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ELEMENTS OF INFINITE FILTRATION IN COMPLEX COBORDISM

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We study complex cobordism $MU^*(X)$ for CW-complexes with finite skeletons, with particular interest in $MU^*(BG)$ where G is a compact Lie group. In case G is *abelian* (the product of a torus and a finite abelian group) we computed $MU^*(BG)$ in [5] and found the result to be similar to the K-theory $K^*(BG)$. However in general (even for finite groups) one has little understanding of $MU^*(BG)$, even though one has a lot of information about $K^*(BG)$ and one knows that K-theory and complex cobordism are closely related. We show here that concerning elements of infinite filtration this relation works quite well.

THEOREM 1. *If G is a compact Lie group and $\alpha \in MU^*(BG)$ has infinite filtration (that is, α restricts to zero on each skeleton of BG), then $\alpha = 0$.*

The corresponding result for $K^*(BG)$ was proved by V. M. Buhstaber and A. S. Miscenko [2, Theorem 3], and is required for the proof of Theorem 1 (see also [1]). In fact we prove

THEOREM 2. *If X is a CW-complex with finite skeletons then the following conditions are equivalent:*

(a) *$MU^*(X)$ has no non-zero elements of infinite filtration;*

(b) *$K^*(X)$ has no non-zero elements of infinite filtration;*

(c) *for each $x \in H^*(X;Q)$ there is a torsion-free skeleton-finite CW-complex Y, a map $f\colon X \to Y$ and $y \in H^*(Y;Q)$ such that $f^*(y) = x$;*

(d) *there is a torsion-free skeleton-finite CW-complex Y and a map $f\colon X \to Y$ such that $f^*\colon H^*(Y;Q) \to H^*(X;Q)$ is onto.*

We collect some preliminary results in section 1 and then prove the theorems in the following sections. We give a direct proof of Theorem 1 in section 2 in order to comment on a possible approach for further study of $MU^*(BG)$. In the final section we suggest a route that might lead to information about the complex bordism modules $MU_*(BG)$.

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1. Preliminaries.

Throughout we restrict attention to skeleton-finite CW-complexes. We recall the main theorem of [2] and its companion for complex cobordism [4]:

PROPOSITION 1. *$K^*(X)$ has no non-zero elements of infinite filtration if and only if for each $x \in H^*(X;Q)$ there exists $\alpha \in K^*(X)$ such that* $\mathrm{ch}(\alpha)$ *leads off with nx for some $n \neq 0$.*

PROPOSITION 2. *$MU^*(X)$ has no non-zero elements of infinite filtration if and only if for each $x \in H^*(X;Z)$ there exists $\alpha \in MU^*(X)$ such that $\mu(\alpha) = nx$ for some $n \neq 0$, where μ is the Thom homomorphism.*

COROLLARY 1. *If $H^*(X;Z)$ has no torsion then $K^*(X)$ and $MU^*(X)$ have no non-zero elements of infinite filtration.*

For when $H^*(X;Z)$ has no torsion it is a well-known consequence of the collapsing of the Atiyah–Hirzebruch spectral sequences that μ maps $MU^*(X)$ onto $H^*(X;Z)$ and that each element of $H^*(X;Z)$ is the lead term of a Chern character.

COROLLARY 2. *Let E denote either K or MU. If $f: X \to Y$ induces an epimorphism $H^*(Y;Q) \to H^*(X;Q)$ and if $E^*(Y)$ has no non-zero elements of infinite filtration, then this is also true for $E^*(X)$.*

2. Proof of Theorem 1.

Let $R(G)$ denote the representation ring of G. By [9, Corollary (3.3)] $R(G)$ is a finitely generated ring, hence is generated as a ring by the classes of a finite set of unitary representations $\varrho_i: G \to U_i$. The product of the ϱ_i gives a homomorphism $\varrho: G \to H$ into a product of unitary groups for which $\varrho^*: R(H) \to R(G)$ is onto. Then clearly the image of the augmentation ideal $I(H)$ under ϱ^* is $I(G)$, and so the $I(G)$-adic and $I(H)$-adic topologies on $R(G)$ coincide. From [7, p. 397] it follows that ϱ^* induces an epimorphism $(R(H))^\wedge \to (R(G))^\wedge$ of completions (in fact $R(G)$ is finitely generated over the Noetherian ring $R(H)$ [8, Proposition (3.2)], and for finitely generated modules over a Noetherian ring I-adic completion is exact [7, p. 419]). In view of the natural isomorphism α: $(R(G))^\wedge \to K^*(BG)$ of [1] we have shown that $K^*(BH) \to K^*(BG)$ is onto.

We next observe that also $H^*(BH;Q) \to H^*(BG;Q)$ is onto; this follows from Proposition 1 and the fact [2] (see also [1]) that $K^*(BG)$ has no non-zero elements of infinite filtration. Since BH has no torsion we conclude from Corollaries 1 and 2 that $MU^*(BG)$ has no non-zero elements of infinite filtration.

Remarks. We conjecture that in fact $(B_\varrho)^*: MU^*(BH) \to MU^*(BG)$ is onto, but are able to prove this only if G is abelian so that we may choose H to be a torus [5]. We further conjecture that $MU^*(BG)$ is a flat $\pi_*(MU)$-module, so that in particular it has no torsion as an $\pi_*(MU)$-module and as an abelian group; again this is true in the abelian case.

3. Proof of Theorem 2.

It is evident that (d) $\Rightarrow$ (c), and the implications (c) $\Rightarrow$ (a), (c) $\Rightarrow$ (b) follow from Propositions 1 and 2 and Corollary 1.

We show that (a) $\Rightarrow$ (b). If $MU^*(X)$ has no non-zero elements of infinite filtration and $x \in H^n(X;Z)$, then we choose α in $MU^n(X)$ such that $\mu(\alpha)=nx$ with $n \neq 0$. Now recall the transformation μ_c: $MU^*(X) \to K^*(X)$ [3]; we obtain $\mu_c(\alpha)$ in $K^*(X)$ for which $\mathrm{ch}\,\mu_c(\alpha)$ leads off with the image in $H^n(X;Q)$ of $\mu(\alpha)=nx$. From Proposition 1 it follows that $K^*(X)$ has no non-zero elements of infinite filtration.

Finally, if $K^*(X)$ has no non-zero elements of infinite filtration then by Proposition 1, for each $x \in H^*(X;Q)$ there exists $\alpha \in K^*(X)$ such that $\mathrm{ch}(\alpha)$ leads off with nx for some $n \neq 0$. For such a complex, the first step of the proof of Theorem 1 of [2] provides a map f: $X \to Y$ into a torsion-free skeleton-finite CW-complex Y such that

$$f^*: \quad H^*(Y;Q) \to H^*(X;Q)$$

is onto. Thus (b) $\Rightarrow$ (d) and the proof is complete.

4. Further comments.

Let X be a CW-complex with finite skeletons X^n, and suppose that $MU^*(X)$ has no non-zero elements of infinite filtration. By Theorem 2 of [4], for each t the inverse system $\{MU^t(X^n)\}$ satisfies a strong version of the Mitlag–Leffler condition: for each n there exists $m \geqq n$ such that

$$\mathrm{Im}\,\{MU^t(X) \to MU^t(X^n)\} = \mathrm{Im}\,\{MU^t(X^m) \to MU^t(X^n)\}\,.$$

In fact, this holds uniformly in t in view of D. Quillen's theorem [8] which asserts that, for a finite complex K, $MU^*(K)$ is generated by elements of non-negative degree. We conclude that for each n there exists $m \geqq n$ such that

$$\operatorname{Im}\{MU^*(X) \to MU^*(X^n)\} = \operatorname{Im}\{MU^*(X^m) \to MU^*(X^n)\}.$$

It would be very interesting to have the dual statement hold for complex bordism $MU_*(X)$ of classifying spaces and Eilenberg–MacLane spaces, as was indicated in Problem 4 of [6]. The dual statement we have in mind asserts that for each n there exists $m \geqq n$ such that

$$\operatorname{Ker}\{MU_*(X^n) \to MU_*(X)\} = \operatorname{Ker}\{MU_*(X^n) \to MU_*(X^m)\}.$$

One shows easily that this is equivalent to the complex bordism module $MU_*(X)$ being a pseudo-coherent $\pi_*(MU)$-module (each finitely generated submodule is finitely presented), and has the pleasant consequence that all annihilator ideals of elements of $MU_*(X)$ are finitely generated. There might be a sufficient duality between $MU_*(BG)$ and $MU^*(BG)$, at least for finite groups or even finite abelian groups, to yield this sort of information about $MU_*(BG)$ from what is presently known about $MU^*(BG)$.

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