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GROUP EXTENSIONS AND PRINCIPAL FIBRATIONS

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1. Introduction.

We consider in this paper the following extension problem. Suppose we have a principal fibration

$$\Omega C \xrightarrow{i} E \xrightarrow{\pi} B ,$$

with classifying map $\theta: B \to C$. One then has an extended sequence of fibrations

$$\ldots \to \Omega^{m+1} C \xrightarrow{{}^m i} \Omega^m E \xrightarrow{{}^m \pi} \Omega^m B \to \ldots B \xrightarrow{\theta} C .$$

(For any map f we write ${}^m f = \Omega^m f$, $m \geqq 1$.) Given a space X a standard problem in topology is to compute the group $[X, \Omega^m E]$. We have the exact sequence

$$(1.1) \qquad e\colon 0 \to (\operatorname{Coker}^{m+1}\theta_*) \to [X, \Omega^m E] \to (\operatorname{Ker}^m \theta_*) \to 0 ,$$

and our problem is:

PROBLEM 1. *Compute the extension e.*

Note, in particular, that if B and C are products of Eilenberg–MacLane spaces, then E (and hence $\Omega^m E$) is a 2-stage Postnikov system.

We assume now that all the groups in (1.1) are abelian — e.g., take $m > 1$, or take $m = 1$ with B, C loop spaces and θ a loop map. If $\operatorname{Ker}^m \theta_*$ is finitely generated, it is known (see section 5) that the extension e is completely determined by a set of homomorphisms $\Phi(p,k)$, defined for each prime p and positive integer k. Here

$$\Phi(p,k)\colon (\operatorname{Ker}^m \theta_*) \cap (\text{elements of order } p^k) \to (\operatorname{Coker}^{m+1}\theta_*)/p^k(\operatorname{Coker}^{m+1}\theta_*) ,$$

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and is defined as follows. Let u be an element in $[X,\Omega^m B]$ such that ${}^m\theta_* u=0$, $p^k u=0$. Choose v in $[X,\Omega^m E]$ such that ${}^m\pi_*(v)=u$. Then, ${}^m\pi_*(p^k v)=0$, and so there is a class w in Cokernel ${}^{m+1}\theta_*$ such that ${}^m j_*(w)=p^k v$. Set

$$\Phi(p,k)(u) = w \mod p^k(\mathrm{Coker}\,{}^{m+1}\theta_*) .$$

Thus we have

PROBLEM 2. *Compute the operations* $\Phi(p,k)$.

In sections 2 and 3 we give a solution to this problem for the case E is a 2-stage Postnikov system. In section 4 we illustrate our theory with three examples: stable cohomotopy, complex K-theory, and immersion groups for manifolds.

REMARK. We emphasize that we consider here only abelian group extensions. In a subsequent paper we will develop an analogous theory for non-abelian, central extensions. This corresponds in (1.1) to taking $m=1$, B and C loop spaces, but θ *not* a loop map.

2. Functional operations.

The morphism Φ, defined in section 1, is an example of a functional operation. In general, consider the following commutative diagram of Abelian groups and homomorphisms; we assume that each row is exact:

(2.1)
$$\begin{array}{ccccccc} & & B_1 & \xrightarrow{j_1} & C_1 & \xrightarrow{k_1} & D_1 & \xrightarrow{l_1} & E_1 \\ & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \\ A_2 & \xrightarrow{i_2} & B_2 & \xrightarrow{j_2} & C_2 & \xrightarrow{k_2} & D_2 . & & \end{array}$$

Define

$$\Phi\colon \mathrm{Ker}\,\delta \cap \mathrm{Ker}\,l_1 \to B_2/i_2A_2+\beta B_1$$

by

(2.2)
$$\Phi = j_2{}^{-1}\circ\gamma\circ k_1{}^{-1} .$$

Following Steenrod [13] we call Φ the functional operation at D_1 associated with (2.1). Note that the operation $\Phi(p,k)$ in section 1, fits into this context by taking β,γ,δ to be multiplication by p^k.

For use in subsequent papers it is desirable to consider a slightly more general version of a functional operation. That is, in diagram (2.1) we now no longer assume that C_1, C_2 are abelian, and we drop the requirement that γ be a homomorphism. We simply require that, for $b \in B_1$, $c \in C_1$,

$$(2.3) \qquad \gamma(j_1(b)c) = j_2(\beta(b))\gamma(c) .$$

Definition (2.2) continues to make sense, and one easily checks that Φ continues to have the same indeterminacy.

We now consider a geometric setting in which one has two functional operations, which we will show are equal.

Suppose we are given a cofibration sequence

$$(2.4) \qquad S \xrightarrow{\gamma} Q \xrightarrow{\delta} P \xrightarrow{\varrho} \Sigma S \to \Sigma Q \to \dots ,$$

where P is the cofiber of γ, and Σ denotes (reduced) suspension. We assume that S and Q are themselves suspensions, though γ is *not* assumed to be the suspension of a map.

We work in the category of pointed spaces and maps.

For spaces X, Y let Y^X denote the function space of (pointed) maps with the compact-open topology. Using (2.4) in conjunction with the fibration

$$\Omega C \xrightarrow{i} E \xrightarrow{\pi} B ,$$

given in section 1, we obtain the following commutative diagram; each long column is a fibration sequence, as is each long row (by Borsuk [11; section 2.8.2]).

$$(2.5) \qquad \begin{array}{ccccccccc}
 & & & & & & & & (\Omega B)^S \\
 & & & & & & & & \downarrow{\scriptstyle (1_\theta)^S} \\
 & & & & & & (\Omega C)^Q & \xrightarrow{\gamma} & (\Omega C)^S \\
 & & & & & & \downarrow{\scriptstyle i^Q} & & \downarrow{\scriptstyle i^S} \\
 & & & & & & E^Q & \xrightarrow{\gamma} & E^S \\
 & & & & & & \downarrow{\scriptstyle \pi^Q} & & \downarrow{\scriptstyle \pi^S} \\
 & & B^{\Sigma S} & \xrightarrow{\varrho} & B^P & \xrightarrow{\delta} & B^Q & \xrightarrow{\gamma} & B^S \\
 & & \downarrow{\scriptstyle \theta^{\Sigma S}} & & \downarrow{\scriptstyle \theta^P} & & \downarrow{\scriptstyle \theta^Q} & & \\
C^{\Sigma Q} & \xrightarrow{\Sigma\gamma} & C^{\Sigma S} & \xrightarrow{\varrho} & C^P & \xrightarrow{\delta} & C^Q . & &
\end{array}$$

(By an abuse of notation, if $f\colon K \to L$ we also write $f\colon Y^L \to Y^K$ for the induced map on function spaces.)

We now assume that B and C are loop spaces, but θ is not assumed to be a loop map. Applying the functor $[X, \cdot]$ to diagram (2.5) we obtain a commutative diagram of groups, Fig. 1, where each long row and column is exact. Moreover, all groups are abelian except $[X,E^Q]$, $[X,E^S]$, $[X,B^P]$, $[X,C^P]$. Finally, all maps in the diagram are homomorphisms except

$$\gamma_*\colon\ [X,E^Q]\to[X,E^S], \qquad \theta_*{}^P\colon\ [X,B^P]\to[X,C^P]\,.$$

However, one can easily check that these two morphisms enjoy property (2.3). Thus the two rows and the two columns in Fig. 1 are examples

$$\begin{array}{ccccccccc}
 & & & & & & & & [X,(\Omega B)^S] \\
 & & & & & & & & \downarrow{\scriptstyle (1\theta)_*{}^S} \\
 & & & & & & [X,(\Omega C)^Q] & \xrightarrow{\gamma_*} & [X,(\Omega C)^S] \\
 & & & & & & \downarrow{\scriptstyle i_*{}^Q} & & \downarrow{\scriptstyle i_*{}^S} \\
 & & & & & & [X,E^Q] & \xrightarrow{\gamma_*} & [X,E^S] \\
 & & & & & & \downarrow{\scriptstyle \pi_*{}^Q} & & \downarrow{\scriptstyle \pi_*{}^S} \\
 & & [X,B^{\Sigma S}] & \xrightarrow{\varrho_*} & [X,B^P] & \xrightarrow{\delta_*} & [X,B^Q] & \xrightarrow{\gamma_*} & [X,B^S] \\
 & & \downarrow{\scriptstyle (\theta^{\Sigma S})_*} & & \downarrow{\scriptstyle \theta_*{}^P} & & \downarrow{\scriptstyle \theta_*{}^Q} & & \\
[X,C^{\Sigma Q}] & \xrightarrow{\Sigma\gamma_*} & [X,C^{\Sigma S}] & \xrightarrow{\varrho_*} & [X,C^P] & \xrightarrow{\delta_*} & [X,C^Q]\,. & &
\end{array}$$

Figure 1.

$$\begin{array}{ccccccccc}
 & & & & & & & & [X\wedge S,\Omega B] \\
 & & & & & & & & \downarrow{\scriptstyle 1\theta_*} \\
 & & & & & & [X\wedge Q,\Omega C] & \xrightarrow{(1\wedge\gamma)_*} & [X\wedge S,\Omega C] \\
 & & & & & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\
 & & & & & & [X\wedge Q,E] & \xrightarrow{(1\wedge\gamma)^*} & [X\wedge S,E] \\
 & & & & & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\
 & & [X\wedge\Sigma S,B] & \xrightarrow{(1\wedge\varrho)^*} & [X\wedge P,B] & \xrightarrow{(1\wedge\delta)^*} & [X\wedge Q,B] & \xrightarrow{(1\wedge\gamma)^*} & [X\wedge S,B] \\
 & & \downarrow{\scriptstyle \theta_*} & & \downarrow{\scriptstyle \theta_*} & & \downarrow{\scriptstyle \theta_*} & & \\
[X\wedge\Sigma Q,C] & \xrightarrow{(1\wedge\Sigma\gamma)^*} & [X\wedge\Sigma S,C] & \xrightarrow{(1\wedge\varrho)^*} & [X\wedge P,C] & \xrightarrow{(1\wedge\delta)^*} & [X\wedge Q,C]\,. & &
\end{array}$$

Figure 2.

of (2.1), in the more general setting described prior to (2.3), and so if we set

$$K(\theta^Q,\gamma) = \mathrm{Ker}\gamma_* \cap \mathrm{Ker}(\theta^Q)_* \subset [X,B^Q],$$

by (2.2) we obtain two functional operations, each based at $[X,B^Q]$:

$$\Phi_1\colon\ K(\theta^Q,\gamma) \to [X,(\Omega C)^S]/\gamma_*[X,(\Omega C)^Q] + ({}^1\theta)^S{}_*[X,(\Omega B)^S],$$
$$\Phi_2\colon\ K(\theta^Q,\gamma) \to [X,C^{\Sigma S}]/(\Sigma\gamma)_*[X,C^{\Sigma Q}] + (\theta^{\Sigma S})_*[X,B^{\Sigma S}].$$

Now for any space Y, $Y^{\Sigma S} \equiv (\Omega Y)^S$, and so we may identify $[X,Y^{\Sigma S}] = [X,(\Omega Y)^S]$. With this identification we have:

THEOREM 2.5. $\Phi_1 = \Phi_2$.

We shall prove this shortly, but first we relate the theorem to the material given in section 1.

Let $S = Q = S^1$, the 1-sphere, and define $\gamma\colon S \to S$ to be a map of degree p^k. Then $P(=P(k))$, the cofibre of γ, is the space $S^1 \cup_{p^k} e^2$. If we assume that θ is a loop map, then $\Phi_1 = \Phi(p,k)$ as defined in section 1; we will solve problem 2 by computing the operation Φ_2.

PROOF OF THEOREM 2.5. Let X, Y, Z be spaces with basepoint. One has a natural transformation, the adjoint,

$$a\colon\ [X,Y^Z] \leftrightarrow [X\wedge Z, Y],$$

which gives a 1–1 correspondence between the two sets. Applying a to every group in Fig. 1, we obtain Fig. 2, again a commutative diagram with exact rows and columns. The operations Φ_1,Φ_2 go over to operations $\tilde{\Phi}_1,\tilde{\Phi}_2$ based at $[X\wedge Q,B]$. Moreover, using a,

$$[X\wedge\Sigma S,C] = [X\wedge S,\Omega C].$$

To prove Theorem 2.5 we show that, with the above identification, $\tilde{\Phi}_1 = \tilde{\Phi}_2$.

Let u be a class in $[X\wedge Q,B]$ such that $(1\wedge\gamma)_* u = 0$ and $\theta_* u = 0$; that is, u is in the domain of $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$. Consider the following sequence of spaces and maps:

$$X\wedge S \xrightarrow{1\wedge\gamma} X\wedge Q \xrightarrow{u} B \xrightarrow{\theta} C. \tag{2.6}$$

Notice that, by hypothesis, the compositions $u\circ(1\wedge\gamma)$ and $\theta\circ u$ are null-homotopic. But this is precisely the situation considered by Spanier in [12], and so one can associate with (2.6), elements

$$\varphi_1 \in [X\wedge S,\Omega C]/(1\wedge\gamma)_*[X\wedge Q,\Omega C] + ({}^1\theta)_*[X\wedge S,\Omega B],$$

and

$$\varphi_2 \in [X \wedge \Sigma S, C]/(1 \wedge \Sigma\gamma)^*[X \wedge \Sigma Q, C] + \theta_*[X \wedge \Sigma S, B] .$$

Moreover, as remarked in [12], the adjoint carries φ_1 to φ_2. But by the definition given by Spanier [12],

$$\varphi_1 = \tilde{\Phi}_1(u), \quad \varphi_2 = \tilde{\Phi}_2(u) ,$$

and so, $\tilde{\Phi}_1 = \tilde{\Phi}_2$, as claimed.

Remark. Theorem 2.5 is related to work of Peterson [9], [10].

3. The functor *P*.

For the rest of the paper we restrict attention to the cofibration

$$S \xrightarrow{\gamma} S \xrightarrow{\delta} P \xrightarrow{\varrho} S^2 \to \dots ,$$

where $S = S^1$ and γ is a map of degree p^k, p a fixed prime, $k > 0$. Our goal is to compute the operation

$$\Phi_2 = \varrho_*{}^{-1} \circ \theta^P{}_* \circ \delta_*{}^{-1} ,$$

(see Fig. 1). Now in applications we presume that δ_* and ϱ_* are known, and so to compute Φ_2 we need only know the operation

$$\theta^P{}_*: \ [X, B^P] \to [X, C^P] .$$

In this section we compute this operation, assuming that B and C are products of Eilenberg–MacLane spaces and θ a loop map.

We think of P as a contravariant functor

$$X \to X^P, \quad f \to f^P .$$

Notice that $(X_1 \times X_2)^P = X_1{}^P \times X_2{}^P$, and so if X is an H-space, so is X^P. Moreover, if X is an H-space, and if α_i: $B \to X$ are given for $i = 1, 2$, with $\alpha = \alpha_1 + \alpha_2 : B \to X$, we then have $\alpha^P = \alpha_1{}^P + \alpha_2{}^P$. If $B = K_1 \times \dots \times K_r$, C is an H-space, and θ: $B \to C$ has the property that

$$(*) \qquad \theta = \textstyle\sum_{i=1}^r \theta_i \circ \pi_i ,$$

where θ_i: $K_i \to C$ and π_i is the projection of B onto K_i, then

$$\theta^P = \textstyle\sum_i \theta_i{}^P \circ \pi_i{}^P ,$$

and so to compute θ^P we need only compute each $\theta_i{}^P$.

Suppose, finally, that B and C are finite products of Eilenberg–MacLane spaces (each with cyclic homotopy group) and θ a stable map, in the sense that it can be delooped. Then θ has property (*) above, and so to compute θ^P we need only consider the case

$$\theta\colon\ K(G,q) \to K(\Lambda,r)\ ,$$

where G and Λ are cyclic groups.

Now any such operation is formed by adding together compositions of the following operations:

(1) elements of the mod p Steenrod algebra, $\mathcal{A}_p$.

(2) the Bockstein homomorphism δ_k, $k \geqq 1$, associated with the exact sequence

$$\mathsf{Z} \to \mathsf{Z} \to \mathsf{Z}_{p^k}\ .$$

(3) coefficient homomorphisms.

Since $(\theta_1 \circ \theta_2)^P = \theta_1{}^P \circ \theta_2{}^P$, we need compute θ^P only for θ an operation of the above types. For simplicity we will do out only type (1) together with the Bockstein $\delta_1(=\delta)$ and the coefficient homomorphism $\varrho_1(=\varrho)$, where in general ϱ_k is induced by the canonical epimorphism $\mathsf{Z} \to \mathsf{Z}_{p^k}$, $k \geqq 1$.

The space $K(\pi,n)^P$, $\pi = \mathsf{Z}$ *or* Z_p, $P = P(k)$:

We adopt the following notation: $\iota_n \in H^n(K(\pi,n);\pi)$ and $s_i \in H^i(S^i;\mathsf{Z})$, $i=1,2$, will denote fundamental classes. We choose generators $e_i \in H^i(P;\mathsf{Z}_{p^k})$, $i=1,2$, by requiring that

$$\delta^* e_1 \;=\; s_1 \bmod p^k, \quad \varrho^*(s_2 \bmod p^k) \;=\; e_2\ .$$

Note that $\beta_k(e_1)=e_2$, where β_k is the Bockstein coboundary associated with the exact sequence $\mathsf{Z}_{p^k} \to \mathsf{Z}_{p^{2k}} \to \mathsf{Z}_{p^k}$.

Let $f_{n,i}\colon K(\pi,n-i) \to K(\pi,n)^{S^i}$, $i=1,2$, denote the homotopy equivalence whose adjoint

$$f'_{n,i}\colon\ K(\pi,n-i) \wedge S^i \to K(\pi,n)$$

is given by

$$(f'_{n,i})^* \iota_n \;=\; \iota_{n-i} \otimes s_i\ .$$

In a similar fashion we describe $K(\pi,r)^P$ by taking adjoints. Set

$$K_n \;=\; K(\mathsf{Z}_p,n),\ K^*{}_n \;=\; K(\mathsf{Z},n),\ K^k{}_n \;=\; K(\mathsf{Z}_{p^k},n), \quad k>1,\ n \geqq 3\ .$$

Define $f'_n\colon (K_{n-2} \times K_{n-1}) \wedge P \to K_n$ by

$$\text{(3.1)} \qquad f'_n{}^* \iota_n \;=\; (\iota_{n-2} \otimes 1) \otimes e_2 + (1 \otimes \iota_{n-1}) \otimes e_1\ ,$$

and define

$$g'_n\colon\ K^k{}_{n-2} \wedge P \to K^*{}_n$$

by

(3.2) $$g'_n{}^* \iota_n = \delta_k(\iota_{n-2} \otimes e_1)\,.$$

PROPOSITION 3.3. *Let f_n denote the adjoint of f'_n and g_n the adjoint of g'_n; that is,*

$$f_n\colon\ K_{n-2} \times K_{n-1} \to K_n{}^P, \qquad g_n\colon\ K^k{}_{n-2} \to K^*{}_n{}^P\,.$$

Then, f_n and g_n are homotopy equivalences.

PROOF. Consider the following diagram, where i denotes inclusion and r projection. Note that each row is a fibre sequence.

$$\begin{array}{ccccccccc}
K_{n-2} & \xrightarrow{\times p^k} & K_{n-2} & \xrightarrow{i} & K_{n-2} \times K_{n-1} & \xrightarrow{r} & K_{n-1} & \xrightarrow{\times p^k} & K_{n-1} \\
\downarrow{\scriptstyle f_{n,2}} & & \downarrow{\scriptstyle f_{n,2}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n,1}} & & \downarrow{\scriptstyle f_{n,1}} \\
K_n{}^{S^2} & \xrightarrow{\times p^k} & K_n{}^{S^2} & \xrightarrow{\varrho} & K_n{}^P & \xrightarrow{\delta} & K_n{}^S & \xrightarrow{\times p^k} & K_n{}^S
\end{array}$$

CLAIM. *The above diagram homotopy-commutes.*

Assuming this the proof of 3.3 follows at once (for f_n), by applying the 5-lemma to the corresponding diagram of homotopy groups.

The extreme squares in the diagram are obviously commutative, since here the horizontal maps are simply multiplication by p^k. The (homotopy) commutativity of the middle squares follows at once from the (homotopy) commutativity of the following diagram, using the fact that $(f_n \circ i)' = f'_n \circ (i \wedge 1)$, etc. We leave the details to the reader.

$$\begin{array}{ccc}
K_{n-2} \wedge P & \xrightarrow{1 \wedge \varrho} & K_{n-2} \wedge S^2 \\
\downarrow{\scriptstyle i \wedge 1} & & \downarrow{\scriptstyle f'_{n,2}} \\
(K_{n-2} \times K_{n-1}) \wedge P & \xrightarrow{f'_n} & K_n \\
\uparrow{\scriptstyle 1 \wedge \delta} & & \uparrow{\scriptstyle f'_{n,1}} \\
(K_{n-2} \times K_{n-1}) \wedge S & \xrightarrow{r \wedge 1} & K_{n-1} \wedge S\,.
\end{array}$$

The proof that g_n is a homotopy equivalence is similar and is omitted.

The morphism ε.

Let p be a fixed prime ($\geqq 2$) and let $\mathcal{A}$ denote the mod p Steenrod algebra. We define a morphism $\varepsilon: \mathfrak{A} \to \mathfrak{A}$ of degree -1 which will be used to compute θ^P.

Recall that $\mathfrak{A}$ is a Hopf algebra with diagonal map, say, $\psi: \mathfrak{A} \to \mathfrak{A}\otimes\mathfrak{A}$. Given $\alpha \in \mathfrak{A}$ define α_1 by the equation

$$\psi(\alpha) = \alpha\otimes 1 + \alpha_1\otimes\beta_1 + \dots ,$$

and define $\varepsilon: \mathfrak{A} \to \mathfrak{A}$ by the rule

$$\alpha \mapsto (-1)^{n+1}\alpha_1, \quad n = \deg\alpha . \tag{3.4}$$

PROPOSITION 3.5. *ε is characterized by the following properties*:

(i) $\varepsilon(\alpha_1\alpha_2) = \varepsilon(\alpha_1)\alpha_2 + (-1)^d\alpha_1\varepsilon(\alpha_2), \quad d = \text{degree}\,\alpha_1$.
(ii) *If* $p=2$, *then* $\varepsilon(\mathrm{Sq}^n) = \mathrm{Sq}^{n-1}, \quad n \geqq 1$.
If $p>2$, *then* $\varepsilon(\beta_1) = 1, \ \varepsilon(P^i) = 0, \quad i \geqq 0$.

The proof follows at once from the fact that ψ is an algebra morphism. Note that for $p=2$, ε is the map $\varkappa$ considered by Kristensen [4]. For convenience we will write $\varepsilon\alpha = \tilde{\alpha}$ for an element $\alpha \in \mathfrak{A}$.

Computation of θ^P.

Suppose that a (stable) operation $\theta: K(G,q) \to K(\Lambda,r)$ is given. We now compute the operation

$$\theta^P: \ K(G,q)^P \to K(\Lambda,r)^P$$

in the following cases:

I. $G=\Lambda=\mathrm{Z}_p, \quad \theta \in \mathfrak{A}$,
II. $G=\mathrm{Z}, \ \Lambda=\mathrm{Z}_p, \quad \theta=\varrho$,
III. $G=\mathrm{Z}_p, \ \Lambda=\mathrm{Z}, \quad \theta=\delta$.

If $\Lambda=\mathrm{Z}_p$, then $K(\Lambda,r)^P \equiv K_{r-2}\times K_{r-1}$, and so θ^P is a 2-valued operation. If u is in domain θ^P, we write

$$[\theta^P(u)_{r-2}, \theta^P(u)_{r-1}] ,$$

for the two values of the operation. Also, if $G=\mathrm{Z}_p$, then $K(G,q)^P \equiv K_{q-2}\otimes K_{q-1}$, and we compute the values of θ^P on the two fundamental classes $\iota_{q-2}\otimes 1$ and $1\otimes\iota_{q-1}$.

THEOREM 3.6. *Case I. $G=\Lambda=\mathrm{Z}_p$, $\theta \in \mathfrak{A}$. Then,*

$$\theta^P(\iota_{q-2}\otimes 1) = [(\theta\iota_{q-2}\otimes 1)_{r-2}, \ (0)_{r-1}] ,$$
$$\theta^P(1\otimes\iota_{q-1}) = [(-1)^r\lambda_k(1\otimes\tilde{\theta}\iota_{q-1})_{r-2}, \ (1\otimes\theta\iota_{q-1})_{r-1}] .$$

Here $\lambda_k = 0$ *if* $k > 1$, $\lambda_1 = 1$.

Case II. $G = \mathbf{Z}$, $\Lambda = \mathbf{Z}_p$, $\theta = \varrho$, *Then,*

$$\theta^P(\iota_{q-2}) = [(-1)^q(\iota_{q-2} \bmod p)_{q-2}, \ (\varrho\,\delta_k\,\iota_{q-2})_{q-1}] .$$

Case III. $G = \mathbf{Z}_p$, $\Lambda = \mathbf{Z}$, $\theta = \delta$. *Then,*

$$\theta^P(\iota_{q-2} \otimes 1) = (-1)^{q-1}\varrho_k\,\delta(\iota_{q-2}) ,$$
$$\theta^P(1 \otimes \iota_{q-1}) = s_k(\iota_{q-1}) .$$

Here s_k *is the cohomology operation induced by the coefficient homomorphism* $\mathbf{Z}_p \to \mathbf{Z}_{p^k}$, *given by*

$$a \bmod p \mapsto p^{k-1}a \bmod p^k .$$

PROOF. We do out the details only for Cases I and III.

Case I. For $n \geqq 3$, set $L_n = K_{n-2} \times K_{n-1}$, and let $f_n \colon L_n \to K_n{}^P$ be defined as in (3.3). Consider the following diagrams:

$$\begin{array}{ccc} L_q & \xrightarrow{\varphi} & L_r \\ {\scriptstyle f_q}\downarrow & & \downarrow{\scriptstyle f_r} \\ K_q{}^P & \xrightarrow{\theta^P} & K_r{}^P \end{array} , \qquad \begin{array}{ccc} L_q \wedge P & \xrightarrow{\varphi\wedge 1} & L_r \wedge P \\ {\scriptstyle f'_q}\downarrow & & \downarrow{\scriptstyle f'_r} \\ K_q & \xrightarrow{\theta} & K_r \end{array} .$$

Here, φ, in the left hand diagram, is chosen to be any map making the diagram homotopy-commute. (Recall that f_q, f_r are homotopy equivalences.) The right hand diagram is obtained by taking the adjoint of the left hand diagram, and hence also homotopy-commutes. By (3.1) and (3.4),

$$(*) \quad (\theta \circ f'_q)^*\iota_r = \theta(f'_q{}^*\iota_q) = \theta(\iota_{q-2} \otimes 1 \otimes e_2 + 1 \otimes \iota_{q-1} \otimes e_1)$$
$$= \theta(\iota_{q-2}) \otimes 1 \otimes e_2 + 1 \otimes \theta(\iota_{q-1}) \otimes e_1 + \lambda_k(-1)^r 1 \otimes \tilde{\theta}(\iota_{q-1}) \otimes e_2 .$$

And similarly,

$$(\varphi \wedge 1)^* f'_r{}^*\iota_r = (\varphi \wedge 1)^*(\iota_{r-2} \otimes 1 \otimes e_2 + 1 \otimes \iota_{r-1} \otimes e_1)$$
$$= \varphi^*(\iota_{r-2} \otimes 1) \otimes e_2 + \varphi^*(1 \otimes \iota_{r-1}) \otimes e_1 .$$

Comparing the coefficients of e_1 and e_2, we find

$$\varphi^*(1 \otimes \iota_{r-1}) = 1 \otimes \theta(\iota_{q-1}) ,$$
$$\varphi^*(\iota_{r-2} \otimes 1) = \theta(\iota_{q-2}) \otimes 1 + \lambda_k(-1)^r 1 \otimes \tilde{\theta}(\iota_{q-1}) ,$$

as claimed.

Case III. As above we may choose a map $\varphi: K_{q-2}\times K_{q-1} \to K^k{}_{q-1}$ so that the left hand diagram, given below, homotopy-commutes. The right hand diagram is obtained by taking adjoints, and so homotopy-commutes also.

$$\begin{array}{ccc} K_{q-2}\times K_{q-1} & \xrightarrow{\varphi} & K^k{}_{q-1} \\ {\scriptstyle f_q}\downarrow & & \downarrow{\scriptstyle g_{q+1}} \\ K_q^P & \xrightarrow{(\delta)^P} & K^*{}_{q+1}{}^P , \end{array} \qquad \begin{array}{ccc} K_{q-2}\times K_{q-1}\wedge P & \xrightarrow{\varphi\wedge 1} & K^k{}_{q-1}\wedge P \\ {\scriptstyle f'_q}\downarrow & & \downarrow{\scriptstyle g'_{q+1}} \\ K_q & \xrightarrow{\delta} & K^*{}_{q+1} . \end{array}$$

Our aim is to compute $\varphi^*\iota_{q-1}$, a mod p^k class. Thus it suffices to do our calculations mod p^k, rather than with integer coefficients. By (3.3),

$$\begin{aligned} f'_q{}^*\delta^*(\iota_{q+1}) \bmod p^k &= \varrho_k\delta f'_q{}^*(\iota_{q+1}) \\ &= \varrho_k\delta(\iota_{q-2}\otimes 1\otimes e_2 + 1\otimes\iota_{q-1}\otimes e_1) \\ &= \varrho_k\delta(\iota_{q-2})\otimes 1\otimes e_2 + 1\otimes\varrho_k\delta(\iota_{q-1})\otimes e_1 \\ &\qquad + (-1)^{q-1} 1\otimes s_k(\iota_{q-1})\otimes e_2 . \end{aligned}$$

On the other hand

$$\begin{aligned} (\varphi\wedge 1)^* g'_{q+1}{}^*(\iota_{q+1}) \bmod p^k &= \varrho_k(\varphi\wedge 1)^* g'_{q+1}{}^*(\iota_{q+1}) \\ &= (\varphi\wedge 1)^*\varrho_k\delta_k(\iota_{q-1}\otimes e_1) \\ &= (\varphi\wedge 1)^*\beta_k(\iota_{q-1}\otimes e_1) \\ &= (\varphi\wedge 1)^*(\beta_k\iota_{q-1}\otimes e_1 + (-1)^{q-1}\iota_{q-1}\otimes e_2) . \end{aligned}$$

Comparing coefficients of e_2, we obtain

$$\varphi^*(\iota_{q-1}) = (-1)^{q-1}\varrho_k\delta(\iota_{q-2})\otimes 1 + 1\otimes s_k(\iota_{q-1}) .$$

This completes the proof. The proof for Case II is similar and is left to the reader.

REMARK. For simplicity we have considered only stable operations θ. A similar treatment also handles the case θ non-stable; one simply now defines $\tilde{\theta}$ to be the operation required in equation (*) given above.

We now relate Theorem 3.6 to our original Problem 1, given in section 1. We take $m=1$ and suppose that our spaces B, C are Eilenberg–MacLane spaces of type $K(G,q), K(\Lambda,r)$, as above. For simplicity we do only the cases $G,\Lambda = \mathsf{Z}$ or Z_p. The operation $\Phi(p,k)$, which determines extension (1.1), is then given as follows. (Note that if G or $\Lambda = \mathsf{Z}_p$, we need consider only the operation $\Phi(p,1)$, which we write simply as Φ.)

COROLLARY 3.7.

Case I. Let $B=K_q$, $C=K_r$, $\theta \in \mathcal{A}_p$. *Then,*

$$\Phi = (-1)^r \tilde{\theta} .$$

Case II. Let $B=K^*_q$, $C=K_r$, $\theta=\psi\varrho$, $\psi \in \mathcal{A}_p$. *Then,*

$$\Phi = (-1)^q \psi\delta^{-1} + (-1)^r \tilde{\psi}\varrho .$$

Case III. Let $B=K_q$, $C=K^*_r$, $\theta=\delta\psi$, $\psi \in \mathcal{A}_p$. *Then,*

$$\Phi = \varrho^{-1}\psi - \delta\tilde{\psi} .$$

Case IV. Let $B=K^*_q$, $C=K^*_r$, $\theta=\delta\psi\varrho$, $\psi \in \mathcal{A}_p$. *Then,*

$$\Phi(p,k) = p^{k-1}(\varrho^{-1}\psi\varrho) - \lambda_k(\delta\tilde{\psi}\varrho) + (-1)^{q+r}\delta\psi\delta_k{}^{-1} .$$

(Recall that $\lambda_1 = 1$, $\lambda_k = 0$ if $k > 0$.)

4. Examples.

Let $\{h^i\}$, $i \geqq 0$, be a representable cohomology theory [16], [2]. In this section we consider the problem of determining the order of elements in $h^i X$, X a finite-dimensional complex. This is essentially the same as considering elements of $[X,B]$, where B is an x-fold loop space, x large. Two ways have been developed for studying $[X,B]$, both based on the following diagram.

(4.1)
$$\begin{array}{ccccc}
\Omega K_n & \xrightarrow{j_n} & Q_n & & \\
 & & \downarrow{\scriptstyle p_n} & & \\
\Omega K_{n-1} & \xrightarrow{j_{n-1}} & Q_{n-1} & \xrightarrow{\theta_n} & K_n \\
 & & \downarrow{\scriptstyle p_{n-1}} & & \\
 & & \vdots & & \\
 & & \downarrow & & \\
\Omega K_{i+1} & \xrightarrow{j_{i+1}} & Q_{i+1} & \xrightarrow{\theta_{i+2}} & K_{i+2} \\
 & & \downarrow{\scriptstyle p_{i+1}} & & \\
\Omega K_i & \xrightarrow{j_i} & Q_i & \xrightarrow{\theta_{i+1}} & K_{i+1} \\
 & & \downarrow{\scriptstyle p_i} & & \\
 & & \vdots & & \\
 & & \downarrow & & \\
\Omega K_1 & \xrightarrow{j_1} & Q_1 & \xrightarrow{\theta_2} & K_2 \\
 & & \downarrow{\scriptstyle p_1} & & \\
 & & Q_0 & \xrightarrow{\theta_1} & K_1 .
\end{array}$$

Here p_i: $Q_i \to Q_{i-1}$ is the principal fibration with θ_i: $Q_{i-1} \to K_i$ as classifying map. We assume that the entire diagram is the 2-fold loop of an analogous diagram: that is, there are spaces R_{i+2}, L_{i+3}, $0 \leqq i \leqq n$, and maps ψ_{i+3}: $R_{i+2} \to L_{i+3}$ such that

$$Q_i = \Omega^2 R_{i+2}, \quad K_{i+1} = \Omega^2 L_{i+3}, \quad \theta_{i+1} = \Omega^2 \psi_{i+3}, \quad \text{etc.}$$

We use the diagram in two ways. First, the diagram may represent a Postnikov resolution [7], [15] of the space B. Then there will be a map q: $B \to Q_n$ such that for complexes X with dimension less than some integer N,

$$q_*: [X,B] \approx [X,Q_n] .$$

Moreover, Q_0 and K_i, $i \geqq 1$, will be products of Eilenberg–MacLane spaces. In the second way, $B = Q_0$, and the Q_i's represent the connective coverings of B [17], [14]; the K_i's again are products of Eilenberg–MacLane spaces. We consider now the problem of computing the order of elements in $[X,Q_n]$, respectively $[X,Q_0]$.

Case I. $[X,Q_n]$. Let π denote the composition $Q_n \to Q_0$. Suppose that $u \in [X,Q_n]$, and set $v = \pi_* u \in [X,Q_0]$. We consider the question: if we know the order of v what can be said about the order of u?

For $1 \leqq r \leqq n$, consider the following diagram:

$$\begin{array}{ccc}
\Omega K_r & = & \Omega K_r \\
\downarrow{\scriptstyle k_r} & & \downarrow{\scriptstyle j_r} \\
Z_r & \xrightarrow{l_r} & Q_r \\
\downarrow{\scriptstyle s_r} & & \downarrow{\scriptstyle p_r} \\
\Omega K_{r-1} & \xrightarrow{j_{r-1}} & Q_{r-1} \\
\downarrow{\scriptstyle \varphi_r} & & \downarrow{\scriptstyle \theta_r} \\
K_r & = & K_r
\end{array}$$

Here $\varphi_r = \theta_r \circ j_{r-1}$, and s_r is the principal fibration induced by j_{r-1} from p_r. (Let $\Omega K_0 = Q_0$, $j_0 =$ identity.)

We now apply the theory of sections 1–3 to the sequence

$$\ldots \to \Omega^2 K_{r-1} \xrightarrow{1\varphi_r} \Omega K_r \xrightarrow{k_r} Z_r \xrightarrow{s_r} \Omega K_{r-1} \xrightarrow{\varphi_r} K_r;$$

by hypothesis K_r and K_{r-1} are products of Eilenberg–MacLane spaces, while Z_r is a loop space. Let X be a complex and assume that there is a prime p such that $p(\text{cokernel}\,{}^1\varphi_{r*})=0$, $1 \leqq r \leqq n$. By section 1 we then have a homomorphism

$$\Phi_r\colon\ (\operatorname{Ker}\varphi_{r*}) \cap (\text{elements of order } p) \to \operatorname{coker}{}^1\varphi_{r*}\,.$$

From the definition of Φ_r we have at once:

PROPOSITION 4.2. *Let $u \in [X,Q_r]$ and set $v=p_{r*}u \in [X,Q_{r-1}]$. Suppose there is a class $x \in [X,\Omega K_{r-1}]$ such that $v=j_{r-1*}(x)$ and $px=0$. Then,*

$$pu = j_{r*}\Phi_r(x)\,.$$

Notice that by exactness, $\varphi_{r*}x=0$. Of course x may vary by Image ${}^1\theta_{r-1*}$. We prove

PROPOSITION 4.3. $\Phi_r(\text{Image}\,{}^1\theta_{r-1*}) \subset \text{Image}\,{}^1\theta_{r*}$.

The proof is immediate by the following commutative diagram:

$$\begin{array}{ccccccc}
\Omega Q_{r-1} & \xrightarrow{{}^1p_{r-1}} & \Omega Q_{r-2} & \xrightarrow{{}^1\theta_{r-1}} & \Omega K_{r-1} & \xrightarrow{j_{r-1}} & Q_{r-1} \\
\downarrow{\scriptstyle {}^1\theta_r} & & \downarrow{\scriptstyle t_r} & & \| & & \downarrow{\scriptstyle \theta_r} \\
\Omega K_r & \xrightarrow{k_r} & Z_r & \xrightarrow{s_r} & \Omega K_{r-1} & \xrightarrow{\varphi_r} & K_r\,.
\end{array}$$

(We regard ${}^1\theta_{r-1}$ as the principal fibration with j_{r-1} as classifying map, and s_r as principal fibration with φ_r as classifying map; thus the map t_r exists making the diagram commute.)

Thus Φ_r induces a morphism

$$\tilde\Phi_r\colon\ (\operatorname{Ker}\varphi_{r*})/(\text{Image}\,{}^1\theta_{r-1*}) \to \operatorname{Coker}{}^1\theta_{r*}\,.$$

By Proposition 4.2 we then have:

THEOREM 4.4. *Let $u \in [X,Q_n]$ and set $v=\pi_* u \in [X,Q_0]$. Suppose that for some integer $r(\geqq 0)$, $p^{r+1}v=0$. Then,*

(a) *for* $1 \leqq i < n$,

$$p^{r+i}u=0 \ \Rightarrow\ \tilde\Phi_i\circ\dots\circ\tilde\Phi_1(p^r v)=0\,,$$

(b) $p^{r+n}u=0 \ \Leftrightarrow\ \tilde\Phi_n\circ\dots\circ\tilde\Phi_1(p^r v)=0\,.$

(c) $p^{r+n+1}u=0$.

Later in the section we give two examples illustrating the Theorem.

Case II. $[X,Q_0]$. Let $v \in [X,Q_0]$ and set $w=\theta_{1*}v \in [X,K_1]$. If we know the order of w, what can be said about the order of v? We sketch an approach dual to that given in Case I.

By hypothesis there are spaces L_{r+1}, $1 \leqq r \leqq n$, and maps $\psi_{r+1}: \Omega L_r \to L_{r+2}$ such that $\Omega L_{r+1} = K_r$ and $\varphi_r = \Omega \psi_{r+1}$. (Also, L_r is a loop space and ψ_r a loop map, $r \geqq 1$.) Define

$$q_{r+2}\colon\ Y_{r+2} \to K_r$$

to be the principal fibration with ψ_{r+2} as classifying map. It is easily seen that there is a map

$$t_{r+2}\colon\ Q_{r-1} \to Y_{r+2}$$

so that the following diagram commutes:

$$\begin{array}{ccccccccc}
\Omega K_r & \xrightarrow{j_r} & Q_r & \xrightarrow{p_r} & Q_{r-1} & \xrightarrow{\theta_r} & K_r & & \\
\| & & \downarrow{\scriptstyle \theta_{r+1}} & & \downarrow{\scriptstyle t_{r+2}} & & \| & & \\
\Omega K_r & \xrightarrow{\varphi_{r+1}} & K_{r+1} & \longrightarrow & Y_{r+2} & \xrightarrow{q_{r+2}} & K_r & \xrightarrow{\psi_{r+2}} & L_{r+2}\,.
\end{array}$$

Assume now:

$$p(\operatorname{Coker} \varphi_{r+1*}) = 0, \qquad 1 \leqq r \leqq n-1\,.$$

As in section 1 define the functional operation

(4.5) $\quad \Phi_r$: $(\operatorname{Ker} \psi_{r+2*}) \cap$ (elements of order p) $\to \operatorname{Coker} \varphi_{r+1}{}^*$.

We then have

(4.6) *Let $v \in [X, Q_{r-1}]$ be a class such that $p(\theta_{r*} v) = 0$, and let $u \in [X, Q_r]$ be chosen so that $p_{r*}(u) = pv$. Then,*

$$\theta_{r+1*}(u) \in \Phi_r(\theta_r(v))\,.$$

The analogue of Theorem 4.4 is:

THEOREM 4.7. *Let X be a complex and p a prime. Suppose that*

$$p[X, K_r] = 0, \quad {}^1\varphi_{r*}[X, \Omega^2 K_{r-1}] = [X, \Omega K_r], \quad 1 \leqq r \leqq n\,.$$

Let $v \in [X, Q_0]$ and set $w = \theta_{1*} v \in [X, K_1]$. Suppose there is an integer $s (\geqq 0)$ such that $p^{s+1} w = 0$. Then, for $1 \leqq i < n$,

(a) $$p^{s+i} v = 0 \ \Rightarrow\ \Phi_i \circ \ldots \circ \Phi_1(p^s w) = 0\,.$$

If $[X, Q_n] = 0$, then

(b) $$p^{s+n-1}v = 0 \iff \Phi_{n-1}\circ \dots \circ \Phi_1(p^s w) = 0 .$$

(c) $$p^{s+n}v = 0 .$$

We leave the proof to the reader.

We turn now to examples illustrating Theorems (4.4) and (4.7).

EXAMPLE I. Cohomotopy.

As usual, we write $\pi^n X = [X, S^n]$. Consider the following (2-primary) Postnikov resolution of S^n:

$$\begin{array}{ccccc}
K_{n+3} & \xrightarrow{j_3} & Q_3 & & \\
 & & \downarrow & & \\
K_{n+2}\times K_{n+3} & \xrightarrow{j_2} & Q_2 & \xrightarrow{\beta^4} & K_{n+4} \\
 & & \downarrow & & \\
K_{n+1}\times K_{n+3} & \xrightarrow{j_1} & Q_1 & \xrightarrow{(\alpha^3,\alpha^4)} & K_{n+3}\times K_{n+4} \\
 & & \downarrow & & \\
 & & K(\mathbf{Z},n) & \xrightarrow{(Sq^2,Sq^4)} & K_{n+2}\times K_{n+4} .
\end{array}$$

Here

$$j_1{}^*\alpha^3 = \mathrm{Sq}^2\iota_{n+1}\otimes 1,\quad j_1{}^*\alpha^4 = \mathrm{Sq}^2\,\mathrm{Sq}^1\iota_{n+1}\otimes 1 + 1\otimes \mathrm{Sq}^1\iota_{n+3} ,$$
$$j_2{}^*\beta^4 = \mathrm{Sq}^2\iota_{n+2}\otimes 1 + 1\otimes \mathrm{Sq}^1\iota_{n+3} .$$

The classes α^3, α^4 represent secondary cohomology operations, the class β^4 a tertiary operation. Since the four and five stems are zero, we have

$$\pi^n X/(\text{odd torsion}) \approx [X, Q_3] \quad \text{for } \dim X \leqq n+5;$$

we assume that $n \geqq 5$.

Applying Theorem 4.4 and using Corollary (3.7) to compute the operations Φ_r we have:

THEOREM 4.8. *Let X be a complex of dimension $\leqq n+5$, with $n \geqq 5$. Let $\alpha \in \pi^n X$, and set $v = \alpha^* s_n \in H^n(X; \mathbf{Z})$, where s_n generates $H^n(S^n; \mathbf{Z})$. Suppose that $2v = 0$, and let $w \in H^{n-1}(X; \mathbf{Z}_2)$ be a class such that $\delta w = v$. Then,*

(a) $2\alpha = 0$ *implies* $(\mathrm{Sq}^2 w, \mathrm{Sq}^4 w) \in (\mathrm{Sq}^2, \mathrm{Sq}^4) H^{n-1}(X; \mathbf{Z})$.

(b) $4\alpha = 0$ *implies* $(\mathrm{Sq}^3 w, \mathrm{Sq}^4 w) \in (\alpha^3, \alpha^4) H^{n-1}(X; \mathbf{Z})$.

(c) $8\alpha = 0$ *if and only if*, $\mathrm{Sq}^4 w \in \beta^4 H^{n-1}(X; \mathbf{Z})$.

(d) $16\alpha = 0$.

We omit the details, noting only that in (b), we use the fact that

$$\mathrm{Sq}^2\mathrm{Sq}^2 w = \mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1 w = \mathrm{Sq}^1\mathrm{Sq}^2 v = 0\,,$$

since $v=\alpha^* s_n$.

EXAMPLE II. Immersions of Manifolds.

Let M be a smooth orientable manifold of dimension n. We consider the problem of enumerating the set of immersions of M in R^{n+k}, $k>0$. Call this set $\mathrm{Imm}[M, \mathsf{R}^{n+k}]$. Let ν_M denote the stable normal bundle of M. Then Hirsch has shown that $\mathrm{Imm}[M, \mathsf{R}^{n+k}]$ is in 1–1 correspondence with the set of homotopy classes of sections of the bundle with fibre $SO/SO(k)$, associated to ν_M. But Becker [1] has shown that, in the stable range, the set of sections of a bundle can be given a natural affine group structure; in particular, if one chooses some immersion as basepoint then $\mathrm{Imm}[M, \mathsf{R}^{n+k}]$ has an abelian group structure, provided $k>\frac{1}{2}n$; and the isomorphism class of this group is independent of the choice of basepoint.

Suppose now that M is a spin manifold ($W_1 M = W_2 M = 0$) and that $k \geqq n-2$. Then, ν_M is a principal bundle, and hence,

$$\mathrm{Imm}[M, \mathsf{R}^{n+k}] \approx [M, V_k]\,,$$

where $V_k = SO/SO(k)$ ($=$ Spin/Spin(k)).

For our second example, we calculate some groups $[M, V_k]$, interpreted as groups of immersions.

(i) $k\equiv 1 \bmod 4$, $k\geqq 5$. A Postnikov resolution of V_k, through dimension $k+2$, is given below:

$$\begin{array}{ccccc}
K_{k+2} & \xrightarrow{j_2} & Q_2 & & \\
 & & \downarrow{\scriptstyle p_2} & & \\
K_{k+1}\times K_{k+2} & \xrightarrow{j_1} & Q_1 & \xrightarrow{\beta} & K_{k+3} \\
 & & \downarrow{\scriptstyle p_1} & & \\
 & & K_k\times K_{k+2} & \xrightarrow{\alpha} & K_{k+2}\times K_{k+3}\,.
\end{array}$$

Here

$$\begin{aligned}
\alpha^*\iota_{k+2} &= \mathrm{Sq}^2\iota_k\otimes 1\,,\\
\alpha^*\iota_{k+3} &= \mathrm{Sq}^2\mathrm{Sq}^1\iota_k\otimes 1 + 1\otimes \mathrm{Sq}^1\iota_{k+2}\,,\\
\beta^*\iota_{k+3} &= \mathrm{Sq}^2\iota_{k+1}\otimes 1 + 1\otimes \mathrm{Sq}^1\iota_{k+2}\,.
\end{aligned}$$

Now let M be a $(k+2)$-dimensional spin manifold. Using the Wu formulae and the fact that $B\,\mathrm{Spin}(k)$ is 3-connected, one shows that

$$\mathrm{Image\,Sq}^1 = \mathrm{Image\,Sq}^2 = \mathrm{Image}\,\beta = 0\,,$$

in $H^{k+2}(M; Z_2)$. Thus the group $[M, Q_2]$ $(=[M, V_k])$ can be calculated by using two exact sequences. (We set $n=k+2$):

$$e_2\colon\ 0 \to H^nM \xrightarrow{j_2^*} [M,Q_2] \xrightarrow{p_2^*} [M,Q_1] \to 0\,,$$

$$e_1\colon\ 0 \to H^{n-1}M \otimes H^nM \xrightarrow{j_1^*} [M,Q_1] \xrightarrow{p_1^*} H^{n-2}M \otimes H^nM \to 0\,.$$

Let $u \in H^{n-2}M$, $v \in H^nM$. Then e_1 is determined by the homomorphism Φ_1, defined by

$$(u,v) \mapsto (\mathrm{Sq}^1 u \otimes v) \in H^{n-1}M \otimes H^nM\,.$$

Suppose now that $\mathrm{Sq}^1\colon H^{n-2}M \to H^{n-1}M$ is injective. Then a class $\alpha \in [M,Q_1]$ has order 2 if and only if $\alpha \in \mathrm{Image}\, j_1^*$. Thus e_2 (in this case) is determined by the morphism Φ_2, defined by

$$j_1^*(x,y) \to \mathrm{Sq}^1 x + y\,,$$

where $x \in H^{n-1}M$, $y \in H^nM$. But M is orientable and so $\mathrm{Sq}^1 x = 0$. Computing the extensions e_1, e_2 explicitly, one then has:

THEOREM 4.9. *Let M be an n-dimensional spin-manifold, with $n \geqq 7$, $n \equiv 3 \bmod 4$. Suppose that $\mathrm{Sq}^1\colon H^{n-2}M \to H^{n-1}M$ is injective. Then,*

$$\mathrm{Imm}[M, R^{2n-2}] \approx Z_8 \oplus (\oplus_{i=1}^{a} Z_4) \oplus (H^{n-1}M/\mathrm{Sq}^1 H^{n-2}M)\,,$$

where $a = \dim H^{n-2}M$. In particular, if P^n denotes real projective n-space, then

$$\mathrm{Imm}[P^n, R^{2n-2}] \approx Z_8 \oplus Z_4, \quad n \equiv 3 \bmod 4\,.$$

(ii) $k \equiv 2 \bmod 4$, $k \geqq 6$. A Postnikov resolution of V_k, through dimension $k+2$ is given below:

$$\begin{array}{ccc} K_{k+1} & \xrightarrow{j_1} & Q_1 \\ & & \big\downarrow p_1 \\ & & K^*_k \times K_{k+1} \xrightarrow{\alpha} K_{k+2}\,, \end{array}$$

with

$$\alpha^* \iota_{k+2} = \mathrm{Sq}^2 \iota_k \otimes 1 + 1 \otimes \mathrm{Sq}^1 \iota_{k+1}\,.$$

Thus we have an exact sequence (setting $n=k+2$)

$$e_1\colon\ 0 \to H^{n-1}M/\mathrm{Sq}^1(H^{n-2}M) \xrightarrow{j_1^*} [M,Q_1] \xrightarrow{p_1^*} H^{n-2}(M; Z) \otimes H^{n-1}M \to 0\,.$$

Let $u \in H^{n-2}(M; \mathsf{Z})$ be a class such that $2u=0$; choose $x \in H^{n-3}(M; \mathsf{Z}_2)$ such that $\delta x = u$. Then e_1 is given by:

$$(u,v) \to \mathrm{Sq}^2 x + v \in H^{n-1}M ,$$

where $v \in H^{n-1}M$. But M is a spin manifold and hence $\mathrm{Sq}^2 H^{n-3}M = 0$ (cf. [8]). Thus we have:

THEOREM 4.10. *Let M be a spin manifold of dimension n, where $n \equiv 0 \bmod 4$ and $n \geqq 8$. Then,*

$$\mathrm{Imm}[M, \mathsf{R}^{2n-2}] \approx H^{n-2}(M; \mathsf{Z}) \oplus H^{n-1}(M; \mathsf{Z}_4) .$$

REMARK. In a subsequent paper, by using a "twisted" version of the theory presented here, we will compute $\mathrm{Imm}[M^n, \mathsf{R}^{2n-1}]$, for all manifolds M^n, $n \geqq 5$.

EXAMPLE III. Complex K-Theory.

Our final example falls under Case II above; that is, diagram (4.1) is taken to be a sequence of connective coverings. Specifically, set $Q_0 = BU$, and let Q_i denote the $(2i+1)$-connective covering of BU. Thus, $K_i = K(\mathsf{Z}, 2i)$ and θ_i is a generator of $H^{2i}(Q_{i-1}; \mathsf{Z}) \approx \mathsf{Z}$. Also,

$$\varphi_{i+1} = \theta_{i+1} \circ j_i = \delta\, \mathrm{Sq}^2 \varrho_2(\iota_{2i-1}) , \tag{4.11}$$

where ι_{2i-1} generates $H^{2i-1}(\Omega K_i; \mathsf{Z})$. Notice that $\theta_1 = C_1$, the first Chern class in $H^2(BU; \mathsf{Z})$.

We compute the operation Φ_i, given in (4.5). By Theorem (3.6), taking $P = S^1 \cup_2 e^2$,

$$(\delta\, \mathrm{Sq}^2 \varrho)^P = \varepsilon(\mathrm{Sq}^1\mathrm{Sq}^2)\mathrm{Sq}^1 + \mathrm{Sq}^1\mathrm{Sq}^2 = \mathrm{Sq}^2\mathrm{Sq}^1 + \mathrm{Sq}^1\mathrm{Sq}^2 .$$

Let $x \in H^{2i}(X; \mathsf{Z})$ be a class such that $2x = 0$ and $\delta\,\mathrm{Sq}^2\varrho(x) = 0$. Choose classes

$$y \in H^{2i-1}(X; \mathsf{Z}_2), \quad z \in H^{2i+2}(X; \mathsf{Z})$$

such that

$$\delta y = x, \quad \varrho z = \mathrm{Sq}^2 x .$$

Then,

$$\begin{aligned} \Phi_i(x) &= \varrho^{-1}((\mathrm{Sq}^2\mathrm{Sq}^1 + \mathrm{Sq}^1\mathrm{Sq}^2)y) \\ &= \varrho^{-1}(\mathrm{Sq}^2 x + \mathrm{Sq}^1\mathrm{Sq}^2 y) \equiv z + \delta\,\mathrm{Sq}^2 y \bmod \Phi^*_{i+1} . \end{aligned} \tag{4.12}$$

By (4.7) we have:

THEOREM 4.13. *Let X be a complex and n an integer such that*

(i) $2H^{2i}(X; \mathsf{Z}) = 0, \quad$ for $2 \leqq i \leqq n$, *and suppose further that*

(ii) $\delta \mathrm{Sq}^2 \varrho H^{2i-2}(X; \mathsf{Z}) = H^{2i+1}(X; \mathsf{Z}), \quad 0 \leqq i \leqq n-1$.

Let $\eta \in \widetilde{\mathrm{KU}}(X)$, *and set* $w = C_1(\eta) \in H^2(X; \mathsf{Z})$. *Suppose that* $2^{r+1}w = 0$, *for some integer* $r \geqq 0$. *Then, for* $1 \leqq i < n$,

(a) $2^{r+i}\eta = 0 \Rightarrow \Phi_i \circ \ldots \circ \Phi_1(2^r w) = 0$.

If $\dim X \leqq 2n+1$, *then*

(b) $2^{r+n-1}\eta = 0 \Leftrightarrow \Phi_{n-1} \circ \ldots \circ \Phi_1(2^r w) = 0$,
(c) $2^{r+n}\eta = 0$.

As an example we have:

(4.14) Let X be a complex of dimension $2n+\varepsilon$ ($\varepsilon = 0$ or 1), satisfying (4.13) (i) and (ii). Suppose there is a class x in $H^1(X; \mathsf{Z}_2)$ such that $x^{2n} \neq 0$. Let η be the complex line bundle over X with $C_1(\eta) = \delta x \in H^2(X; \mathsf{Z})$. Then η generates a cyclic subgroup of order 2^n in $\widetilde{\mathrm{KU}}(X)$.

COROLLARY 4.15. $\widetilde{\mathrm{KU}}(P^{2n+\varepsilon}) = \mathsf{Z}_{2^n}$, $\varepsilon = 0$ or 1.

PROOF OF 4.14. We show that

$$\Phi_i(\delta x^{2i-1}) = \delta x^{2i+1} ,$$

and hence $\Phi_{n-1} \circ \ldots \circ \Phi_1(\delta x) = \delta x^{2n-1} \neq 0$. (By 4.14 (ii), $\varphi_{i+1*} = 0$, $i \geqq 0$.) Now

$$\mathrm{Sq}^2(\delta x^{2i-1}) = i x^{2i+2} = i(\varrho \delta x^{2i+1})$$

and

$$\delta \mathrm{Sq}^2 x^{2i-1} = (i-1)\delta x^{2i+1} .$$

Thus, by (4.12),

$$\Phi_i(\delta x^{2i-1}) = i(\delta x^{2i+1}) + (i-1)\delta x^{2i+1} = \delta x^{2i+1}$$

as claimed.

REMARK. By precisely the same technique one can compute the order of elements in *real* K-Theory. Of course, here one must determine four operations. See [5].

5. Appendix.

Let A, B, and C be Abelian groups, and let $e: 0 \to A \xrightarrow{\iota} B \xrightarrow{\lambda} C \to 0$ be an extension of C by A. For each integer n, let $K(n) \subset C$ be the kernel of multiplication by n. If $x \in K(n)$, define $\mu_e(x,n)$ to be $\iota^{-1}n\lambda^{-1}x$, a coset of nA. Thus $\mu_e(\cdot,n)$ is a homomorphism from $K(n)$ to A/nA. If m is another integer, and if $x \in K(nm)$, then $mx \in K(n)$ and $\mu_e(x,nm) \subset \mu_e(mx,n)$; the following diagram is thus commutative, where q is the quotient map:

$$\begin{array}{ccc} K(n) & \xrightarrow{\mu_e(\cdot,n)} & A/nA \\ {\scriptstyle m}\uparrow & & \downarrow{\scriptstyle q} \\ K(nm) & \xrightarrow{\mu_e(\cdot,nm)} & A/nmA . \end{array}$$

As the following theorem shows, knowledge of the homomorphisms $\mu_e(\cdot,p^k)$ for all primes p and all integers $k>0$ is sufficient to determine e as an element of $\operatorname{Ext}(C,A)$ if C is finitely generated.

THEOREM 5.1. *Let A and C be Abelian groups, where C is finitely generated. Let $K(n) \in C$ be the kernel of multiplication by n, for each integer n. If a homomorphism $\mu(\cdot,p^k)\colon K(p^k) \to A/p^kA$ is given for each prime p and each positive integer k, and if the following diagram is always commutative:*

$$\begin{array}{ccc} K(p^k) & \xrightarrow{\mu(\cdot,p^k)} & A/p^kA \\ {\scriptstyle p}\uparrow & & \downarrow{\scriptstyle q} \\ K(p^{k+1}) & \xrightarrow{\mu(\cdot,p^{k+1})} & A/p^{k+1}A , \end{array}$$

then there exists a unique $e \in \operatorname{Ext}(C,A)$ such that $\mu_e(\cdot,p^k)=\mu(\cdot,p^k)$ for all p and all k.

We leave the proof to the reader (cf. [6, p. 76], [3, p. 63]).

REFERENCES

1. J. Becker, *Cohomology and the classification of liftings*, Trans. Amer. Math. Soc. 133 (1968), 447–475.
2. E. Brown, *Cohomology theories*, Ann. of Math. 75 (1962), 467–484.
3. P. Hilton, *Homotopy Theory and Duality*, Gordon and Breach, New York · London · Paris, 1965.
4. L. Kristensen, *On a Cartan formula for secondary cohomology operations*, Math. Scand. 16 (1965), 97–115.
5. L. L. Larmore, *Computation of* $[X;Y]$ *and application to K-theory*, 1969, mimeographed preprint, California State College at Dominguez Hills.
6. S. MacLane, *Homology*, Springer-Verlag, Berlin · Göttingen · Heidelberg, 1963.
7. M. Mahowald, *On obstruction theory in orientable fibre bundles*, Trans. Amer. Math. Soc. 110 (1964), 315–349.

8. W. Massey, *Normal vector fields on manifolds*, Proc. Amer. Math. Soc. 12 (1961), 33–40.
9. F. P. Peterson, *Functional cohomology operations*, Trans. Amer. Math. Soc. 86 (1957), 197–211.
10. F. P. Peterson, *Functional higher order cohomology operations*, Symposium International de Topologia Algebraica, La Universidad National Autónoma de México y la UNESCO, 1958, 159–164.
11. E. Spanier, *Algebraic Topology*, McGraw–Hill, New York · Toronto · London, 1966.
12. E. Spanier, *Secondary operations on mappings and cohomology*, Ann. of Math. 75 (1962), 260–282.
13. N. Steenrod, *Cohomology invariants of mappings*, Ann. of Math. 50 (1949), 954–988.
14. E. Thomas, *A spectral sequence for K-theory*, Appendix in lecture notes of R. Bott, Harvard Univ., 1962.
15. E. Thomas, *Postnikov invariants and higher order cohomology operations*, Ann. of Math. 85 (1967), 184–217.
16. G. Whitehead, *Generalized homology theories*, Trans. Amer. Math. Soc. 102 (1962), 227–283.
17. G. Whitehead, *Fiber spaces and the Eilenberg homology groups*, Proc. Nat. Acad. Sci U.S.A., 38 (1952), 426–430.

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