y}) - \varphi(x - \overline{y}) - \varepsilon(x')\|x' - x\|.\end{aligned}$$

This shows that $x^* \in \partial_F(-\varphi)(x - \overline{y})$, which ends the proof of the inclusion $\partial_F(-f^\varphi)(x) \subset \partial_F(-\varphi)(x - \overline{y})$.

(iii) In view of (i) and (ii), the non-emptiness of $\partial_F\varphi(x-\overline{y})$ and $\partial_F(-f^\varphi)(x)$ implies the non-emptiness of $\partial_F f^\varphi(x)$ and $\partial_F(-\varphi)(x - \overline{y})$, which implies in turn the Fréchet differentiability of f^φ at x and the Fréchet differentiability of φ at $x - \overline{y}$. The equality $D_F f^\varphi(x) = D_F\varphi(x - \overline{y})$ then follows immediately from the inclusion of (i).

(iv) Since the convex function φ is finite and continuous at $x - \overline{y}$ by (1), it ensues that $\partial\varphi(x - \overline{y}) \ne \emptyset$, thus (iv) follows from (iii).

Finishing this preliminary section, we give a formula connecting the subdifferential of φ for maximizing sequences with the Fréchet derivative of f^φ.

PROPOSITION 2.2. *Let* $(X, \|\cdot\|)$ *be a normed space, let* $\varphi\colon X \to \mathbb{R} \cup \{+\infty\}$ *be a proper convex function and let* $f\colon X \to \overline{\mathbb{R}}$ *be a function such that* $f^{\varphi}(x)$ *is finite and the Fréchet derivative* $D_F f^{\varphi}(x)$ *exists. Let* $\{y_i\}_{i\in\mathbb{N}}$ *be any maximizing sequence for* $f^{\varphi}(x)$*, that is,*

$$\lim_{i\to\infty}\big(\varphi(x-y_i) - f(y_i)\big) = f^{\varphi}(x). \tag{7}$$

Then the following properties hold.

(a) *The function* φ *is continuous at* $x - y_i$ *for* $i \in \mathbb{N}$ *sufficiently large. If* $x_i^* \in \partial_{\delta_i}\varphi(x-y_i)$ *for all* $i \in \mathbb{N}$*, then*

$$\lim_{i\to\infty}\|x_i^* - D_F f^{\varphi}(x)\| = 0, \tag{8}$$

where $\delta_i \geq 0$ *for all* $i \in \mathbb{N}$ *and* $\lim_{i\to\infty}\delta_i = 0$.

(b) *Moreover, if* $\{y_{i_k}\}_{k\in\mathbb{N}}$ *is a subsequence converging weakly to* $\overline{y}$*, then the sequences* $\{\varphi(x-y_{i_k})\}_{k\in\mathbb{N}}$ *and* $\{f(y_{i_k})\}_{k\in\mathbb{N}}$ *converge to* $\varphi(x-\overline{y}) \in \mathbb{R}$ *and* $(\overline{\text{co}}\, f)(\overline{y}) \in \mathbb{R}$*, respectively. The limit point* $\overline{y}$ *satisfies* $\overline{y} \in M_{\varphi}(\overline{\text{co}}\, f)(x)$.

PROOF. (a) Since $f^{\varphi}(x)$ is finite, then we may suppose that all $f(y_i)$ and $\varphi(x-y_i)$ are finite, thus the continuity of φ at each $x - y_i$ is a consequence of Lemma 2.1(1). Consider any sequence $\{\delta_i\}_{i\in\mathbb{N}}$ with $\delta_i \geq 0$ and $\lim_{i\to\infty}\delta_i = 0$, and consider also, for each $i \in \mathbb{N}$, any $x_i^* \in \partial_{\delta_i}\varphi(x-y_i)$. Put

$$\varepsilon_i := \sqrt{\max\{f^{\varphi}(x) - (\varphi(x-y_i) - f(y_i)), \delta_i, i^{-1}\}}$$

and $x^* := D_F f^{\varphi}(x)$ as well. For all $h \in \mathbb{B}_X$ and all $i \in \mathbb{N}$, we have

$$\begin{aligned}\langle x_i^*, h\rangle &\leq \frac{\varepsilon_i^2 + \varphi(x+\varepsilon_i h - y_i) - \varphi(x-y_i)}{\varepsilon_i}\\ &\leq \frac{\varepsilon_i^2 + \varphi(x+\varepsilon_i h - y_i) - f(y_i) - (\varphi(x-y_i) - f(y_i))}{\varepsilon_i}\\ &\leq \frac{f^{\varphi}(x+\varepsilon_i h) - f^{\varphi}(x) + 2\varepsilon_i^2}{\varepsilon_i} \leq \langle x^*, h\rangle + 2\varepsilon_i + \theta_i,\end{aligned}$$

where $\theta_i \to 0$ uniformly with respect to $h \in \mathbb{B}_X$. This ensures that the sequence of subgradients is strongly convergent to the Fréchet derivative of f^{φ} at x.

(b) Note that, by Lemma 2.1, the function φ is continuous on $x - \text{dom}\,\overline{\text{co}}\, f$, and, by Proposition 2.1, $(\overline{\text{co}}\, f)^{\varphi}(x) = f^{\varphi}(x)$. Thus relation (7) ensures that

$$(\overline{\text{co}}\, f)^{\varphi}(x) = f^{\varphi}(x) = \lim_{i\to\infty}\big(\varphi(x-y_i) - \overline{\text{co}}\, f(y_i)\big). \tag{9}$$

Now, let $\{y_{i_k}\}_{k\in\mathbb{N}}$ be a subsequence converging weakly to $\overline{y}$. For large k, say $k \geq k_0$, by (a) the convex function φ is finite and continuous at $x - y_{i_k}$, so we can choose $x^*_{i_k} \in \partial\varphi(x - y_{i_k})$. By (a) again, we know that $\{x^*_{i_k}\}_{k\in\mathbb{N}}$ converges strongly to x^*. Thus for $k \geq k_0$ we have

$$\langle x^*_{i_k}, u + y_{i_k} - x\rangle + \varphi(x - y_{i_k}) \leq \varphi(u), \quad \forall u \in X,$$

which allows us to say (by taking, respectively, $u \in \operatorname{dom}\varphi$ and $u = x - \overline{y}$) that

$$\limsup_{k\to+\infty} \varphi(x - y_{i_k}) < +\infty \quad \text{and} \quad \limsup_{k\to+\infty} \varphi(x - y_{i_k}) \leq \varphi(x - \overline{y}).$$

On the other hand, relation (9) implies that

$$\begin{aligned}\liminf_{k\to+\infty} \varphi(x - y_{i_k}) &\geq f^\varphi(x) + \liminf_{k\to+\infty} \overline{\text{co}}\, f(y_{i_k})\\ &\geq f^\varphi(x) + \overline{\text{co}}\, f(\overline{y})\\ &\geq \varphi(x - \overline{y}).\end{aligned}$$

Thus we have $\lim_{k\to+\infty} \varphi(x - y_{i_k}) = \varphi(x - \overline{y}) \in \mathbb{R}$. We then deduce from the last inequalities above that $f^\varphi(x) + \overline{\text{co}}\, f(\overline{y}) = \varphi(x - \overline{y})$. By using relations (7) and (9), we finally obtain $\lim_{k\to+\infty} f(y_{i_k}) = \lim_{k\to+\infty} \overline{\text{co}}\, f(y_{i_k}) = \overline{\text{co}}\, f(\overline{y})$, which completes the proof.

3. NSLUC sets

We begin this section by recalling the definition of the NSLUC property, which was introduced in [12].

Definition 3.1. Let S be a subset of the normed space $(X, \|\cdot\|)$. We say that S has the *norm subdifferential local uniform convexity property*, NSLUC property for short, if for every bounded subset $S' \subset S$ with $0 \notin \text{cl}_{\|\cdot\|} S'$ and every $u \in \mathbb{S}_X$ for which there is a continuous linear functional $u^* \in \partial\|\cdot\|(u)$ satisfying

$$\inf_{s'\in S'} \|s' - \langle u^*, s'\rangle u\| > 0,$$

one can find a real $\beta > 0$ such that

$$\forall s' \in S', \quad \|s'\| \geq |\langle u^*, s'\rangle| + \beta\|s' - \langle u^*, s'\rangle u\|.$$

Of course, the NSLUC property holds for any subset of a set having this property. Several characterizations of the NSLUC property were established in [12]. To cite some of them, given a norm $\|\cdot\|$ on a vector space X we need to recall some definitions. The norm $\|\cdot\|$ is *strictly convex on a subset S of X* if the following implication holds true:

$$x, y \in S,\ \|x + y\| = \|x\| + \|y\|,\ x \neq 0, y \neq 0 \implies x = \frac{\|x\|}{\|y\|} y. \quad (10)$$

If (10) holds true for $S = \mathbb{S}_X$, then one just says that the norm is *strictly convex* or the space $(X, \|\cdot\|)$ is *strictly convex*.

We say that a set $S \subset X$ has the *Kadec-Klee property* with respect to the norm $\|\cdot\|$ whenever any sequence $\{x_i\}_{i\in\mathbb{N}}$ of elements of S converging weakly to $x \in X$ along with $\lim_{i\to\infty} \|x_i\| = \|x\|$ converges strongly to x (that is, $\|x_i - x\| \to 0$ as $i \to \infty$). So, the norm $\|\cdot\|$ has the Kadec-Klee property if and only if the whole set X has the Kadec-Klee property with respect to $\|\cdot\|$. As recalled in the next proposition, see [12] for details, the NSLUC property entails both the strict convexity and the Kadec-Klee property.

PROPOSITION 3.1. *Let S be a set in the normed space $(X, \|\cdot\|)$ having the NSLUC property. Then the norm $\|\cdot\|$ is strictly convex on S and S has the Kadec-Klee property with respect to $\|\cdot\|$.*

We recall (see [7]) that the norm $\|\cdot\|$ of X is *locally uniformly rotund* (LUR, for short) at a given point $x \in X\setminus\{0\}$ if $\lim_{i\to\infty} \|x_i - x\| = 0$, whenever $\{x_i\}_{i\in\mathbb{N}}$ is a sequence in X such that $\lim_{i\to\infty} \|x_i\| = \|x\|$ and $\lim_{i\to\infty} \|x_i + x\| = 2\|x\|$. Clearly (see, e.g., [8]), the norm $\|\cdot\|$ is locally uniformly rotund (LUR) or simply $(X, \|\cdot\|)$ is LUR if it is LUR at each $x \in X \setminus \{0\}$. The following proposition on the NSLUC property of every LUR space was obtained in [12].

PROPOSITION 3.2. *If the normed space $(X, \|\cdot\|)$ is LUR, then X has the NSLUC property.*

In the next propositions the notion of relative ball compactness is used to get the NSLUC properties. We say that a subset S of the normed space $(X, \|\cdot\|)$ is *relatively ball compact* (resp. *relatively weakly sequentially ball compact*) whenever the intersection of S with any closed ball is relatively compact (resp. relatively weakly sequentially compact). The following result was obtained in [12]. There it is stated that a weakly relatively ball compact set has the NSLUC property whenever the norm is strictly convex and has the Kadec-Klee property.

PROPOSITION 3.3. *Let S be a relatively weakly sequentially ball compact subset of the normed space $(X, \|\cdot\|)$. If the norm $\|\cdot\|$ is strictly convex (or equivalently the norm is strictly convex on $\mathbb{S}_X$) and the set S has the Kadec-Klee property, then S has the NSLUC property.*

The strict convexity of a normed space can be also characterized through the NSLUC property for some class of sets, we refer to [12] for details.

PROPOSITION 3.4. *A normed space* $(X, \|\cdot\|)$ *is strictly convex if and only if every relatively ball compact subset* S *of* X *has the NSLUC property.*

The NSLUC property can be obtained by the use of the dual norm. The Gâteaux differentiability of the dual norm with the Kadec-Klee property of the norm guarantees the NSLUC property of the space, see [12].

PROPOSITION 3.5. *Let* $(X, \|\cdot\|)$ *be a Banach space whose norm has the Kadec-Klee property and the dual norm is Gâteaux differentiable off the origin. Then* X *has the NSLUC property.*

Now we present some new results using the NSLUC property and which will be crucial in the development in Section 4. The first one is related to the limit superior of approximate subdifferentials of the norm. Recall that, for a topological space (Z, τ) and a sequence of subsets $\{S_i\}_{i\in\mathbb{N}}$, the τ-limits inferior and superior are defined as

$$
\begin{aligned}
{}^{\tau}\operatorname*{Lim\,inf}_{i\to\infty} S_i &:= \Big\{z \in Z : z = \tau - \lim_{i\to\infty} z_i,\ \ z_i \in S_i, \forall i \in \mathbb{N}\Big\},\\
{}^{\tau}\operatorname*{Lim\,sup}_{i\to\infty} S_i &:= \Big\{z \in Z : \text{there is a subsequence } \{i_k\}_{k\in\mathbb{N}} \text{ such}\\
&\qquad\qquad \text{that } z = \tau - \lim_{k\to\infty} z_k, z_k \in S_{i_k}, \forall k \in \mathbb{N}\Big\}.
\end{aligned}
$$

As was mentioned before, the LUR property can be useful to derive the NSLUC property. However, when we limit ourselves to a sequence, a weaker condition can be used to guarantee the NSLUC property on the sequence, namely the LUR property at a given point. Indeed, as illustrated in the next proposition, there is an interesting relationship between the LUR property at a given point x and the NSLUC property of a sequence $\{x_i\}_{i\in\mathbb{N}}$ such that

$$
{}^{\|\cdot\|}\operatorname*{Lim\,sup}_{i\to+\infty} \partial_{\alpha_i}\|\cdot\|(x_i) \subset \partial\|\cdot\|(x), \tag{11}
$$

for any sequence $\{\alpha_i\}_{i\in\mathbb{N}}$ of non-negative numbers with $\lim_{i\to+\infty}\alpha_i = 0$. Let us note that if $x \neq 0$ then the inclusion (11) implies immediately that the sequence $\{x_i\}_{i\in\mathbb{N}}$ does not have any subsequence which converges to the origin. In order to see this, observe that $0 \in \partial_{\alpha_i}\|\cdot\|(x_i)$ with $\alpha_i = \|x_i\|$, so if a subsequence converges to the origin, then by (11) we get $0 \in \partial\|\cdot\|(x)$, but this is impossible.

PROPOSITION 3.6. *Let* $(X, \|\cdot\|)$ *be a normed space and* $x \in X \setminus \{0\}$ *be given such that the norm* $\|\cdot\|$ *of the space has the LUR property at* x. *Then each* *bounded sequence* $\{x_i\}_{i\in\mathbb{N}}$ *satisfying* (11) *for every sequence* $\{\alpha_i\}_{i\in\mathbb{N}}$ *such that* $\alpha_i \geq 0$ *and* $\lim_{i\to\infty} \alpha_i = 0$, *has the NSLUC property.*

PROOF. If the LUR property of the norm holds at $x \neq 0$, then it obviously holds at $\|x\|^{-1}x$. This and the equality

$$\partial\|\cdot\|(\|x\|^{-1}x) = \partial\|\cdot\|(x)$$

allows us to suppose without loss of generality that $\|x\| = 1$. Let $\{x_i\}_{i\in\mathbb{N}}$ be a bounded sequence satisfying (11) for every sequence $\{\alpha_i\}_{i\in\mathbb{N}}$ of non-negative reals such that $\lim_{i\to\infty} \alpha_i = 0$.

(a) Suppose first that $\|x_i\| = 1$ for all $i \in \mathbb{N}$. To show the NSLUC property, let us argue by contradiction. So let us assume that there exist $u \in \mathbb{S}_X$ and $u^* \in \partial\|\cdot\|(u)$ such that for some subsequence $\{x_{i_k}\}_{k\in\mathbb{N}}$ we have

$$\inf_{k\in\mathbb{N}} \|x_{i_k} - \langle u^*, x_{i_k}\rangle u\| > 0 \tag{12}$$

and

$$\lim_{k\to\infty} |\langle u^*, x_{i_k}\rangle| = 1.$$

Putting $\delta_k := 1 - |\langle u^*, x_{i_k}\rangle|$, we have $\delta_k \geq 0$ and $\delta_k \to 0$ as $k \to +\infty$. Taking into account the boundedness of the sequence $\{x_i\}_{i\in\mathbb{N}}$, then, without loss of generality we may suppose that either $\langle u^*, x_{i_k}\rangle > 0$ for all $k \in \mathbb{N}$ or $\langle u^*, x_{i_k}\rangle < 0$ for all $k \in \mathbb{N}$.

If $\langle u^*, x_{i_k}\rangle > 0$ for all k, then $u^* \in \partial_{\delta_k}\|\cdot\|(x_{i_k})$ and therefore, by (11), $u^* \in \partial\|\cdot\|(x)$. Thus, using the definition of the subdifferential combined with the equality $\langle u^*, u\rangle = \|u\|$ (due to $u^* \in \partial\|\cdot\|(u)$), and using the triangle inequality, we see that $\|u + x\| = \|u\| + \|x\| = 2\|x\|$, hence $u = x$ (because of the LUR property at x applied with the sequence $z_i := u$ for all $i \in \mathbb{N}$). Similarly, using the inclusion $u^* \in \partial_{\delta_{i_k}}\|\cdot\|(x_{i_k})$ combined with the equality $\langle u^*, x\rangle = \|x\|$ and using the triangle inequality, we obtain that $\lim_{k\to\infty} \|x_{i_k} + x\| = 2\|x\|$.

If $\langle u^*, x_{i_k}\rangle < 0$ for all k, then $-u^* \in \partial_{\delta_k}\|\cdot\|(x_{i_k})$ and thus, by (11), $-u^* \in \partial\|\cdot\|(x)$. Consequently, as above, on the one hand $\|u - x\| = \|u\| + \|x\| = 2\|-x\|$, hence $u = -x$, and on the other hand $\lim_{k\to\infty} \|x_{i_k} - x\| = 2\|x\|$. Using the LUR property again, in both cases we obtain $\lim_{k\to\infty} \|x_{i_k} - u\| = 0$, which contradicts (12).

(b) Remove now the condition $\|x_i\| = 1$ for all $i \in \mathbb{N}$. As it is observed in comments concerning (11) the inclusion ${}^{\|\cdot\|}\mathrm{Lim\,sup}_{i\to+\infty}\, \partial_{\alpha_i}\|\cdot\|(x_i) \subset \partial\|\cdot\|(x)$ implies that

$$\liminf_{i\to+\infty} \|x_i\| > 0.$$

Then put $\tilde{x}_i := x_i/\|x_i\|$ and observe that the sequence $\{\tilde{x}_i\}_{i\in\mathbb{N}}$ satisfies (11). We deduce from (a) that $\{\tilde{x}_i\}_{i\in\mathbb{N}}$ has the NSLUC property, which entails in turn that $\{x_i\}_{i\in\mathbb{N}}$ has the NSLUC property. This finishes the proof.

The second result, which is needed in the next section, is quite technical.

PROPOSITION 3.7. *Assume that* $(X, \|\cdot\|)$ *is a normed space,* $S \subset X$ *is a bounded subset such that the set* $x - S$ *has the NSLUC property for some* $x \in X$, *and let* $d \in X \setminus \{x\}$. *Suppose that for each* $n \in \mathbb{N}$ *there exist* $m_n \in \mathbb{N}$, $t_1^n, \ldots, t_{m_n}^n \in]0, 1]$, *with* $\sum_{i=1}^{m_n} t_i^n = 1$, *and* $y_1^n, \ldots, y_{m_n}^n \in S$ *such that*

$$\lim_{n\to\infty} \sum_{i=1}^{m_n} t_i^n y_i^n = d \quad \textit{and} \quad \lim_{n\to\infty} \sum_{i=1}^{m_n} t_i^n \|x - y_i^n\| = \|x - d\|.$$

Then for every $\varepsilon > 0$, $u^* \in \partial\|\cdot\|(u)$ *with* $u := \|x - d\|^{-1}(x - d)$ *and*

$$C_\varepsilon^n := \left\{ j \in \{1, \ldots, m_n\} : \|x - y_j^n - \langle u^*, x - y_j^n\rangle u\| \geq \varepsilon \right\},$$

one has

$$\lim_{n\to\infty} \sum_{i\in C_\varepsilon^n} t_i^n = 0, \tag{13}$$

and consequently for every sequence $\{\varepsilon_i\}_{i\in\mathbb{N}}$, *such that* $\varepsilon_i > 0$ *for all* $i \in \mathbb{N}$ *and* $\lim_{i\to\infty} \varepsilon_i = 0$, *we are able to choose* $n_i := n(\varepsilon_i)$, *such that* $\lim_{i\to\infty} n_i = \infty$ *and*

$$\lim_{i\to\infty} \max_{j\in\{1,\ldots,m_{n_i}\}\setminus C_{\varepsilon_i}^{n_i}} d_{x+\operatorname{span}\{x-d\}}(y_j^{n_i}) = 0,$$

and

$$\lim_{i\to\infty} \sum_{j\in\{1,\ldots,m_{n_i}\}\setminus C_{\varepsilon_i}^{n_i}} t_j^{n_i} = 1.$$

PROOF. For each $n \in \mathbb{N}$, take $m_n \in \mathbb{N}$, $t_1^n, \ldots, t_{m_n}^n \in]0, 1]$, with $\sum_{i=1}^{m_n} t_i^n = 1$, and $y_1^n, \ldots, y_{m_n}^n \in S$ such that

$$\lim_{n\to\infty} \sum_{i=1}^{m_n} t_i^n y_i^n = d \quad \text{and} \quad \lim_{n\to\infty} \sum_{i=1}^{m_n} t_i^n \|x - y_i^n\| = \|x - d\|. \tag{14}$$

Note that the set $\bigcup_{n\in\mathbb{N}}\{y_1^n, \ldots, y_{m_n}^n\}$ is bounded according to the boundedness of S. Let us fix $\mu > 0$ and consider the sets

$$M_\mu^n := \Big\{(i, j) \in \mathbb{N} \times \mathbb{N} : i, j \in \{1, \ldots m_n\} \text{ and } \|x - y_i^n\| + \|x - y_j^n\| \geq \mu + \|x - y_i^n + x - y_j^n\|\Big\}$$

and $(M_\mu^n)^c$ the complement of M_μ^n in $\{1, \ldots, m_n\} \times \{1, \ldots, m_n\}$. We claim that

$$\lim_{n\to\infty} \sum_{(i,j)\in M_\mu^n} t_i^n t_j^n = 0. \tag{15}$$

If not, that is, $\limsup_{n\to\infty}\sum_{(i,j)\in M_\mu^n} t_i^n t_j^n > 0$, then, by (14), we have

$$\begin{aligned}2\|x-d\| &= \lim_{n\to\infty}\sum_{(i,j)\in\{1,\ldots,m_n\}\times\{1,\ldots,m_n\}} t_i^n t_j^n(\|x-y_i^n\|+\|x-y_j^n\|)\\
&= \lim_{n\to\infty}\Bigg(\sum_{(i,j)\in M_\mu^n} t_i^n t_j^n(\|x-y_i^n\|+\|x-y_j^n\|)\\
&\quad + \sum_{(i,j)\in (M_\mu^n)^c} t_i^n t_j^n(\|x-y_i^n\|+\|x-y_j^n\|)\Bigg)\\
&\geq \limsup_{n\to\infty}\Bigg(\sum_{(i,j)\in M_\mu^n} t_i^n t_j^n(\mu+\|x-y_i^n+x-y_j^n\|)\\
&\quad + \sum_{(i,j)\in (M_\mu^n)^c} t_i^n t_j^n(\|x-y_i^n+x-y_j^n\|)\Bigg)\\
&\geq \limsup_{n\to\infty}\sum_{(i,j)\in M_\mu^n} t_i^n t_j^n\mu + 2\|x-d\| > 2\|x-d\|,\end{aligned}$$

a contradiction. Given $\varepsilon > 0$, u and u^* as in the statement of the proposition, we are able to construct a discrete (infinite, denumerable) subset Ξ of $]0, 1[$ and a sequence $\{n(\mu)\}_{\mu\in\Xi}$ such that for all $\mu_1, \mu_2 \in \Xi$ we have

$$\mu_1 > \mu_2 \implies n(\mu_1) < n(\mu_2)$$

and $0 \in \operatorname{cl}\Xi$, and (keep in mind (15))

$$\lim_{\mu\in\Xi,\mu\to 0^+}\sum_{(i,j)\in (M_\mu^{n(\mu)})^c} t_i^{n(\mu)} t_j^{n(\mu)} = 1, \tag{16}$$

$$\lim_{\mu\in\Xi,\mu\to 0^+}\sum_{i\in C_\varepsilon^{n(\mu)}} t_i^{n(\mu)} = \limsup_{n\to\infty}\sum_{i\in C_\varepsilon^n} t_i^n, \tag{17}$$

and such that for all $(i, j) \in \{1, \ldots, m_{n(\mu)}\} \times \{1, \ldots, m_{n(\mu)}\} \setminus M_\mu^{n(\mu)}$ we have

$$\|x-y_i^{n(\mu)}\| + \|x-y_j^{n(\mu)}\| \leq \mu + \|x-y_i^{n(\mu)}+x-y_j^{n(\mu)}\|. \tag{18}$$

Since the set $x - S$ has the NSLUC property, there exists $\beta > 0$ (not depending on μ) such that for all $(i, j) \in \{1, \ldots, m_{n(\mu)}\}\cap C_\varepsilon^{n(\mu)} \times \{1, \ldots, m_{n(\mu)}\}\cap C_\varepsilon^{n(\mu)} \setminus M_\mu^{n(\mu)}$ we get

$$\|x-y_i^{n(\mu)}\| \geq |\langle u^*, x-y_i^{n(\mu)}\rangle| + \beta\|x-y_i^{n(\mu)} - \langle u^*, x-y_i^{n(\mu)}\rangle u\|,$$

$$\|x-y_j^{n(\mu)}\| \geq |\langle u^*, x-y_j^{n(\mu)}\rangle| + \beta\|x-y_j^{n(\mu)} - \langle u^*, x-y_j^{n(\mu)}\rangle u\|.$$

Putting $\mathbf{1}_{C_\epsilon^{n(\mu)}}(i) = 1$ if $i \in C_\epsilon^{n(\mu)}$ and $\mathbf{1}_{C_\epsilon^{n(\mu)}}(i) = 0$ otherwise, it follows from (14), (16), (17) and (18) that

$$\begin{aligned}
&2\|x-d\| \\
&\quad= \lim_{\mu\in\Xi,\mu\to 0^+} \sum_{(i,j)\in(M_\mu^{n(\mu)})^c} t_i^{n(\mu)} t_j^{n(\mu)} \big(\mu + \|x - y_i^{n(\mu)} + x - y_j^{n(\mu)}\|\big) \\
&\quad= \lim_{\mu\in\Xi,\mu\to 0^+} \sum_{(i,j)\in(M_\mu^{n(\mu)})^c} t_i^{n(\mu)} t_j^{n(\mu)} \big(\|x - y_i^{n(\mu)}\| + \|x - y_j^{n(\mu)}\|\big) \\
&\quad\geq \lim_{\mu\in\Xi,\mu\to 0^+} \sum_{(i,j)\in(M_\mu^{n(\mu)})^c} t_i^{n(\mu)} t_j^{n(\mu)} \big(|\langle u^*, x - y_i^{n(\mu)}\rangle| \\
&\qquad\quad + \mathbf{1}_{C_\epsilon^{n(\mu)}}(i)\beta\|x - y_i^{n(\mu)} - \langle u^*, x - y_i^{n(\mu)}\rangle u\| \\
&\qquad\quad + |\langle u^*, x - y_j^{n(\mu)}\rangle| + \mathbf{1}_{C_\epsilon^{n(\mu)}}(j)\beta\|x - y_j^{n(\mu)} - \langle u^*, x - y_j^{n(\mu)}\rangle u\|\big) \\
&\quad= 2\|x-d\| + \lim_{\mu\in\Xi,\mu\to 0^+} \sum_{(i,j)\in C_\varepsilon^{n(\mu)}\times C_\varepsilon^{n(\mu)}\cap(M_\mu^{n(\mu)})^c} t_i^{n(\mu)} t_j^{n(\mu)} \beta\big(\|x - y_i^{n(\mu)} \\
&\qquad\quad - \langle u^*, x - y_i^{n(\mu)}\rangle u\| + \|x - y_j^{n(\mu)} - \langle u^*, x - y_j^{n(\mu)}\rangle u\|\big),
\end{aligned}$$

where the second equality is due to (15) and the last equality is due to the fact that $\langle u^*, x-d\rangle = \|x-d\|$ since $u^* \in \partial\|\cdot\|(u)$. Then, we obtain by definition of C_ε^n that

$$\begin{aligned}
0 &\geq \limsup_{\mu\in\Xi,\mu\to 0^+} \sum_{(i,j)\in C_\varepsilon^{n(\mu)}\times C_\varepsilon^{n(\mu)}\cap(M_\mu^{n(\mu)})^c} t_i^{n(\mu)} t_j^{n(\mu)} \beta 2\varepsilon \\
&\geq 2\beta\varepsilon \lim_{\mu\in\Xi,\mu\to 0^+} \Bigg(\sum_{(i,j)\in C_\varepsilon^{n(\mu)}\times C_\varepsilon^{n(\mu)}} t_i^{n(\mu)} t_j^{n(\mu)} - \sum_{(i,j)\in M_\mu^{n(\mu)}} t_i^{n(\mu)} t_j^{n(\mu)} \Bigg) \\
&= 2\beta\epsilon \lim_{\mu\in\Xi,\mu\to 0^+} \Bigg(\sum_{i\in C_\varepsilon^{n(\mu)}} t_i^{n(\mu)} \Bigg)^2 = 2\beta\epsilon \limsup_{n\to\infty} \Bigg(\sum_{i\in C_\varepsilon^n} t_i^n \Bigg)^2.
\end{aligned}$$

So, it ensues as desired that

$$\limsup_{n\to\infty} \sum_{i\in C_\epsilon^n} t_i^n = 0. \tag{19}$$

Now, let $\{\varepsilon_i\}_{i\in\mathbb{N}}$ be a sequence such that $\varepsilon_i > 0$ for all $i \in \mathbb{N}$ and $\lim_{i\to\infty} \varepsilon_i = 0$.

By (19) we can choose $n_i := n(\varepsilon_i) \in \mathbb{N}$ such that $\lim_{i\to\infty} n_i = \infty$ and

$$\lim_{i\to\infty} \sum_{j\in\{1,\ldots,m_{n_i}\}\setminus C_{\varepsilon_i}^{n_i}} t_j^{n_i} = 1.$$

Then, for each $i \in \mathbb{N}$ from the definition of $C_{\varepsilon_i}^{n_i}$ it results that

$$d_{x+\operatorname{span}(d-x)}(y_j^{n_i}) \le \|x - y_j^{n_i} - \langle u^*, x - y_j^{n_i}\rangle u\| < \epsilon_i, \quad \forall i \in \mathbb{N}, \forall j \in (C_{\epsilon_i}^{n_i})^c,$$

which justifies that

$$\lim_{i\to\infty} \max_{j\in\{1,\ldots,m_{n_i}\}\setminus C_{\varepsilon_i}^{n_i}} d_{x+\operatorname{span}\{x-d\}}(y_j^{n_i}) = 0$$

and finishes the proof.

The last result of this section is related to a sequence $\{y_i\}_{i\in\mathbb{N}}$ such that the set $\{x - y_i : i \in \mathbb{N}\}$ satisfies the NSLUC property along with a condition on the convergence of the sequence of subdifferential $\{\partial\varphi(x - y_i)\}_{i\in\mathbb{N}}$.

PROPOSITION 3.8. *Let $(X, \|\cdot\|)$ be a normed space, Q be a convex set of X and $g\colon [0, +\infty[\to [0, +\infty[$ be an increasing convex function with $g(0) = 0$. Suppose that the function $\varphi\colon X \to \mathbb{R} \cup \{+\infty\}$ defined by*

$$\varphi(x) := g(\|x\|) + \delta_Q(x), \quad \forall x \in X$$

is lower semicontinuous. Let us fix $x, y \in X$, with $x \ne y$, $x - y \in Q$, $N(Q, x - y) = \{0\}$ and consider a bounded sequence $\{y_i\}_{i\in\mathbb{N}}$ in X such that the NSLUC property holds for the set $\{x - y_i : i \in \mathbb{N}\}$ and there is $x^* \in \partial\varphi(x - y)$ such that

$$\lim_{i\to\infty} d_{\partial\varphi(x-y_i)}(x^*) = 0, \tag{20}$$

which means $x^* \in {}^{\|\cdot\|}\operatorname{Lim\,inf}_{i\to\infty} \partial\varphi(x - y_i)$. Then

$$\lim_{i\to\infty} \big\|x - y_i - \|x - y_i\|u\big\| = 0, \tag{21}$$

with $u = (x - y)/\|x - y\|$.

PROOF. Suppose that our conclusion is false, that is, there exist $\varepsilon > 0$ and a subsequence of $\{y_i\}_{i\in\mathbb{N}}$ (which is not relabeled for simplicity) such that

$$\big\|x - y_i - \|x - y_i\|u\big\| \ge \varepsilon. \tag{22}$$

Let us notice that, by the subdifferential calculus, there are $a \in \partial g(\|x - y\|)$, with $a > 0$ (because $x \ne y$ and g is increasing), $u^* \in \partial\|\cdot\|(x - y)$, such that $x^* = au^* \neq 0$. Of course, $\|u^*\| = 1$ and $u^* \in \partial\|\cdot\|(u)$. By (20) there is a sequence $\{x_i^*\}_{i\in\mathbb{N}}$ of subgradients $x_i^* \in \partial\varphi(x - y_i)$, which converges in norm to x^* and satisfies $x_i^* = a_i u_i^*$, with $a_i \in \partial g(\|x - y_i\|)$ as well as $a_i > 0$ (since, because of (22), $x \neq y_i$), and $u_i^* \in \partial\|\cdot\|(u_i)$, where $u_i = \frac{x-y_i}{\|x-y_i\|}$. Since $a_i = \|x_i^*\| \to \|x^*\| = a$, from the equality $x_i^* = a_i u_i^*$ we see that $\|u_i^* - u^*\| \to 0$ as $i \to \infty$.

Now, taking into account (22), we obtain for i large enough,

$$\begin{aligned}\|x - y_i - \langle u^*, x - y_i\rangle u\| &\geq \|x - y_i - \langle u_i^*, x - y_i\rangle u\| - \|u_i^* - u^*\|\|x - y_i\| \\ &\geq \big\|x - y_i - \|x - y_i\|u\big\| - \varepsilon/2 \qquad (23)\\ &\geq \varepsilon/2.\end{aligned}$$

From the NSLUC property of the set $S' := \{x - y_i : i \in \mathbb{N}\}$, there exists $\beta > 0$ such that

$$\|x - y_i\| \geq |\langle u^*, x - y_i\rangle| + \beta\|x - y_i - \langle u^*, x - y_i\rangle u\|, \quad \text{for all } i \in \mathbb{N}.$$

Using (23) and taking i sufficiently large it ensues that

$$\begin{aligned}\|x - y_i\| &\geq |\langle u^*, x - y_i\rangle| + \beta\varepsilon/2 \\ &\geq |\langle u_i^*, x - y_i\rangle| + \beta\varepsilon/4 \\ &= \|x - y_i\| + \beta\varepsilon/4,\end{aligned}$$

and this contradiction completes the proof.

Let us point out that (21) implies that

$$\lim_{i\to+\infty} d_{x+\mathbb{R}_+(y-x)}(y_i) = 0.$$

Because of the boundedness of the sequence $\{y_i\}_{i\in\mathbb{N}}$, this is in turn equivalent to

$$\emptyset \neq {}^{\|\cdot\|}\operatorname*{Lim\,sup}_{i\to\infty}\{y_i, y_{i+1}, \ldots\} \subset x + \mathbb{R}_+(y - x).$$

This ensures the existence of cluster points of the sequence $\{y_i\}_{i\in\mathbb{N}}$ with respect to the strong topology. The key tool in getting this is the NSLUC property assumed on the set $\{x - y_i : i \in \mathbb{N}\}$.

4. Properties of attainment sets

We start this section with the following question: when does the non-emptiness of $M_\varphi(\overline{\text{co}}\, f)(x)$ imply the non-emptiness of $M_\varphi f(x)$? An answer to that question was given in the paper [12] for φ of the form $\varphi = \|\cdot\|^p$ with $p \in [1, +\infty[$.

Below this result is extended to a more general case, namely instead of the norm to a power $p \geq 1$, a function φ of the form as $\varphi(\cdot) = g(\|\cdot\|) + \delta_Q(\cdot)$ is investigated, where Q is a convex subset and $g\colon [0, \infty[\to [0, \infty[$ is an increasing convex function with $g(0) = 0$. The function f is also assumed to be lower semicontinuous with bounded domain and such that $x - \operatorname{dom} f$ has the NSLUC property. Under such conditions, the inclusion $M_\varphi(\overline{\text{co}}\, f)(x) \subset \text{co}\, M_\varphi f(x)$ is still preserved, see Theorem 4.1(a). The non-emptiness of $M_\varphi(\overline{\text{co}}\, f)(x)$ implies the non-emptiness of $M_\varphi f(x)$ in this case. Moreover, assuming Fréchet differentiability of f^φ at x and strict convexity of g, the equality $M_\varphi(\overline{\text{co}}\, f)(x) = M_\varphi f(x)$ is obtained, see Theorem 4.1(b). Additionally, the non-emptiness of $M_\varphi f(x)$ implies that the problem is well posed, that is, any maximizing sequence is strongly convergent.

The theorem concerning the inclusion $M_\varphi(\overline{\text{co}}\, f)(x) \subset \text{co}\, M_\varphi f(x)$, Theorem 4.1 below, involves the Painlevé-Kuratowski limit superior of sets. Let G be the Painlevé-Kuratowski limit superior of a sequence of sets in the normed space $(X, \|\cdot\|)$, that is, $G = {}^{\|\cdot\|}\text{Lim sup}_{i\to\infty} G_i$. Of course, there are several questions concerning properties of the limit superior of sets. It is not the aim of our work to investigate those questions in detail. However, in our reasoning below, we need the inclusion ${}^{\|\cdot\|}\text{Lim sup}_{i\to\infty} \text{co}\, G_i \subset \text{co}\, G$, which holds in some important cases. In order to clarify the necessity of conditions for the validity of such an inclusion, we provide a counter-example showing that the inclusion is not valid even in the case where G and G_i are compact. Next, we give a lemma where the property is true for a particular sequence of sets which will be involved in Theorem 4.1. Let us start with the counter-example.

EXAMPLE 4.1. Let $X := \ell_2(\mathbb{N})$, $G_i := \{0, e_1, \ldots, i^{-1}e_i\}$, and

$$G := \{0, e_1, 2^{-1}e_2, 3^{-1}e_3, \ldots\} = \{0\} \cup \{i^{-1}e_i : i \in \mathbb{N}\},$$

where $e_i := \{\delta_i(j)\}_{j\in\mathbb{N}}$, with $\delta_i(j) = 0$, whenever $i \neq j$, and $\delta_i(j) = 1$, for $i = j$. We have $G = {}^{\|\cdot\|}\text{Lim sup}_{i\to\infty} G_i$, however ${}^{\|\cdot\|}\text{Lim sup}_{i\to\infty} \text{co}\, G_i \not\subset \text{co}\, G$. In order to see this, put $\theta_j := 2^{-j}$, $y_i := \sum_{j=1}^{i} \theta_j j^{-1} e_j$. Then, $\lim_{i\to\infty} y_i \in {}^{\|\cdot\|}\text{Lim sup}_{i\to\infty} \text{co}\, G_i$, however $\lim_{i\to\infty} y_i \notin \text{co}\, G$, since there are infinitely many positive coefficients in the limit.

LEMMA 4.1. *Let $(X, \|\cdot\|)$ be a normed space and let $A_i \subset \mathbb{N}$ be finite subsets for $i \in \mathbb{N}$. Suppose that a bounded set $\{y_j^i \in X : i \in \mathbb{N},\ j \in A_i\}$ is such that*

$$\lim_{i\to+\infty} \max_{j\in A_i} d_L(y_j^i) = 0, \tag{24}$$

for some finite-dimensional affine subspace L of X. Then, for $G :=$

${}^{\|\cdot\|}\mathrm{Lim\,sup}_{i\to+\infty} G_i$, *with* $G_i := \{y^i_j : j \in A_i\}$, *we have*

$$^{\|\cdot\|}\operatorname*{Lim\,sup}_{i\to+\infty} \operatorname{co} G_i \subset \operatorname{co} G.$$

PROOF. First note that by the definition of G and relation (24), we have $G \subset L$, and hence $\overline{\operatorname{co}}\, G = \operatorname{co} G$. Let $v \in {}^{\|\cdot\|}\mathrm{Lim\,sup}_{i\to\infty} \operatorname{co} G_i$ and suppose that $v \notin \operatorname{co} G$. By the Hahn-Banach separation theorem, there exist $x^* \in X^*$, with $x^* \neq 0$, and $\alpha \in \mathbb{R}$ such that

$$\langle x^*, v\rangle < \alpha < \langle x^*, z\rangle, \quad \forall z \in G. \tag{25}$$

By the definition of G_i, there are sequence $\{i_k\}_{k\in\mathbb{N}}$ of integers, with $\lim_{k\to+\infty} i_k = +\infty$, and non-negative numbers $\{t_j : j \in A_{i_k}\}$, with $\sum_{j\in A_{i_k}} t_i = 1$, such that

$$\lim_{k\to\infty} \sum_{j\in A_{i_k}} t_j y^{i_k}_j = v. \tag{26}$$

Relations (24), (25) and (26) ensure, for k sufficiently large, the following strict inequalities

$$\left\langle x^*, \sum_{j\in A_{i_k}} t_j y^{i_k}_j \right\rangle < \alpha < \langle x^*, y^{i_k}_\ell\rangle, \quad \forall \ell \in A_{i_k},$$

and hence

$$\left\langle x^*, \sum_{j\in A_{i_k}} t_j y^{i_k}_j \right\rangle < \alpha < \left\langle x^*, \sum_{j\in A_{i_k}} t_j y^{i_k}_j \right\rangle$$

whence the contradiction. The proof is then complete.

THEOREM 4.1. *Assume that* $(X, \|\cdot\|)$ *is a normed space and* $f\colon X \to \mathbb{R} \cup \{+\infty\}$ *is a lower semicontinuous function such that* $x - \operatorname{dom} f$ *is a non-empty bounded set which has the NSLUC property for some* $x \in X$. *Let* $\varphi\colon X \to \mathbb{R} \cup \{+\infty\}$ *be a lower semicontinuous function of the form*

$$\varphi(\cdot) = g(\|\cdot\|) + \delta_Q(\cdot),$$

where Q *is a non-empty convex subset with* $0 \in Q$ *and* $g\colon [0, \infty[\to [0, \infty[$ *is an increasing convex function with* $g(0) = 0$. *Suppose that* f^φ *is finite at* x. *Then the following hold.*

(a) *One has the inclusions*

$$M_\varphi f(x) \subset M_\varphi(\overline{\operatorname{co}}\, f)(x) \subset \operatorname{co} M_\varphi f(x).$$

(b) *Moreover if the Fréchet derivative of f^φ at x exists, then*
- *the set $M_\varphi(\overline{\text{co}}\, f)(x)$ is convex, and hence $M_\varphi(\overline{\text{co}}\, f)(x) = \text{co}\, M_\varphi f(x)$;*
- *the set $M_\varphi f(x)$ is at most a singleton provided that g is strictly convex;*
- *each maximizing sequence $\{y_i\}_{i\in\mathbb{N}}$ for $f^\varphi(x)$ is strongly convergent provided that the set $M_\varphi f(x)$ is a singleton (which holds true if g is strictly convex and $M_\varphi f(x) \neq \emptyset$).*

PROOF. (a) First observe that the inclusion $M_\varphi f(x) \subset M_\varphi(\overline{\text{co}}\, f)(x)$ is immediate. To prove the second inclusion, let us fix any $d \in M_\varphi(\overline{\text{co}}\, f)(x)$ (if any). By Proposition 2.1 we have

$$f^\varphi(x) = (\overline{\text{co}}\, f)^\varphi(x) = \varphi(x-d) - (\overline{\text{co}}\, f)(d). \tag{27}$$

Case 1: If $d = x$, then since the equality $\inf_{y\in X} \overline{\text{co}}\, f(y) = \inf_{y\in X} f(y)$ is obvious, by (27) we get

$$\begin{aligned} f^\varphi(d) &= (\overline{\text{co}}\, f)^\varphi(d) = -\overline{\text{co}}\, f(d) \\ &= \sup_{y\in X} -\overline{\text{co}}\, f(y) = \sup_{y\in X} -f(y) \\ &= -\inf_{y\in X} f(y). \end{aligned}$$

Take a sequence $\{y_n\}_{n\in\mathbb{N}}$ in $\text{dom}\, f$ such that $\lim_{n\to\infty} f(y_n) = \inf_{y\in X} f(y)$ and note that

$$-f(y_n) \le \varphi(d-y_n) - f(y_n) \le f^\varphi(d) = -\inf_{y\in X} f(y).$$

It follows that $\lim_{n\to\infty} \varphi(d-y_n) = 0$, and consequently $\lim_{n\to\infty} y_n = d$. So using the lower semicontinuity of f, we get

$$f(d) \le \liminf_{n\to\infty} f(y_n) = \inf_{y\in X} f(y) \le f(d)$$

and this implies that $f^\varphi(d) = -f(d)$, and consequently $d \in M_\varphi f(x)$.

Case 2: If $d \neq x$, then define $u := \|x-d\|^{-1}(x-d)$ and pick $u^* \in \partial\|\cdot\|(x-d)$.

For each $n \in \mathbb{N}$ there exist $m_n \in \mathbb{N}$, $t_1^n, \dots, t_{m_n}^n \in\,]0,1]$, with $\sum_{i=1}^{m_n} t_i^n = 1$, and $y_1^n, \dots, y_{m_n}^n \in \text{dom}\, f$ such that

$$\lim_{n\to\infty} \sum_{i=1}^{m_n} t_i^n y_i^n = d, \quad \overline{\text{co}}\, f(d) = \lim_{n\to\infty} \sum_{i=1}^{m_n} t_i^n f(y_i^n). \tag{28}$$

Note that the set $\bigcup_{n\in\mathbb{N}}\{y_1^n, \ldots, y_{m_n}^n\}$ is bounded according to the boundedness of dom f. On the other hand, as by (27) for all $n \in \mathbb{N}$ and $i = 1, \ldots, m_n$,

$$\varphi(x - y_i^n) - f(y_i^n) \leq \varphi(x - d) - \overline{\text{co}}\, f(d),$$

which by the choice of $\{t_i^n\}_{n\in\mathbb{N},\, i\in\{1,\ldots,m_n\}}$ implies

$$\sum_{i=1}^{m_n} t_i^n[\varphi(x - y_i^n) - f(y_i^n)] \leq \varphi(x - d) - \overline{\text{co}}\, f(d),$$

and hence, by the convexity of φ,

$$\varphi\left(\sum_{i=1}^{m_n} t_i^n(x - y_i^n)\right) \leq \sum_{i=1}^{m_n} t_i^n\varphi(x - y_i^n) \leq \varphi(x - d) - \overline{\text{co}}\, f(d) + \sum_{i=1}^{m_n} t_i^n f(y_i^n).$$

This combined with (28) and the lower semicontinuity of φ at $x - d$ entails

$$\lim_{n\to\infty} \sum_{i=1}^{m_n} t_i^n\varphi(x - y_i^n) = \varphi(x - d). \tag{29}$$

Let $\mu > 0$ be arbitrary and consider, for each $n \in \mathbb{N}$, the following sets

$$\Gamma_\mu^n := \left\{i \in \{1, \ldots, m_n\} : \varphi(x - y_i^n) - f(y_i^n) + \mu \leq f^\varphi(x)\right\}$$

and

$$\Upsilon_\mu^n := \{1, \ldots, m_n\} \setminus \Gamma_\mu^n.$$

We have

$$\lim_{n\to\infty} \sum_{i\in\Gamma_\mu^n} t_i^n = 0, \tag{30}$$

because (keep in mind (28) and (29))

$$\begin{aligned}
f^\varphi(x) &= \varphi(x - d) - \overline{\text{co}}\, f(d) \\
&= \lim_{n\to\infty}\left(\sum_{i\in\Gamma_\mu^n} t_i^n[\varphi(x - y_i^n) - f(y_i^n)] + \sum_{i\in\Upsilon_\mu^n} t_i^n[\varphi(x - y_i^n) - f(y_i^n)]\right) \\
&\leq \liminf_{n\to\infty}\left(\sum_{i\in\Gamma_\mu^n} t_i^n[f^\varphi(x) - \mu] + \sum_{i\in\Upsilon_\mu^n} t_i^n f^\varphi(x)\right) \\
&\leq f^\varphi(x) - \mu \limsup_{n\to\infty} \sum_{i\in\Gamma_\mu^n} t_i^n \leq f^\varphi(x).
\end{aligned}$$

Let $\{\mu_i\}_{i\in\mathbb{N}}$ be a decreasing sequence in the set $]0, 1[$ such that $\lim_{i\to\infty} \mu_i = 0$, and consider a sequence of integers $\{n(\mu_i)\}_{i\in\mathbb{N}}$ satisfying

$$i < j \implies \mu_i > \mu_j \implies n(\mu_i) < n(\mu_j).$$

To simplify, we put $n_i := n(\mu_i)$ and $m_i := m_{n(\mu_i)}$. Keeping in mind (30), we may suppose that the sequence satisfies the following properties

$$\sum_{j=1}^{m_i} t_j^{n_i} = 1 \quad \text{and} \quad \lim_{i\to\infty} \sum_{j\in\Upsilon_{\mu_i}^{n_i}} t_j^{n_i} = 1. \tag{31}$$

For all $j \in \Upsilon_{\mu_i}^{n_i}$ we have

$$f^{\varphi}(x) = \varphi(x-d) - \overline{\text{co}}\, f(d) \le \varphi(x - y_j^{n_i}) - f(y_j^{n_i}) + \mu_i.$$

We claim that

$$\lim_{i\to\infty} \sum_{j=1}^{m_i} t_j^{n_i} \|x - y_j^{n_i}\| = \|d - x\|.$$

In fact, if

$$\limsup_{i\to\infty} \sum_{j=1}^{m_i} t_j^{n_i} \|x - y_j^{n_i}\| > \|d - x\|, \tag{32}$$

then it follows from (29) that

$$\begin{aligned}
\varphi(x-d) &= \lim_{i\to\infty} \sum_{j=1}^{m_i} t_j^{n_i} \varphi(x - y_j^{n_i}) \\
&= \lim_{i\to\infty} \sum_{j=1}^{m_i} t_j^{n_i} g(\|x - y_j^{n_i}\|) \\
&\ge \limsup_{i\to\infty} g\left(\sum_{j=1}^{m_i} t_j^{n_i} \|x - y_j^{n_i}\|\right) \\
&\ge g\left(\limsup_{i\to\infty} \sum_{j=1}^{m_i} t_j^{n_i} \|x - y_j^{n_i}\|\right) \\
&> \varphi(x-d),
\end{aligned}$$

which is impossible (note that the second inequality is due to the continuity of g while the last one is due to relation (32) and the fact that g is increasing). Let

$C_{\varepsilon_i}^{n_i}$ be as in Proposition 3.7 and let $(C_{\varepsilon_i}^{n_i})^c$ be its complement in $\{1, \dots, m_{n_i}\}$. To simplify notation put $A_i := \Upsilon_{\mu_i}^{n_i} \cap (C_{\varepsilon_i}^{n_i})^c$. We claim that

$$\lim_{i\to\infty} \sum_{j\in A_i} t_j^{n_i} = 1, \quad \text{and} \quad d = \lim_{i\to\infty} \sum_{j\in A_i} t_j^{n_i} y_j^{n_i}.$$

The first equality follows from the following decomposition formula

$$\sum_{j\in\Upsilon_{\mu_i}^{n_i}\cup(C_{\varepsilon_i}^{n_i})^c} t_j^{n_i} = \sum_{j\in\Upsilon_{\mu_i}^{n_i}} t_j^{n_i} + \sum_{j\in(C_{\varepsilon_i}^{n_i})^c} t_j^{n_i} - \sum_{j\in A_i} t_j^{n_i}$$

and the equalities (13) and (31), while the second equality follows from the first one and the first equality in (28). Put $G_i := \{y_j^{n_i} : j \in A_i\}$, for all $i \in \mathbb{N}$, $G := \operatorname{Lim\,sup}_{i\to\infty} G_i$ and $L := x + \operatorname{span}\{d - x\}$. Proposition 3.7 again ensures that relation (24) is satisfied. Applying Lemma 4.1, we obtain

$$d \in \operatorname*{Lim\,sup}_{i\to\infty} \operatorname{co} G_i \subset \operatorname{co} G.$$

Observe by the definition of G that $G \subset M_\varphi f(x)$. Indeed, for any $y \in G$ there exists a sequence $\{y_{j_k}^{n_{i_k}}\}_{k\in\mathbb{N}}$, with $y_{j_k}^{n_{i_k}} \in G_{i_k}$ for all $k \in \mathbb{N}$, such that $\lim_{k\to+\infty} y_{j_k}^{n_{i_k}} = y$. By the definition of $\Upsilon_{\mu_i}^{n_i}$, we have $\lim_{k\to+\infty}[\varphi(x - y_{j_k}^{n_{i_k}}) - f(y_{j_k}^{n_{i_k}})] = f^\varphi(x)$ and hence

$$\liminf_{k\to\infty} \varphi(x - y_{j_k}^{n_{i_k}}) \geq f^\varphi(x) + f(y).$$

By the definition of f^φ, we have $x - y_{j_k}^{n_{i_k}} \in Q$ and hence $\liminf_{k\to\infty} \varphi(x - y_{j_k}^{n_{i_k}}) = \liminf_{k\to\infty} g(\|x - y_{j_k}^{n_{i_k}}\|) = g(\|x - y\|)$ (the later equality is due to the continuity of g), so the lower semicontinuity of φ ensures that $x - y \in Q$. Consequently,

$$\liminf_{k\to\infty} \varphi(x - y_{j_k}^{n_{i_k}}) = \varphi(x - y).$$

Combining this equality with the last inequality, we obtain

$$\varphi(x - y) \geq f^\varphi(x) + f(y)$$

and hence $y \in M_\varphi f(x)$. It follows that $d \in \operatorname{co} M_\varphi f(x)$ as desired, which finishes the proof of assertion (a).

(b) Now suppose that f^φ is Fréchet differentiable at x, and let $y_1, y_2 \in M_\varphi(\overline{\operatorname{co}}\, f)(x)$. Using the fact $x^* \in \partial\varphi(x - y_1) \cap \partial\varphi(x - y_2)$ for $x^* := D_F f^\varphi(x)$, we easily see that

$$\varphi\big(x - (ty_1 + (1 - t)y_2)\big) = t\varphi(x - y_1) + (1 - t)\varphi(x - y_2), \quad \forall t \in [0, 1],$$

and consequently, by simple algebra, we obtain $ty_1+(1-t)y_2 \in M_\varphi(\overline{\mathrm{co}}\, f)(x)$.

Let us now prove that $M_\varphi f(x)$ is at most a singleton when g is strictly convex. So let us suppose that $z_1, z_2 \in M_\varphi f(x)$, with $z_1 \neq z_2$. Then $x - z_1, x - z_2 \in Q$ and Lemma 2.1(i) together with the subdifferential calculus imply that

$$\partial(g \circ \|\cdot\|)(x - z_i) + N(Q, x - z_i) = \partial\varphi(x - z_i) \subset \{x^*\}, \quad i = 1, 2.$$

Thus $N(Q, x - z_i) = \{0\}$ and $\partial(g \circ \|\cdot\|)(x - z_i) = \{x^*\}$, for $i = 1, 2$, or equivalently,

$$\bigcup_{a\in\partial g(\|x-z_i\|)} a\partial\|\cdot\|(x - z_i) = \{x^*\}.$$

Since $z_1 \neq z_2$, we may suppose without loss of generality that $x \neq z_2$. Applying Lemma 2.1(iv) with $\overline{y} = z_2$ and Proposition 3.8 with $y_i = z_1$ for all i and $y = z_2$, we deduce that

$$x - z_1 = \frac{\|x - z_1\|}{\|x - z_2\|}(x - z_2). \tag{33}$$

Since $\partial\varphi(x - z_1) = \partial\varphi(x - z_2) = \{x^*\}$ by Lemma 2.1 again, we have

$$x^* = a_1u_1^* = a_2u_2^*, \tag{34}$$

with $a_k \in \partial g(\|x - z_k\|)$ and $u_k^* \in \partial\|\cdot\|(x - z_k)$, $k = 1, 2$. Using the fact that $\|u_1^*\| = \|u_2^*\| = 1$, the relation (34) implies that $a_1 = a_2$. Then, by the strict convexity of g, we get $\|x - z_1\| = \|x - z_2\|$. From the equality (33), it follows that $z_1 = z_2$, a contradiction.

Finally, assume that $M_\varphi f(x)$ is a singleton, say $\{y\} = M_\varphi f(x)$. It follows from Lemma 2.1(iv) that $D_F\varphi(x-y) = D_F f^\varphi(x)$, so $N(Q, x-y) = \{0\}$, see Lemma 2.1(i). Let us take any maximizing sequence $\{y_i\}_{i\in\mathbb{N}}$, that is, satisfying (7). It follows from Proposition 2.2 that (8) holds true, which implies that the condition in (20) is fulfilled. Applying Proposition 3.8 we get that (21) holds true, which ensures the existence of cluster points of the sequence $\{y_i\}_{i\in\mathbb{N}}$. On the other hand, each such cluster point belongs to $M_\varphi f(x)$ according to the reasoning at the end of the proof of assertion (a). Since $M_\varphi f(x)$ is a singleton, the set of cluster points of the sequence has to be singleton too, in other words the sequence $\{y_i\}_{i\in\mathbb{N}}$ is strongly convergent. The proof is then complete.

REMARK 4.1. Let us notice that to get Theorem 4.1(b) the NSLUC property is only needed for the set $\{x - y_i : i \in \mathbb{N}\}$, where $\{y_i\}_{i\in\mathbb{N}}$ is a maximizing sequence (in the proof of Proposition 3.8). We mean that for each bounded maximizing sequence $\{y_i\}_{i\in\mathbb{N}}$, the set $\{x-y_i : i \in \mathbb{N}\}$ has the NSLUC property. The strict convexity of the function g seems to be essential to guarantee that the set $M_\varphi f(x)$ is at most a singleton. An example showing that the convexity alone is not enough to yield the property is given below.

EXAMPLE 4.2. Assume that $X = \mathbb{R}$. Let us define $\varphi(t) = |t|$ and $f(t) = t + \delta_{\{0,1\}}(t)$, for $t \in \mathbb{R}$, where $\delta_{\{0,1\}}$ is the indicator function of the set $\{0, 1\}$. By an immediate computation, we get $f^\varphi(x) = |x|$ for every $x \in \mathbb{R}$. The Fréchet derivative of f^φ exists at every $x \neq 0$, but $M_\varphi f(x) = \{0, 1\}$ for every $x \leq 0$. Then the set $M_\varphi f(x)$ is not a singleton because the function $g = \mathrm{Id}_{\mathbb{R}_+}$ is not strictly convex.

REMARK 4.2. In Theorem 4.1(b) the convergence of maximizing sequences is obtained for a function φ which is the composition of a strictly convex function and the norm of the space. Below it is shown that the convergence can be also obtained whenever φ is uniformly convex on bounded sets of a Banach space. We recall the following notion, see for example [4, p. 241]: a proper convex function $h : X \to \mathbb{R} \cup \{+\infty\}$ is called *uniformly convex on bounded sets* if for every bounded sequence $\{z_i\}_{i\in\mathbb{N}}$ we have

$$\lim_{m,n\to\infty} \|z_m - z_n\| = 0,$$

whenever

$$\lim_{m,n\to\infty} h\bigl(\tfrac{1}{2}(z_m + z_n)\bigr) - \tfrac{1}{2}\bigl(h(z_m) + h(z_n)\bigr) = 0.$$

For any normed space $(X, \|\cdot\|)$ and any proper convex lower semicontinuous function φ: $X \to \mathbb{R} \cup \{\infty\}$, which is uniformly convex, if $\{y_i\}_{i\in\mathbb{N}}$ is a bounded maximizing sequence of $f^\varphi(x) \in \mathbb{R}$ and the Fréchet derivative of f^φ at x exists, then the maximizing sequence is a Cauchy sequence. In order to prove that, for each $i \in \mathbb{N}$ choose $x_i^* \in \partial\varphi(x - y_i)$ (recall that the convex function φ is continuous at $x - y_i$ by Proposition 2.2(a)). Observe by Proposition 2.2(a) again that

$$\begin{aligned} 0 &= \lim_{m,n\to\infty} \bigl\langle x_m^* - x_n^*, \tfrac{1}{4}\bigl(x - y_n - (x - y_m)\bigr)\bigr\rangle \\ &\leq \lim_{m,n\to\infty} \Bigl(\varphi\bigl(\tfrac{1}{2}(x - y_m + x - y_n)\bigr) - \tfrac{1}{2}\bigl(\varphi(x - y_m) + \varphi(x - y_n)\bigr)\Bigr) \leq 0. \end{aligned}$$

Hence

$$\lim_{m,n\to\infty} \Bigl(\varphi\bigl(\tfrac{1}{2}(x - y_m + x - y_n)\bigr) - \tfrac{1}{2}\bigl(\varphi(x - y_m) + \varphi(x - y_n)\bigr)\Bigr) = 0,$$

so by the uniform convexity we get the statement. We point out that the strong convergence, under the assumption of the uniform convexity, is a virtue which is obtained without assuming a priori the non-emptiness of $M_\varphi f(x)$, only the differentiability is supposed. When we admit additionally, that $M_\varphi f(x)$ is a non-empty set then the assumption on the function φ can be relaxed. Following [5] let us recall that a proper convex function $h: X \to \mathbb{R} \cup \{+\infty\}$ is called *totally convex at a point* z belonging to the algebraic interior (also called core) of $\operatorname{dom} h$ if, for every $t > 0$, we have

$$\inf\big\{h(v) - h(z) - h'(z; v - z) : \|v - z\| = t\big\} > 0,$$

where $h'(z; u) := \lim_{t\downarrow 0} t^{-1}\big(h(z+tu) - h(z)\big)$. If the function is totally convex at any point of the algebraic interior of $\operatorname{dom} h$, then the function is called *totally convex*. For any normed space $(X, \|\cdot\|)$ and any proper convex lower semicontinuous function $\varphi: X \to \mathbb{R}\cup\{+\infty\}$, which is totally convex, if $\{y_i\}_{i\in\mathbb{N}}$ is a bounded maximizing sequence of $f^\varphi(x) \in \mathbb{R}$ and the Fréchet derivative of f^φ at x exists, then this maximizing sequence is a convergent sequence, whenever $M_\varphi f(x) \neq \emptyset$. In order to justify that, choose $y \in M_\varphi f(x)$ and, for each $i \in \mathbb{N}$, choose also as above $x_i^* \in \partial\varphi(x - y_i)$. Put $x^* = D_F f^\varphi(x)$ and note by Lemma 2.1(iv) that φ is Fréchet differentiable with $D_F\varphi(x - y) = x^*$. Then, by Proposition 2.2(a) we have

$$0 \le \lim_{i\to\infty} \big(\varphi(x - y_i) - \varphi(x - y) - \langle x^*, y - y_i\rangle\big) \le \lim_{i\to\infty} \langle x_i^* - x^*, y - y_i\rangle = 0,$$

thus by the total convexity of φ at $x - y$ we get $\lim_{i\to\infty} \|y_i - y\| = 0$.

COMMENTS ON THE BOUNDEDNESS OF THE DOMAIN OF f. For the Klee-envelope, the condition on the boundedness of the domain of f in the above theorem is quite natural, for example, when f is the indicator function of a subset of X. The case when the domain is not a bounded set, can be reduced to the first one; for example if the coercivity of f is assumed, then we can restrict our considerations to a bounded set. A way of ensuring that reduction is to require the condition

$$\liminf_{\|z\|\to\infty} \frac{f(z)}{\|z\|^p} > 1, \tag{35}$$

with a real $p \ge 1$ and, without loss of generality, we may assume that $f(0) = 0$. This condition allows us to reduce the study to the bounded domain case, whenever $\varphi := \|\cdot\|^p$. Indeed, fix $x \in X$ and take $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $A > 0$ such that

$$f(z) \ge \big((1 + \varepsilon_1)^{p-1} + \varepsilon_2\big)\|z\|^p, \quad \forall z \in X \text{ with } \|z\| > A, \tag{36}$$

and

$$\|x\|^p\left(\left(\frac{1 + \varepsilon_1}{\varepsilon_1}\right)^{p-1} - 1\right) < \varepsilon_2 A^p. \tag{37}$$

Then

$$f^{\varphi}(x) = \sup_{\|z\| \le A} \big(\|x - z\|^p - f(z)\big). \tag{38}$$

Indeed, using the convexity of $\|\cdot\|^p$, we obtain the inequality

$$\|x - z\|^p \le (1 + \varepsilon_1)^{p-1}\|z\|^p + \left(\frac{1 + \varepsilon_1}{\varepsilon_1}\right)^{p-1} \|x\|^p, \quad \forall z \in X.$$

Hence, subtracting $f(z)$ in both sides and taking the supremum over $z \in X$, with $\|z\| > A$, we obtain according to (36) that

$$\sup_{\|z\| > A} \big(\|x - z\|^p - f(z)\big) \le -\varepsilon_2 A^p + \left(\frac{1 + \varepsilon_1}{\varepsilon_1}\right)^{p-1} \|x\|^p.$$

Then, (37) yields

$$\sup_{\|z\| > A} \big(\|x - z\|^p - f(z)\big) \le \|x\|^p \le \sup_{\|z\| \le A} \big(\|x - z\|^p - f(z)\big),$$

where the latter inequality is due to the equality $f(0) = 0$. It results that (38) holds true and $M_{\|\cdot\|^p} f(x) = M_{\|\cdot\|^p}(f + \delta_{A\mathbb{B}_X})(x)$, and the reasoning used in the case of the bounded domain can be applied.

Whence, relying on Remark 4.1, the following corollary can be derived.

COROLLARY 4.1. *Let* $\varphi := \|\cdot\|^p$, *with* $p > 1$, *let* $f\colon X \to \mathbb{R} \cup \{+\infty\}$ *be a lower semicontinuous function and* $x \in X$ *be a point where* f^{φ} *is Fréchet differentiable. Suppose that condition* (35) *holds. Then the set* $M_{\varphi} f(x)$ *is at most a singleton and each maximizing sequence* $\{y_i\}_{i\in\mathbb{N}}$ *for* $f^{\varphi}(x)$ *is strongly convergent provided that the set* $\{x - y_i : i \in \mathbb{N}\}$ *has the NSLUC property and* $M_{\varphi} f(x) \neq \emptyset$.

PROOF. Using the previous comments (see relation (38)), we see that each maximizing sequence $\{y_i\}_{i\in\mathbb{N}}$ for $f^{\varphi}(x)$ is bounded. Then the result follows from Remark 4.1 and Theorem 4.1.

Let us continue with the same setting as in Corollary 4.1, that is, $\varphi := \|\cdot\|^p$, with $p > 1$. In Proposition 2.2 it is stated that the Fréchet differentiability of f^{φ} at x ensures the equality

$$^{\|\cdot\|}\limsup_{i\to\infty} \partial_{\delta_i}\varphi(x - y_i) = \{D_F f^{\varphi}(x)\}$$

for each maximizing sequence $\{y_i\}_{i\in\mathbb{N}}$ for $f^{\varphi}(x)$ and any sequence $\{\delta_i\}_{i\in\mathbb{N}}$ of non-negative real numbers, with $\lim_{i\to\infty} \delta_i = 0$. Thus relation (11) is satisfied for the sequence $\{x - y_i\}_{i\in\mathbb{N}}$ at $x - y$, for any $y \in M_\varphi f(x)$ since $D_F\varphi(x - y) = D_F f^\varphi(x)$ for any such y according to Lemma 2.1. Then, if $\{y_i\}_{i\in\mathbb{N}}$ is bounded and the LUR property holds at $x - y$, applying Proposition 3.6, it is clear that the sequence $\{x - y_i\}_{i\in\mathbb{N}}$ satisfies the NSLUC property. So replacing the NSLUC property by the LUR property at $x - y$, this allows us to extend the result by Cibulka and Fabian [7] to the case where $\varphi := \|\cdot\|^p$, with $p > 1$.

COROLLARY 4.2. *Let the assumptions of Corollary 4.1 be satisfied. Then the set $M_\varphi f(x)$ is at most a singleton and each maximizing sequence $\{y_i\}_{i\in\mathbb{N}}$ for $f^\varphi(x)$ is strongly convergent provided that the LUR property holds at $x - y$, where $y \in M_\varphi f(x)$.*

Besides Theorem 4.1, when the Banach space X is reflexive and the set $x - \operatorname{dom} f$ is NSLUC, we have the following list of equivalences with the Fréchet differentiability of f^φ at x.

THEOREM 4.2. *Assume that $(X, \|\cdot\|)$ is a reflexive Banach space and $f\colon X \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function such that $x - \operatorname{dom} f$ is a non-empty bounded set which has the NSLUC property for some $x \in X$. Let $\varphi\colon X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function in the form*

$$\varphi(\cdot) = g(\|\cdot\|) + \delta_Q(\cdot),$$

where Q is a non-empty convex subset with $0 \in Q$ and $g\colon [0, \infty[\to [0, \infty[$ is an increasing convex function with $g(0) = 0$. Assume that f^φ is finite at x and either g is strictly convex or f is constant on its domain. Then the following conditions are equivalent:

(i) *the Fréchet derivative of f^φ at x exists;*

(ii) *each maximizing sequence $\{y_i\}_{i\in\mathbb{N}}$ for $f^\varphi(x)$ is strongly convergent and φ is Fréchet differentiable at any point of the set $x - M_\varphi f(x)$ (which is a singleton);*

(iii) *the set $M_\varphi f(x)$ is a singleton, say $M_\varphi f(x) = \{\overline{y}\}$ and φ is Fréchet differentiable at $x - \overline{y}$, and*

$$^{\|\cdot\|}\operatorname*{Lim\,sup}_{u\to x,\varepsilon\searrow 0} M_{\varepsilon,\varphi} f(u) = {}^{\|\cdot\|}\operatorname*{Lim\,inf}_{u\to x,\varepsilon\searrow 0} M_{\varepsilon,\varphi} f(u) = M_\varphi f(x).$$

PROOF. In order to prove the implication (i) $\Rightarrow$ (ii) let us first observe that it follows from Lemma 2.1 that φ is Fréchet differentiable at any point of the set $x - M_\varphi f(x)$. Let $\{y_i\}_{i\in\mathbb{N}}$ be a maximizing sequence for $f^\varphi(x)$. If $\{y_{i_k}\}_{k\in\mathbb{N}}$ is a subsequence converging weakly to some $\overline{y}$, then Proposition 2.2 ensures that the sequences $\{\varphi(x - y_{i_k})\}_{k\in\mathbb{N}}$ and $\{f(y_{i_k})\}_{k\in\mathbb{N}}$ converge to $\varphi(x - \overline{y})$ and $(\overline{\text{co}}\, f)(\overline{y})$, respectively and the limit point $\overline{y}$ satisfies $\overline{y} \in M_\varphi(\overline{\text{co}}\, f)(x)$.

By Proposition 2.1 we have the equality $f^\varphi(x) = (\overline{\text{co}}\, f)^\varphi(x)$, thus by the definition of φ we get $\lim_{k\to\infty} \|x-y_{i_k}\| = \|x-\overline{y}\|$. By the Kadec-Klee property of $x - \operatorname{dom} f$ with respect to the norm, see Proposition 3.1, we get the strong convergence of the subsequence $\{y_{i_k}\}_{k\in\mathbb{N}}$ to $\overline{y}$, that is, $\lim_{k\to\infty} \|\overline{y} - y_{i_k}\| = 0$. Suppose that w is a cluster point of the sequence $\{y_i\}_{i\in\mathbb{N}}$ (keep in mind that any weak cluster point of the sequence is the strong limit of a subsequence). If g is strictly convex, it follows from Theorem 4.1 that $w = \overline{y}$. In the case when f is constant on its domain we get $\varphi(x - \overline{y}) = \varphi(x - w)$. Hence, we derive from Theorem 4.1 (we recall that the set $M_\varphi(\overline{\text{co}}\, f)(x)$ is convex) that the function φ is constant on the segment $[x - \overline{y}, x - w]$. Let us recall that $w, \overline{y} \in \operatorname{dom} f$ and the norm is strictly convex on $x - \operatorname{dom} f$, see Proposition 3.1, so using the form of φ we get that $w = \overline{y}$ (keep in mind that φ is a sum of an indicator function and a composition of an increasing convex function with the norm). Thus, in any case the sequence $\{y_i\}_{i\in\mathbb{N}}$ has to be convergent to $\overline{y}$.

The implication (ii) $\Rightarrow$ (iii) is obvious.

To prove the last implication (iii) $\Rightarrow$ (i) put $x^* := D_F\varphi(x - \overline{y})$. For every $h \in X$ choose, by assumption (iii), some $y(h) \in M_{\|h\|^2,\varphi} f(x + h)$ with $y(h) \to \overline{y}$ and choose also some $x_h^* \in \partial\varphi(x + h - y(h))$. We then have

$$
\begin{aligned}
& f^\varphi(x+h) - f^\varphi(x) - \langle x^*, h\rangle \\
&\quad \le \varphi(x + h - y(h)) - f(y(h)) + \|h\|^2 \\
&\qquad\quad - \big(\varphi(x - y(h)) - f(y(h))\big) - \langle x^*, h\rangle \\
&\quad = \varphi(x + h - y(h)) - \varphi(x - y(h)) + \|h\|^2 - \langle x^*, h\rangle \\
&\quad \le \langle x_h^*, h\rangle - \langle x^*, h\rangle + \|h\|^2 \\
&\quad \le \big(\|h\| + \varepsilon(h + \overline{y} - y(h))\big)\|h\|,
\end{aligned}
$$

where $\varepsilon(z) \to 0$ as $z \to 0$ since $\partial\varphi(u) \xrightarrow[u\to x-\overline{y}]{} \{x^*\}$ with respect to the Hausdorff distance (see [2, Corollary 2 of Theorem 3]). It ensues that $-x^* \in \partial_F(-f^\varphi)(x)$, hence Lemma 2.1(iii) yields that f^φ is Fréchet differentiable at x.

Taking $\varphi := \|\cdot\|$ and $f := \delta_S$, the following result is obtained as a direct consequence of Theorem 4.2. According to our notation $\kappa_{\lambda,p} f$ for the Klee function and writing κ_S instead of $\kappa_{1,1}\delta_S$, the farthest distance function from the set S is the function κ_S defined by

$$\kappa_S(x) = \kappa_{1,1}\delta_S(x) = \sup_{y\in S} \|x - y\| = (\delta_S)^{\|\cdot\|}(x).$$

COROLLARY 4.3. *Let* $(X, \|\cdot\|)$ *be a reflexive Banach space and* $S \subset X$ *be a closed bounded set such that* $x - S$ *has the NSLUC property for some* $x \in X$. *Then the following assertions are equivalent:*

(i) κ_S *is Fréchet differentiable at* x*;*

(ii) *each maximizing sequence* $\{y_i\}_{i\in\mathbb{N}}$ *for* $\kappa_S(x)$ *is strongly convergent and* $\|\cdot\|$ *is Fréchet differentiable at any point of the set* $x - M_{\|\cdot\|}\delta_S(x)$ (*which is a singleton*)*;*

(iii) *the set* $M_{\|\cdot\|}\delta_S(x)$ *is a singleton, say* $M_{\|\cdot\|}\delta_S(x) = \{\overline{y}\}$ *and* $\|\cdot\|$ *is Fréchet differentiable at* $x - \overline{y}$*, and*

$$\begin{aligned} {}^{\|\cdot\|}\mathop{\mathrm{Lim\,sup}}_{u\to x,\varepsilon\searrow 0} M_{\varepsilon,\|\cdot\|}\delta_S(u) &= {}^{\|\cdot\|}\mathop{\mathrm{Lim\,inf}}_{u\to x,\varepsilon\searrow 0} M_{\varepsilon,\|\cdot\|}\delta_S(u) \\ &= M_{\|\cdot\|}\delta_S(x). \end{aligned}$$

As a consequence of Corollary 4.3 and Proposition 3.2, we recover a result given by Ivanov, see [11, Theorem 1(a)–(e)]. In order to avoid introducing additional notions, we restrict ourselves to statements (a)–(c) of Ivanov's Theorem.

THEOREM 4.3 (Ivanov [11]). *Suppose that* $(X, \|\cdot\|)$ *is a reflexive, LUR Banach space with Fréchet differentiable norm off the origin. Suppose that a number* $r > 0$ *and a convex closed bounded set* $S \subset X$ *are given. Then, the following assertions are equivalent:*

(a) *the farthest distance function* κ_S *(denoted by* f_S *in [11]) is Fréchet differentiable on the set* $\{u \in X : \kappa_S(u) > r\}$*;*

(b) *for any vector* $x \in \{u \in X : \kappa_S(u) > r\}$*, any sequence* $\{a_k\}_{k\in\mathbb{N}}$ *of elements of* S *such that* $\lim_{k\to\infty} \|a_k - x\| = \kappa_S(x)$ *is strongly convergent;*

(c) *the attainment set* $M_{\|\cdot\|}\delta_S(\cdot)$ (*the farthest point mapping of* S *also called the metric antiprojection*) *is single-valued and continuous on the set* $\{u \in X : \kappa_S(u) > r\}$.

5. Genericity of non-emptiness of the attainment set

In Theorem 4.1 it is assumed that φ is in the form $\varphi = g(\|\cdot\|) + \delta_Q$, where g is strictly convex, in order to derive the strong convergence of the maximizing sequence. In this section, to study the genericity in X of $\mathrm{dom}\, M_\varphi f$ we consider a suitable class of functions φ. The class encompasses the sublinear functions, thus allowing to recover the case $\varphi = \|\cdot\|$ as a particular instance. Let us establish first the following fundamental lemma.

LEMMA 5.1. *Let* $(X, \|\cdot\|)$ *be a normed space and* $\varphi\colon X \to \mathbb{R} \cup \{+\infty\}$ *be a convex function with* $\varphi(0) = 0$. *Let* $f\colon X \to \mathbb{R} \cup \{+\infty\}$ *be a function such that* f^φ *is finite-valued and continuous and assume that for each* $\varepsilon > 0$ *and* $x \in X$, *the set* $M_{\varepsilon,\varphi} f(x)$ *is bounded. Then the set*

$$\left\{x \in X : \exists\, x^* \in \partial f^\varphi(x),\ \sup_{y\in X}\big(\langle x^*, x - y\rangle - f(y)\big) < f^\varphi(x)\right\}$$

is of first category in X, *that is, a countable union of closed sets with empty interior.*

PROOF. For each integer $n \in \mathbb{N}$ denote

$$A_n := \left\{x \in X : \exists\, x^* \in \partial f^\varphi(x),\ \sup_{y\in X}\big(\langle x^*, x - y\rangle - f(y)\big) \le f^\varphi(x) - \frac{1}{n}\right\},$$

so the set of the lemma obviously coincides with $\bigcup_{n\in\mathbb{N}} A_n$.

Let us first fix $n \in \mathbb{N}$ and show that A_n is closed. Consider any sequence $\{x_i\}_{i\in\mathbb{N}}$ in A_n converging to some $x \in X$, and for each $i \in \mathbb{N}$ choose by definition of A_n some $x_i^* \in \partial f^\varphi(x_i)$ satisfying

$$\sup_{y\in X}\big(\langle x_i^*, x_i - y\rangle - f(y)\big) \le f^\varphi(x_i) - \frac{1}{n}.$$

The continuous function f^φ is convex by the convexity of φ, so it is locally Lipschitz on X. This local Lipschitz property of f^φ ensures that the sequence $\{x_i^*\}_{i\in\mathbb{N}}$ is bounded. By the Banach-Alaoglu Theorem, see [16, Theorem 3.15, p. 66] for example, we have

$$\bigcap_{i\in\mathbb{N}} (\mathrm{w}^*\text{-cl})\{x_i^*, x_{i+1}^*, \ldots\} \neq \emptyset,$$

where $(\mathrm{w}^*\text{-cl})$ stands for the weak* closure. Take

$$x^* \in \bigcap_{i\in\mathbb{N}} (\mathrm{w}^*\text{-cl})\{x_i^*, x_{i+1}^*, \ldots\}.$$

In order to prove that $x^* \in \partial f^\varphi(x)$, let us fix any $h \in X$. Choose $i_k \le i_{k+1}$ such that $k \le i_k$ and $|\langle x_{i_k}^* - x^*, h\rangle| \le 2^{-k}$ for all $k \in \mathbb{N}$ (this is possible since $x^* \in (\mathrm{w}^*\text{-cl})\{x_k^*, x_{k+1}^*, \ldots\}$). Thus we have

$$\lim_{k\to\infty} \langle x_{i_k}^*, h\rangle = \langle x^*, h\rangle$$

and observe that, by the continuity of f^φ, we get

$$\langle x^*, h\rangle + f^\varphi(x) = \lim_{k\to\infty}\big(\langle x_{i_k}^*, h\rangle + f^\varphi(x_{i_k})\big) \le \lim_{k\to\infty} f^\varphi(x_{i_k} + h) = f^\varphi(x + h).$$

This implies $x^* \in \partial f^\varphi(x)$, since h is arbitrary. In the same manner as above, for every $y \in X$, using the inequality

$$\langle x_i^*, x_i - y\rangle - f(y) \le f^\varphi(x_i) - \tfrac{1}{n}$$

and the continuity of f^φ again, we get also

$$\langle x^*, x - y\rangle - f(y) \le f^\varphi(x) - \tfrac{1}{n}.$$

It follows that $x \in A_n$, justifying the closedness of A_n.

It remains to prove that all A_n have empty interior. Suppose, for some $n \in \mathbb{N}$, that int $A_n \neq \emptyset$ and take some $\overline{x} \in X$ and $r > 0$ such that $B[\overline{x}, r] \subset A_n$. Since the set of approximate maximizers $M_{1,\varphi} f(\overline{x})$ is non-empty and bounded, we can define the real $\gamma := \sup\big\{\|\overline{x} - y\| : y \in M_{1,\varphi} f(\overline{x})\big\}$. For $\varepsilon := (2n(\gamma + r))^{-1} r$, there exists by (5) and by definition of f^φ some $\overline{y} \in M_{\varepsilon,\varphi} f(\overline{x}) \subset M_{1,\varphi} f(\overline{x}) \subset \operatorname{dom} f$ (with both $f(\overline{y})$ and $\varphi(\overline{x} - \overline{y})$ finite) satisfying

$$f^\varphi(\overline{x}) - \varepsilon \le \varphi(\overline{x} - \overline{y}) - f(\overline{y}) \le f^\varphi(\overline{x}). \tag{39}$$

Define $t := r/\gamma$ and $u := \overline{x} + t(\overline{x} - \overline{y}) \in B[\overline{x}, r]$, so $u \in A_n$. From the definition of A_n there is some $u^* \in \partial f^\varphi(u)$ such that

$$\sup_{y \in X}\big(\langle u^*, u - y\rangle - f(y)\big) \le f^\varphi(u) - \tfrac{1}{n}. \tag{40}$$

On the other hand, by (39) and the equality $(u - \overline{y}) = (1 + t)(\overline{x} - \overline{y})$, we also have, according to the equality $\varphi(0) = 0$,

$$\begin{aligned} f^\varphi(\overline{x}) - f^\varphi(u) &\le \varphi(\overline{x} - \overline{y}) - f(\overline{y}) + \varepsilon - f^\varphi(u) \\ &\le \frac{1}{(1+t)}\varphi(u - \overline{y}) - f(\overline{y}) + \varepsilon - f^\varphi(u) \\ &\le \frac{1}{1+t}\big(\varphi(u - \overline{y}) - f(\overline{y})\big) - \frac{t}{1+t} f(\overline{y}) + \varepsilon - f^\varphi(u), \end{aligned}$$

which ensures by the definition of f^φ and by (40)

$$\begin{aligned} f^\varphi(\overline{x}) - f^\varphi(u) &\le \frac{1}{1+t} f^\varphi(u) - \frac{t}{1+t} f(\overline{y}) + \varepsilon - f^\varphi(u) \\ &= \frac{-t}{1+t} f^\varphi(u) - \frac{t}{1+t} f(\overline{y}) + \varepsilon \\ &\le \frac{t}{1+t}\big(\langle u^*, \overline{y} - u\rangle + f(\overline{y})\big) - \frac{t}{1+t} f(\overline{y}) + \varepsilon - \frac{t}{n(1+t)} \\ &= \frac{t}{1+t}\langle u^*, \overline{y} - u\rangle + \varepsilon - \frac{r}{n(\gamma + r)}. \end{aligned}$$

Hence, taking the equality $\overline{y} - u = \frac{1+t}{t}(\overline{x} - u)$ into account, we obtain

$$f^{\varphi}(\overline{x}) - f^{\varphi}(u) \leq \langle u^*, \overline{x} - u \rangle + \varepsilon - \frac{r}{n(\gamma + r)}.$$

Since $\varepsilon < (n(\gamma + r))^{-1} r$, we deduce that

$$f^{\varphi}(\overline{x}) - f^{\varphi}(u) < \langle u^*, \overline{x} - u \rangle,$$

which contradicts the inclusion $u^* \in \partial f^{\varphi}(u)$, completing the proof.

REMARK 5.1. If $\inf_{y \in X} f^{\varphi}(y) + f(y) \leq 0$, then the set

$$\left\{x \in X : \exists x^* \in \partial f^{\varphi}(x),\ \sup_{y \in X} \big(\langle x^*, x - y \rangle - f(y)\big) < f^{\varphi}(x)\right\}$$

is empty. Indeed, let us argue by contraposition and assume that there exist $x \in X$ and $x^* \in \partial f^{\varphi}(x)$ such that $\sup_{y \in X} \big(\langle x^*, x - y \rangle - f(y)\big) < f^{\varphi}(x)$. Thus the following inequalities hold true

$$\sup_{y \in X} \big(f^{\varphi}(x) - f^{\varphi}(y) - f(y)\big) \leq \sup_{y \in X} \big(\langle x^*, x - y \rangle - f(y)\big) < f^{\varphi}(x),$$

which implies that $\sup_{y \in X}(-f^{\varphi}(y) - f(y)) < 0$ and finally $\inf_{y \in X} f^{\varphi}(y) + f(y) > 0$.

In Lemma 5.1 it is established that the set

$$\left\{x \in X : \exists x^* \in \partial f^{\varphi}(x),\ \sup_{y \in X} \big(\langle x^*, x - y \rangle - f(y)\big) < f^{\varphi}(x)\right\}$$

is of first category, thus its complement is a dense set of G_{δ}-type, whenever the space is complete. Hence if we are able to guarantee the inclusion

$$\emptyset \neq \left\{x \in X : \exists x^* \in \partial \varphi(0),\ \sup_{y \in X} \big(\langle x^*, x - y \rangle - f(y)\big) \geq f^{\varphi}(x)\right\} \subset \operatorname{dom} M_{\varphi} f,$$

on an open set V, then for a generic subset of points from the set, say $x \in V$, we have the non-emptiness of $M_{\varphi} f(x)$. In the lemma below, an inclusion of this type is established and applied in the next theorem to derive the non-emptiness of the attainment set.

LEMMA 5.2. *Let $(X, \|\cdot\|)$ be a normed space and let $\varphi\colon X \to \mathbb{R} \cup \{+\infty\}$ be a convex function with $\varphi(0) = 0$. Let $f\colon X \to \overline{\mathbb{R}}$ be a function such that, for each $x^* \in \partial\varphi(0)$, the infimum of the function $f + \langle x^*, \cdot \rangle$ is attained. Then the following inclusion holds*

$$\left\{x \in X : \exists x^* \in \partial \varphi(0),\ \sup_{y \in X} \big(\langle x^*, x - y \rangle - f(y)\big) \geq f^{\varphi}(x)\right\} \subset \operatorname{dom} M_{\varphi} f.$$

PROOF. Let $x \in X$ and $x^* \in \partial\varphi(0)$ be such that $\sup_{y\in X}\big(\langle x^*, x - y\rangle - f(y)\big) \geq f^{\varphi}(x)$. In view of the attainment assumption, there exists $\overline{y} \in X$ such that

$$\sup_{y\in X}\big(\langle x^*, x - y\rangle - f(y)\big) = \langle x^*, x - \overline{y}\rangle - f(\overline{y}).$$

Since $x^* \in \partial\varphi(0)$ and $\varphi(0) = 0$, it ensues that

$$f^{\varphi}(x) \leq \langle x^*, x - \overline{y}\rangle - f(\overline{y}) \leq \varphi(x - \overline{y}) - f(\overline{y}) \leq f^{\varphi}(x).$$

We conclude that $f^{\varphi}(x) = \varphi(x - \overline{y}) - f(\overline{y})$, hence $\overline{y} \in M_{\varphi} f(x)$ and therefore $x \in \operatorname{dom} M_{\varphi} f$.

A theorem on the genericity of non-emptiness of the attainment set is given below.

THEOREM 5.1. *Let* $(X, \|\cdot\|)$ *be a Banach space and let* $\varphi\colon X \to \mathbb{R} \cup \{+\infty\}$ *be a convex function such that* $\varphi(0) = 0$. *Let* $f\colon X \to \mathbb{R} \cup \{+\infty\}$ *be a function such that* f^{φ} *is finite-valued and continuous and assume that for each* $\varepsilon > 0$ *and* $x \in X$, *the set* $M_{\varepsilon,\varphi} f(x)$ *is bounded. Suppose additionally that for each* $x^* \in \partial\varphi(0)$, *the infimum of the function* $f + \langle x^*, \cdot\rangle$ *is attained and that*

$$\forall x \in X, \quad \partial f^{\varphi}(x) \cap \partial\varphi(0) \neq \emptyset. \tag{41}$$

Then the set dom $M_{\varphi} f$ *contains a dense* G_{δ} *subset of* X.

PROOF. In view of assumption (41), we have

$$\Big\{x \in X : \forall x^* \in \partial f^{\varphi}(x),\ \sup_{y\in X}\big(\langle x^*, x - y\rangle - f(y)\big) \geq f^{\varphi}(x)\Big\}$$
$$\subset \Big\{x \in X : \exists x^* \in \partial\varphi(0),\ \sup_{y\in X}\big(\langle x^*, x - y\rangle - f(y)\big) \geq f^{\varphi}(x)\Big\}.$$

From Lemma 5.1, the first set above is a countable intersection of dense open sets, hence is a dense G_{δ}-set (recall that the space X is complete). On the other hand, Lemma 5.2 shows that the second set above is included in dom $M_{\varphi} f$. The proof is complete.

Now assume that the function $\varphi\colon X \to \mathbb{R}\cup\{+\infty\}$ is subadditive and $f\colon X \to \mathbb{R} \cup \{+\infty\}$ is a proper function. By simple algebra, we get

$$f^{\varphi}(x') \leq f^{\varphi}(x) + \varphi(x' - x). \tag{42}$$

In particular, if $f^{\varphi}(x) \in \mathbb{R}$, this ensures that, for any $x^* \in \partial f^{\varphi}(x)$,

$$\langle x^*, x' - x\rangle \leq f^{\varphi}(x') - f^{\varphi}(x) \leq \varphi(x' - x), \quad \text{for all } x' \in X,$$

so $x^* \in \partial\varphi(0)$, whenever φ is convex and $\varphi(0) = 0$. Consequently,

$$\partial f^{\varphi}(x) \subset \partial\varphi(0), \quad \text{for all } x \in X.$$

COROLLARY 5.1. *Let $(X, \|\cdot\|)$ be a Banach space and let $\varphi: X \to \mathbb{R} \cup \{+\infty\}$ be a sublinear function such that $\varphi(0) = 0$. Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be a function such that f^{φ} is finite-valued and continuous and assume that for each $\varepsilon > 0$ and $x \in X$, the set $M_{\varepsilon,\varphi} f(x)$ is bounded. Suppose additionally that for each $x^* \in \partial\varphi(0)$, the infimum of the function $f + \langle x^*, \cdot\rangle$ is attained. Then the set* dom $M_{\varphi} f$ *contains a dense G_{δ}-subset of X.*

PROOF. The subadditivity of φ implies that $\partial f^{\varphi}(x) \subset \partial\varphi(0)$ for every $x \in X$. Since the function f^{φ} is assumed to be continuous, we have $\partial f^{\varphi}(x) \neq \emptyset$ for every $x \in X$. It ensues immediately that condition (41) is satisfied, hence Theorem 5.1 applies.

The continuity of f^{φ} is a key assumption in Lemma 5.1 and Theorem 5.1. The next proposition gives several conditions on φ and f which ensure the uniform continuity or the global Lipschitz continuity of the function f^{φ}.

Let us first observe that a subadditive function which is Lipschitz continuous around the origin is globally Lipschitz continuous, whenever its value at the origin amounts zero.

LEMMA 5.3. *Let $(X, \|\cdot\|)$ be a normed space and $\varphi: X \to \overline{\mathbb{R}}$ be Lipschitz continuous around* 0*, subadditive on the whole space and such that $\varphi(0) = 0$. Then φ is globally Lipschitz continuous on X.*

PROOF. Note first that $\varphi(x) > -\infty$ for all $x \in X$, since φ is finite on some neighborhood of the origin and subadditive on the whole space. On the other hand, take $k \geq 0$ and $r > 0$ such that $|\varphi(u)| \leq k\|u\|$ for every $u \in B(0, r)$. The subadditivity of φ implies that, for any $x, x' \in X$ with $\|x' - x\| < r$,

$$\varphi(x') \leq \varphi(x) + \varphi(x' - x) \leq \varphi(x) + k\|x' - x\|, \tag{43}$$

and symmetrically

$$\varphi(x') \geq \varphi(x) - k\|x' - x\|. \tag{44}$$

If $\varphi(x) \in \mathbb{R}$ then $\varphi(x') \in \mathbb{R}$ for every $x' \in B(x, r)$, thus the set $\varphi^{-1}(\mathbb{R})$ is open. It follows from (44) that the set $\varphi^{-1}(+\infty)$ is open too (taking $\varphi(x) = +\infty$ we have $\varphi(x') = +\infty$ for $x' \in B(x, r)$). Finally, the space X can be decomposed as the union of the two open disjoint sets $\varphi^{-1}(\mathbb{R})$ and $\varphi^{-1}(+\infty)$. Since the space X is connected, we deduce that each of these sets is either empty or equal to X. Since φ is finite at 0, we deduce that φ is finite-valued on X. In view of (43)–(44), we conclude that $|\varphi(x') - \varphi(x)| \leq k\|x' - x\|$ for all $x, x' \in X$ such that $\|x' - x\| < r$, which implies that φ is globally Lipschitz with the constant k, since φ is finite-valued on the whole space and locally Lipschitz with the same constant invariant under the shifting of arguments.

PROPOSITION 5.1. *Let* $(X, \|\cdot\|)$ *be a normed space and* $f, \varphi: X \to \overline{\mathbb{R}}$ *be such that* $f^{\varphi} \neq \pm\omega_X$.

(i) *If* φ *(resp.* $-f$*) is globally Lipschitz continuous on* X*, then* f^{φ} *is globally Lipschitz continuous on* X.

(ii) *If* φ *(resp.* $-f$*) is Lipschitz continuous around* 0*, subadditive on the whole space and such that* $\varphi(0) = 0$ *(resp.* $f(0) = 0$*), then* f^{φ} *is globally Lipschitz continuous on* X.

(iii) *If* φ *(resp.* $-f$*) is upper semicontinuous at* 0*, subadditive and such that* $\varphi(0) = 0$ *(resp.* $f(0) = 0$*), then* f^{φ} *is uniformly continuous on* X.

PROOF. Noticing that $f^{\varphi} = (-\varphi)^{-f}$ it suffices to proceed with merely the case of the function φ in each one of the assertions of the proposition.

(i) Assume that φ is k-Lipschitz continuous on X, for some $k \geq 0$. Then we have for all $x, x' \in X$ and $y \in X$,

$$\varphi(x' - y) - f(y) \leq \varphi(x - y) - f(y) + k\|x' - x\|$$

and $f(y) > -\infty$ (otherwise $f^{\varphi} = \omega_X$). Taking the supremum over $y \in X$, we find for all $x, x' \in X$

$$f^{\varphi}(x') \leq f^{\varphi}(x) + k\|x' - x\|. \tag{45}$$

By interchanging the roles of x and x', we obtain

$$f^{\varphi}(x') \geq f^{\varphi}(x) - k\|x' - x\|. \tag{46}$$

If $f^{\varphi}(x) = -\infty$ (or $f^{\varphi}(x) = \infty$) for some $x \in X$, we deduce from (45) that $f^{\varphi} = -\omega_X$ (resp. $f^{\varphi} = \omega_X$), a contradiction. It ensues that f^{φ} is finite-valued and we conclude in view of (45)–(46) that f^{φ} is k-Lipschitz continuous.

(ii) This is a simple consequence of Lemma 5.3 and statement (i).

(iii) Let us point out that any subadditive function h, which is upper semi-continuous at the origin with $h(0) = 0$ is finite in a neighborhood of the origin, and hence is continuous on the neighborhood. Thus for all real $\varepsilon > 0$ there is some $\delta > 0$ such that $|h(u)| < \varepsilon$ for any $u \in B(0, \delta)$. Applying this to φ, by a similar reasoning as in (i) we obtain

$$f^{\varphi}(x) - \varepsilon \leq f^{\varphi}(x') \leq f^{\varphi}(x) + \varepsilon, \tag{47}$$

for all $x, x' \in X$ with $\|x' - x\| < \delta$, and $(f^{\varphi})^{-1}(\mathbb{R}) = X$. From (47) we deduce that f^{φ} is uniformly continuous.

6. Links between φ-envelopes and marginal inf-value functions of φ

Given a subset S of X, for convenience and mimicking the notation of the distance function, let us denote by φ_S the function defined by

$$\varphi_S(x) := \inf_{y \in S} \varphi(y - x), \quad \text{for all } x \in X.$$

We say that φ_S is the *marginal inf-value function of* φ *over the set* S. Our objective in this section is to show that the techniques of [12, §7] can be adapted to provide a link between the φ-envelope of a function $f : X \to \overline{\mathbb{R}}$ and the marginal inf-value function of φ over suitable level set of the function f.

Let us start with the following lemma.

LEMMA 6.1. *Let* $\varphi, f : X \to \overline{\mathbb{R}}$ *be two extended real-valued functions. Assume that* $\varphi(x) \geq 0$ *for every* $x \in X$. *Then we have*

$$\inf_X f^\varphi \geq -\inf_X f.$$

Moreover, the following inclusion holds

$$\{x \in X : f^\varphi(x) = -f(x)\} \subset \operatorname{argmin} f \cap \operatorname{argmin} f^\varphi.$$

PROOF. Since $\varphi \geq 0$, we have for all $x, y \in X$,

$$\begin{aligned} f^\varphi(x) &\geq \varphi(x - y) \dot{+} (-f(y)) \\ &\geq -f(y). \end{aligned} \tag{48}$$

Taking the infimum over $x \in X$ and then the supremum over $y \in X$, we derive $\inf_X f^\varphi \geq \sup_X(-f) = -\inf_X f$.

Fix $x \in X$ and assume that $f^\varphi(x) = -f(x)$. We deduce from (48) that $f(x) \leq f(y)$ for every $y \in X$, hence $x \in \operatorname{argmin} f$. This proves the inclusion

$$\{x \in X : f^\varphi(x) = -f(x)\} \subset \operatorname{argmin} f.$$

Let us now return to (48) and fix $y \in X$ such that $f^\varphi(y) = -f(y)$. We infer from (48) that $f^\varphi(x) \geq f^\varphi(y)$ for every $x \in X$, hence $y \in \operatorname{argmin} f^\varphi$. The inclusion

$$\{x \in X : f^\varphi(x) = -f(x)\} \subset \operatorname{argmin} f^\varphi$$

is shown, which ends the proof.

Let us notice that assuming $f \neq \omega_X$, under the assumptions of Lemma 6.1, we get $\inf_X f^\varphi > -\infty$. It is easy to notice also that the coercivity assumption on f^φ, that is $\lim_{\|y\| \to +\infty} f^\varphi(y) = +\infty$, implies $f \neq \omega_X$, thus the coercivity of f^φ implies $\inf_X f^\varphi > -\infty$, whenever the assumptions of Lemma 6.1 are fulfilled.

For convenience, given a function $g: X \to \overline{\mathbb{R}}$ and a real β, we will use the notations

$$\{g = \beta\} := \{x \in X : g(x) = \beta\},$$
$$\{g < \beta\} := \{x \in X : g(x) < \beta\},$$
$$\{g \leq \beta\} := \{x \in X : g(x) \leq \beta\},$$

and the corresponding notations for upper level sets. For each $\alpha > 0$, put $m_\alpha = \alpha + \inf_X f^\varphi$ and define the sets

$$D_\alpha = \{f^\varphi \leq m_\alpha\} \quad \text{and} \quad C_\alpha = \{x \in D_\alpha : f^\varphi(x) \neq -f(x)\}. \tag{49}$$

Suppose that $\varphi(x) \geq 0$ for all $x \in X$. It follows from Lemma 6.1 that

$$D_\alpha \setminus \operatorname{argmin} f \subset C_\alpha \quad \text{and} \quad D_\alpha \setminus \operatorname{argmin} f^\varphi \subset C_\alpha.$$

The second inclusion means equivalently that $\{x \in X : \inf_X f^\varphi < f^\varphi(x) \leq m_\alpha\}$ is included in C_α. If $\varphi(x) > 0$ for all $x \neq 0$ and argmin f is not a singleton, it is not difficult to see that $\{x \in X : f^\varphi(x) = -f(x)\} = \emptyset$, thus $C_\alpha = D_\alpha$. In order to see this, fix x and insert two different elements from argmin f, say y_1, y_2, in to the right-hand side of the inequality in (48). For one of them, say y_1, the inequality has to be strict, so $f^\varphi(x) > -f(y_1) = -\inf_{y\in X} f(y)$. The emptiness of the set $\{x \in X : f^\varphi(x) = -f(x)\}$ follows immediately.

Our aim is to show that the sum of the marginal inf-value function of φ and the φ-envelope of f is a constant function on some "large" set. In the proposition below an inequality is obtained and next in Theorem 6.1 the equality is established.

Recall that g is *positively homogeneous of degree* γ if $g(tx) = t^\gamma g(x)$, for all $x \in X$ and $t > 0$.

PROPOSITION 6.1. *Let* $(X, \|\cdot\|)$ *be a normed space and let* $f: X \to \overline{\mathbb{R}}$, $\varphi: X \to [0, \infty]$. *Suppose that* φ *is positively homogeneous of degree* $\gamma \geq 1$ *and proper. Assume that* f^φ *is real-valued and continuous on* X *and satisfies* $\lim_{\|y\|\to+\infty} f^\varphi(y) = +\infty$. *Then the following hold:*

(i) *For all* $x \in C_\alpha$, *we have*

$$m_\alpha \geq f^\varphi(x) + \varphi_{\{f^\varphi = m_\alpha\}}(x). \tag{50}$$

(ii) *The set* C_α *is dense in* D_α. *If the function* $x \mapsto \varphi_{\{f^\varphi = m_\alpha\}}(x)$ *is lower semicontinuous on* D_α, *then the inequality* (50) *holds for all* $x \in D_\alpha$.

PROOF. Keep in mind that m_α is a well-defined real number, see the comments following Lemma 6.1.

(i) Let $x \in C_\alpha$ and $\{y_n\}_{n\in\mathbb{N}}$ be a sequence (see (5)) such that

$$\forall n \geq 1, \quad f^\varphi(x) - \frac{1}{n} < \varphi(x - y_n) - f(y_n) \leq f^\varphi(x).$$

Note that, by the definition of the set C_α, we get $x \neq y_n$ for n large enough, and so we may assume that $x \neq y_n$ for every $n \geq 1$. Using the continuity and the coercivity of f^φ, the intermediate value theorem gives the existence of $t_n \geq 0$ such that

$$f^\varphi(z_n) = m_\alpha, \quad \text{where } z_n := x + t_n(x - y_n).$$

Thus

$$\begin{aligned} m_\alpha - f^\varphi(x) + \frac{1}{n} &= f^\varphi(z_n) - f^\varphi(x) + \frac{1}{n} \\ &\geq f^\varphi(z_n) + f(y_n) - \varphi(x - y_n) \\ &\geq \varphi(z_n - y_n) - \varphi(x - y_n) \\ &= \varphi\big((1 + t_n)(x - y_n)\big) - \varphi(x - y_n). \end{aligned}$$

Since φ is positively homogeneous of degree γ, this implies that

$$\begin{aligned} m_\alpha - f^\varphi(x) + \frac{1}{n} &\geq (1 + t_n)^\gamma \varphi(x - y_n) - \varphi(x - y_n) \\ &\geq (1 + t_n^\gamma)\varphi(x - y_n) - \varphi(x - y_n) \quad \text{because } \gamma \geq 1 \\ &= t_n^\gamma \varphi(x - y_n) \\ &= \varphi(z_n - x). \end{aligned}$$

Recalling that $f^\varphi(z_n) = m_\alpha$, it follows that

$$m_\alpha - f^\varphi(x) + \frac{1}{n} \geq \varphi_{\{f^\varphi = m_\alpha\}}(x).$$

The desired inequality (50) is obtained by taking the limit as $n \to \infty$.

(ii) Now let us show that the set C_α is dense in D_α. So suppose the contrary, that is, there exist $y_0 \in D_\alpha$ and $r > 0$ such that

$$B(y_0, r) \cap D_\alpha \subset \{x \in X : f^\varphi(x) = -f(x)\}. \tag{51}$$

Lemma 6.1 implies that

$$\{x \in X : f^\varphi(x) = -f(x)\} \subset \operatorname{argmin} f^\varphi \subset \{f^\varphi < m_\alpha\} \subset \operatorname{int} D_\alpha,$$

where the last inclusion is a consequence of the continuity of f^φ. In view of (51), we deduce that $y_0 \in \operatorname{int} D_\alpha$, hence there exists a real $0 < r' < r$ such that the inclusion $B(y_0, r') \subset D_\alpha$ holds. This and (51) again yield that

$$B(y_0, r') \subset \{x \in X : f^\varphi(x) = -f(x)\}.$$

Recalling Lemma 6.1, we infer that f and f^φ are constant and finite on $B(y_0, r')$, with respective values $\inf_X f$ and $\inf_X f^\varphi$. Thus we have

$$\begin{aligned} -f(y_0) = f^\varphi(y_0) &\geq \sup_{y \in B(y_0, r')} \big(\varphi(y_0 - y) - f(y)\big) \\ &= \sup_{y \in B(y_0, r')} \varphi(y_0 - y) - f(y_0) \\ &= \sup_{z \in B(0, r')} \varphi(z) - f(y_0). \end{aligned}$$

This implies that $\sup_{z \in B(0,r')} \varphi(z) \leq 0$, and hence $\varphi(z) = 0$ for every $z \in B(0, r')$. Because of the homogeneity of φ, we conclude that $\varphi \equiv 0$. This implies in turn that $f^\varphi \equiv \sup_X(-f) = -\inf_X f \in \mathbb{R}$, but this contradicts the coercivity of f^φ. The density of C_α in D_α is then established. Finally, the second conclusion in (ii) follows directly from (i) and the above density.

REMARK 6.1. The coercivity of φ ensures that of f^φ, whenever $f \neq \omega_X$. Indeed, given $\overline{x} \in \operatorname{dom} f$, we have $f^\varphi(x) \geq \varphi(x - \overline{x}) \dot{+} (-f(\overline{x}))$ for every $x \in X$, hence $\lim_{\|x\| \to +\infty} f^\varphi(x) = +\infty$.

REMARK 6.2. The lower semicontinuity of the function $x \mapsto \varphi_{\{f^\varphi = m_\alpha\}}(x) \in [0, \infty]$ is fulfilled in anyone of the following situations:

- (i) φ is subadditive, continuous at 0 and such that $\varphi(0) = 0$ (see the proof of Theorem 6.1 below);
- (ii) $\varphi = \|\cdot\|^\gamma$ $(\gamma \geq 1)$;
- (iii) the set $\{f^\varphi = m_\alpha\}$ is compact and φ is lower semicontinuous;
- (iv) X is finite dimensional, $\lim_{\|y\| \to +\infty} \varphi(y) = +\infty$ and φ is lower semicontinuous.

Let us now state and prove the theorem establishing the link between the φ-envelope of f and the marginal inf-value function of φ over suitable level set of f.

THEOREM 6.1. *Let $(X, \|\cdot\|)$ be a normed space and let $f: X \to \overline{\mathbb{R}}$. Let $\varphi: X \to [0, \infty[$ be a continuous and sublinear function. Assume that $\lim_{\|y\| \to +\infty} f^\varphi(y) = +\infty$. Then, for all $x \in D_\alpha$ one has*

$$m_\alpha = f^\varphi(x) + \varphi_{\{f^\varphi = m_\alpha\}}(x) = f^\varphi(x) + \varphi_{\{f^\varphi \geq m_\alpha\}}(x),$$

or equivalently

$$m_\alpha = f^\varphi(x) + \inf_{y \in \{f^\varphi = m_\alpha\}} \varphi(y - x) = f^\varphi(x) + \inf_{y \in \{f^\varphi \geq m_\alpha\}} \varphi(y - x).$$

PROOF. First observe that the above equalities are trivially true if $f^\varphi = \omega_X$. From now on, let us assume that $f^\varphi \neq \omega_X$, which implies $f(y) > -\infty$ for all $y \in X$. In order to prove the inequality

$$m_\alpha \geq f^\varphi(x) + \inf_{y \in \{f^\varphi = m_\alpha\}} \varphi(y - x), \tag{52}$$

for every $x \in D_\alpha$, let us check that the hypotheses of Proposition 6.1 are fulfilled. Since the function φ is subadditive and continuous at 0, and since $f^\varphi \neq \pm\omega_X$ (note that $f^\varphi \neq -\omega_X$ by the coercivity assumption of f^φ), Proposition 5.1(iii) ensures that the function f^φ is continuous on X. Let us now prove that the function $x \mapsto \inf_{y \in \{f^\varphi = m_\alpha\}} \varphi(y-x)$ is lower semicontinuous. Let $x \in X$ and $\{x_k\}_{k \in \mathbb{N}}$ be a sequence in X converging to x. Since φ is subadditive (and finite-valued), one has

$$\varphi(y - x) \leq \varphi(y - x_k) + \varphi(x_k - x),$$

and hence

$$\varphi(y - x_k) \geq \varphi(y - x) - \varphi(x_k - x).$$

Taking the infimum over $y \in \{f^\varphi = m_\alpha\}$, we obtain

$$\inf_{y \in \{f^\varphi = m_\alpha\}} \varphi(y - x_k) \geq \inf_{y \in \{f^\varphi = m_\alpha\}} \varphi(y - x) - \varphi(x_k - x).$$

Using the continuity of φ at 0, we deduce that

$$\liminf_{k \to +\infty} \inf_{y \in \{f^\varphi = m_\alpha\}} \varphi(y - x_k) \geq \inf_{y \in \{f^\varphi = m_\alpha\}} \varphi(y - x),$$

which shows that the function $x \mapsto \inf_{y \in \{f^\varphi = m_\alpha\}} \varphi(y - x)$ is lower semicontinuous. We can then apply Proposition 6.1(ii) to obtain that the inequality (52) is valid for every $x \in D_\alpha$.

To finish the proof, observe that the subadditivity of φ implies that, for all $x, y \in X$

$$f^\varphi(y) \leq f^\varphi(x) + \varphi(y - x),$$

see (42). It ensues that, for all $x \in X$ and $y \in \{f^\varphi \geq m_\alpha\}$,

$$m_\alpha \leq f^\varphi(y) \leq f^\varphi(x) + \varphi(y - x),$$

and hence for every $x \in X$

$$m_\alpha \le f^\varphi(x) + \inf_{y\in\{f^\varphi \ge m_\alpha\}} \varphi(y-x) \le f^\varphi(x) + \inf_{y\in\{f^\varphi = m_\alpha\}} \varphi(y-x),$$

and the proof is complete.

REMARK 6.3. Putting $\varphi_-(u) := \varphi(-u)$ for all $u \in X$, the equality $m_\alpha = f^\varphi(x) + \inf_{y\in\{f^\varphi = m_\alpha\}} \varphi(y-x)$ can be rewritten as

$$\begin{aligned} f^\varphi(x) &= m_\alpha + \sup_{y\in\{f^\varphi = m_\alpha\}} -\varphi(y-x) \\ &= m_\alpha + \sup_{y\in X}\big(-\varphi_-(x-y) - \delta_{\{f^\varphi = m_\alpha\}}(y)\big) \\ &= m_\alpha + \big(\delta_{\{f^\varphi = m_\alpha\}}\big)^{-\varphi_-}(x). \end{aligned}$$

It ensues that the result of Theorem 6.1 can be interpreted as follows

$$f^\varphi = m_\alpha + \big(\delta_{\{f^\varphi = m_\alpha\}}\big)^{-\varphi_-} = m_\alpha + \big(\delta_{\{f^\varphi \ge m_\alpha\}}\big)^{-\varphi_-} \quad \text{on } D_\alpha.$$

Note that Theorem 6.1 with $\varphi = \frac{1}{\lambda}\|\cdot\|$, $\lambda > 0$, corresponds to the following result in [12, Theorem 4].

COROLLARY 6.1. *Let* $\lambda > 0$ *and* $f\colon X \to \overline{\mathbb{R}}$. *For every* $x \in D_\alpha = \{x \in X : \kappa_{\lambda,1} f(x) \le m_\alpha\}$, *we have*

$$m_\alpha = \kappa_{\lambda,1} f(x) + \frac{1}{\lambda} d\big(x, \{\kappa_{\lambda,1} f = m_\alpha\}\big) = \kappa_{\lambda,1} f(x) + \frac{1}{\lambda} d\big(x, \{\kappa_{\lambda,1} f \ge m_\alpha\}\big).$$

PROOF. The function $\varphi = \frac{1}{\lambda}\|\cdot\|$ satisfies all the requirements of Theorem 6.1. If $f \ne \omega_X$, the coercivity of $\|\cdot\|$ implies that of $\kappa_{\lambda,1} f$, cf. Remark 6.1. It suffices then to apply Theorem 6.1. If $f = \omega_X$, then $\kappa_{\lambda,1} f = -\omega_X$ and $m_\alpha = -\infty$, hence the announced equalities are satisfied.

Finally, as in [12, Theorem 5], for each $x \in C_\alpha$ we show that the infimum in the definition of $\varphi_{\{f^\varphi = m_\alpha\}}(x)$ is attained whenever the supremum in the definition of $f^\varphi(x)$ is attained.

COROLLARY 6.2. *Let the assumptions of Theorem 6.1 be satisfied, and let* $x \in C_\alpha$ *be such that* $M_\varphi f(x) \ne \emptyset$. *Then there exists* $u \in \{f^\varphi = m_\alpha\}$ *such that*

$$\varphi(u-x) = \inf_{y\in\{f^\varphi \ge m_\alpha\}} \varphi(y-x).$$

PROOF. Let $s \in M_\varphi f(x)$. Then, since $x \in C_\alpha$ and $f^\varphi(x) = \varphi(x-s) - f(s)$, we have $x \ne s$. By the coercivity of f^φ and the inequality $f^\varphi(x) \le m_\alpha$ (by the definition of D_α in (49) and the inclusions $x \in C_\alpha \subset D_\alpha$), we have $\big(x + \mathbb{R}_+(x - s)\big) \cap \{f^\varphi = m_\alpha\} \neq \emptyset$. So let $u \in \big(x + \mathbb{R}_+(x - s)\big) \cap \{f^\varphi = m_\alpha\} \neq \emptyset$. By Theorem 6.1, $f^\varphi(u) = f^\varphi(x) + \inf_{y \in \{f^\varphi \geq m_\alpha\}} \varphi(y - x)$, and hence, because $f^\varphi(x) = \varphi(x - s) - f(s)$ (as seen above),

$$\varphi(x - s) - f(s) + \inf_{y \in \{f^\varphi \geq m_\alpha\}} \varphi(y - x) = f^\varphi(u) \geq \varphi(u - s) - f(s).$$

Thus

$$\inf_{y \in \{f^\varphi \geq m_\alpha\}} \varphi(y - x) \geq \varphi(u - s) - \varphi(x - s)$$

and by taking into account that $u \in x + \mathbb{R}_+(x - s)$, we get $\varphi(u - s) - \varphi(x - s) = \varphi(u - x)$ which allows us to obtain

$$\inf_{y \in \{f^\varphi \geq m_\alpha\}} \varphi(y - x) \geq \varphi(u - x)$$

and to complete the proof.

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