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MINIMAL COMPLEXES OF COTORSION FLAT MODULES

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Abstract

Let R be a commutative noetherian ring. We give criteria for a complex of cotorsion flat R-modules to be minimal, in the sense that every self homotopy equivalence is an isomorphism. To do this, we exploit Enochs' description of the structure of cotorsion flat R-modules. More generally, we show that any complex built from covers in every degree (or envelopes in every degree) is minimal, as well as give a partial converse to this in the context of cotorsion pairs. As an application, we show that every R-module is isomorphic in the derived category over R to a minimal semi-flat complex of cotorsion flat R-modules.

Introduction

One of the most ubiquitous examples of a minimal chain complex is that of a minimal free resolution, introduced by Hilbert in the 1890s. Minimal projective and injective resolutions are an integral part of homological algebra, and there are useful criteria for identifying whether a such a complex is minimal. In particular, a projective resolution P of a finitely generated module over a local ring $(R, \mathfrak{m})$ is minimal if and only if $P \otimes_R R/\mathfrak{m}$ has zero differential; an injective resolution I is minimal if and only if $\operatorname{Hom}_R(R/\mathfrak{p}, I)_\mathfrak{p}$ has zero differential for every prime $\mathfrak{p}$. In [3], Avramov and Martsinkovsky introduced a versatile notion of minimality for chain complexes, which recovers both of these classical notions: a chain complex C is *minimal* if every homotopy equivalence $\gamma\colon C \to C$ is an isomorphism. One of our goals is to give criteria, in the spirit of these classical conditions, for a chain complex of cotorsion flat modules (defined below) to be minimal.

Let R be a commutative noetherian ring. We say an R-module M is *cotorsion flat* if it is both flat and satisfies the added assumption that $\operatorname{Ext}^1_R(F, M) = 0$ for every flat R-module F, i.e., M is also cotorsion. Enochs showed [7] that cotorsion flat R-modules have a unique decomposition, indexed by the prime ideals of R, similar to the decomposition for injective modules given by Matlis [15]. We use this description to characterize minimal complexes of cotorsion

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flat R-modules. In a subsequent paper [20], we show that minimal complexes of cotorsion flat R-modules are useful in computing cosupport, an invariant homologically dual to support that was introduced by Benson, Iyengar, and Krause in [4].

Parallel to the minimality criteria for complexes of projective or injective R-modules, one of our goals is to show (Theorem 3.5) that a complex B of cotorsion flat R-modules is minimal if and only if either of the following criteria hold for every $\mathfrak{p} \in \operatorname{Spec} R$:

- the complex $R/\mathfrak{p} \otimes_R \operatorname{Hom}_R(R_\mathfrak{p}, \Lambda^\mathfrak{p} B)$ has zero differential;
- there is no subcomplex of the form $0 \to \widehat{R_\mathfrak{p}}^\mathfrak{p} \xrightarrow{\cong} \widehat{R_\mathfrak{p}}^\mathfrak{p} \to 0$ that is degreewise a direct summand of B.

The first condition has been studied previously in the context of flat resolutions of cotorsion modules, where the numbers $\dim_{\kappa(\mathfrak{p})} \operatorname{Tor}_i^{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \operatorname{Hom}_R(R_\mathfrak{p}, M))$ were examined by Enochs and Xu in [22], [10] (here, $\kappa(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$). More recently, Dailey showed in [6, Theorem 4.2.8] that a flat resolution F of a cotorsion module is built from flat covers if and only if the complex $\kappa(\mathfrak{p}) \otimes_R \operatorname{Hom}_R(R_\mathfrak{p}, F)$ has zero differential. The notion of minimality studied by Enochs, Xu, and Dailey in this context refers to a resolution being built by flat covers, as opposed to the "homotopic" minimality defined in [3] and as is considered in Theorem 3.5.

One of the ingredients in the proof of Theorem 3.5 is understanding how cotorsion flat R-modules can be broken down. Roughly, we show that $\mathfrak{p}$-adic completion $\Lambda^\mathfrak{p}(-)$ and colocalization $\operatorname{Hom}_R(R_\mathfrak{p}, -)$ allow us to focus on the "$\mathfrak{p}$-component" of a cotorsion flat R-module; see Lemma 2.2. After proving Theorem 3.5, we end Section 3 by remarking that completion and colocalization both preserve minimal complexes of cotorsion flat R-modules; see Proposition 3.6.

We also study the relationship between minimality and covers/envelopes (see §1.5 for definitions) more generally in Section 4. There we show that constructing complexes from covers (or envelopes) leads to a stronger form of minimality than the homotopic version introduced by Avramov and Martsinkovsky. We prove (Theorem 4.1) that a complex of R-modules which is built entirely from covers or from envelopes (in a fixed class of modules) must be minimal.

Although not every minimal complex is built in this way (see Example 4.2), we do give a partial converse in the context of cotorsion pairs, see Proposition 4.3. An immediate consequence is that projective resolutions of modules are minimal if and only if they are built from projective covers in every degree; injective resolutions of modules are minimal if and only if they are built from injective envelopes in every degree. We end Section 4 with an application to characterizing perfect rings by the existence of minimal projective resolutions for all modules.

In Section 5, we prove (Theorem 5.2) the existence of minimal left and right cotorsion flat resolutions for certain modules, as well as show that every R-module is isomorphic in the derived category over R to a minimal semi-flat complex of cotorsion flat R-modules.

1. Preliminaries

Throughout this paper, the ring R is assumed to be commutative and noetherian. We briefly recall some background material and set notation needed throughout.

1.1. Complexes

A *complex* of R-modules (or shorter, *R-complex*) is a sequence of R-modules and R-linear maps

$$C = \cdots \xrightarrow{\partial_C^{i-1}} C^i \xrightarrow{\partial_C^i} C^{i+1} \xrightarrow{\partial_C^{i+1}} \cdots$$

such that $\partial_C^{i+1}\partial_C^i = 0$ for all $i \in \mathbb{Z}$. For R-complexes C and D, a *degree zero chain map* $f\colon C \to D$ is a collection of R-linear maps $\{f^i\colon C^i \to D^i\}_{i\in\mathbb{Z}}$, satisfying $f^{i+1}\partial_C^i = \partial_D^i f^i$. These are the morphisms in the category of R-complexes. We say that an R-complex C is *bounded on the left* (respectively, *right*) if $C^i = 0$ for $i \ll 0$ (respectively, $C^i = 0$ for $i \gg 0$). As is standard, we set $C_i = C^{-i}$. For R-complexes C and D, the total tensor product complex $C \otimes_R D$ is the direct sum totalization of the evident double complex, and the total Hom complex $\operatorname{Hom}_R(C, D)$ is the direct product totalization of the underlying double complex (see [21, 2.7.1 and 2.7.4, respectively]). We say an R-complex C is *exact* (or *acyclic*) if $\mathrm{H}^i(C) = 0$ for all $i \in \mathbb{Z}$.

1.2. Homotopy and derived categories

We say that degree zero chain maps $f, g\colon C \to D$ are *chain homotopic*, denoted by $f \sim g$, if there exists a cohomological degree -1 map (called a chain homotopy) $h\colon C \to D$ such that $f - g = \partial_D h + h\partial_C$. An R-complex C is *contractible* if $1_C \sim 0_C$. For further details on homotopies and complexes in general, see for example [1].

The *homotopy category* $\mathrm{K}(R)$ is the category whose objects are complexes of R-modules and morphisms are degree zero chain maps up to chain homotopy. If we further invert all quasi-isomorphisms between R-complexes (degree zero chain maps that induce an isomorphism on cohomology), we obtain the *derived category* of R, denoted $\mathrm{D}(R)$. We use $\simeq$ to denote isomorphisms in $\mathrm{D}(R)$. For more details on the derived category, see for example [21, Chapter 10].

We say that an R-complex F is *semi-flat* (also called DG-flat, as in [2]) if F^i is flat for all $i \in \mathbb{Z}$ and the functor $F \otimes_R -$ preserves quasi-isomorphisms. For example, any bounded on the right complex of flat R-modules is semi-flat [2, Example 1.1.F].

1.3. Injective modules

Over a commutative noetherian ring R, we have a decomposition of injective R-modules, due to Matlis [15]. In fact, there exists a bijection between prime ideals $\mathfrak{p}$ of Spec R and indecomposable injective modules $E(R/\mathfrak{p})$, the injective hull of $R/\mathfrak{p}$ over R. In this way, for some sets $X_\mathfrak{p}$, every injective R-module can be uniquely (up to isomorphism) expressed as $\bigoplus_{\mathfrak{p}\in \mathrm{Spec}\, R} E(R/\mathfrak{p})^{(X_\mathfrak{p})}$, where for a module N and set X, we set $N^{(X)} = \bigoplus_X N$. The indecomposable injective R-module $E(R/\mathfrak{p})$ is $\mathfrak{p}$-torsion and $\mathfrak{p}$-local [18, p. 354]; a module M is $\mathfrak{p}$-*torsion* if for every $x \in M$, there exists $n \geq 1$ such that $\mathfrak{p}^n x = 0$ and M is $\mathfrak{p}$-*local* if for every $y \in R \setminus \mathfrak{p}$, multiplication by y on M is an automorphism.

1.4. Completions

For an ideal $\mathfrak{a} \subseteq R$ and an R-module M, the $\mathfrak{a}$-*adic completion of* M is

$$\Lambda^{\mathfrak{a}} M = \varprojlim_n (M/\mathfrak{a}^n M),$$

or equivalently, $\Lambda^{\mathfrak{a}} M = \varprojlim_n (R/\mathfrak{a}^n \otimes_R M)$. This is also written as $\widehat{M}^{\mathfrak{a}}$. As $\Lambda^{\mathfrak{a}}(-)$ defines an additive functor on the category of R-modules, it naturally extends to a functor on the homotopy category $\Lambda^{\mathfrak{a}}\colon K(R) \to K(R)$. We say an R-complex M is $\mathfrak{a}$-*complete* if the natural map $M \to \Lambda^{\mathfrak{a}} M$ is an isomorphism. For a nice discussion of $\Lambda^{\mathfrak{a}}$ and its left-derived functor $\mathbf{L}\Lambda^{\mathfrak{a}}$, see [17].

1.5. Covers, envelopes, and $\mathscr{F}$-resolutions

Let $\mathscr{F}$ be a class of R-modules closed under isomorphisms. For an R-module M, a morphism $\phi\colon M \to F$ with $F \in \mathscr{F}$ is an $\mathscr{F}$-*envelope* of M if:

(1) for any map $\phi'\colon M \to F'$ with $F' \in \mathscr{F}$, there exists $f\colon F \to F'$ such that $f \circ \phi = \phi'$, and
(2) if $f\colon F \to F$ is an endomorphism with $f \circ \phi = \phi$, then f must be an isomorphism.

If $\phi\colon M \to F$ satisfies (1) but not necessarily (2), it is called an $\mathscr{F}$-*preenvelope*. If an $\mathscr{F}$-envelope exists, it is unique up to isomorphism. A class $\mathscr{F}$ is *enveloping* (respectively, *preenveloping*) if every R-module has an $\mathscr{F}$-envelope (respectively, an $\mathscr{F}$-preenvelope). If an enveloping class contains all injective R-modules, the envelopes will necessarily be injections.

$\mathscr{F}$*-(pre)covers* and *(pre)covering* classes are defined dually; see [9, Chapter 5] for details. In particular, if the class $\mathscr{F}$ contains the ring R, then $\mathscr{F}$-covers are surjective.

For any ring, Xu showed that the class of cotorsion modules is enveloping if and only if the class of flat modules is covering [23, Theorem 3.4.6]; shortly after, Bican, El Bashir, and Enochs showed that the class of flat modules is covering [5] for any ring (as was shown for a commutative noetherian ring by Xu [23]). Hence the class of cotorsion modules is enveloping. More classically, Fuchs showed that in a noetherian ring, the class of pure-injective modules is enveloping [11].

If $\mathscr{F}$ is an enveloping class, an *enveloping* $\mathscr{F}$*-resolution* of M is an R-complex

$$0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

with each $F^i \in \mathscr{F}$, constructed so that $M \to F^0$, $\operatorname{coker}(M \to F^0) \to F^1$, and $\operatorname{coker}(F^{i-1} \to F^i) \to F^{i+1}$ for $i \geq 1$ are $\mathscr{F}$-envelopes. Dually, if $\mathscr{F}$ is a covering class, a *covering* $\mathscr{F}$*-resolution* of M is an R-complex

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

with each $F_i \in \mathscr{F}$, constructed so that $F_0 \to M$, $F_1 \to \ker(F_0 \to M)$, and $F_{i+1} \to \ker(F_i \to F_{i-1})$ for $i \geq 1$ are $\mathscr{F}$-covers. Observe that the augmented enveloping $\mathscr{F}$-resolution $0 \to M \to F^0 \to F^1 \to \cdots$ and the augmented covering $\mathscr{F}$-resolution $\cdots \to F_1 \to F_0 \to M \to 0$ need not be exact.

REMARK 1.1. Our terminology of enveloping/covering $\mathscr{F}$-resolutions is intentionally non-standard to avoid collision with usage of the term "minimal". What we call enveloping/covering $\mathscr{F}$-resolutions are referred to as *minimal* left/right $\mathscr{F}$-resolutions in [9, Chapter 8] as well as elsewhere in the literature, but we prefer for now to reserve the term "minimal" to mean a minimal complex. We show later, in Theorem 4.1, that enveloping/covering $\mathscr{F}$-resolutions of modules are in fact minimal complexes, which justifies the existing terminology.

We continue to use the un-decorated term *resolution* to mean an honest resolution in the sense that the augmented sequence is exact. That is, an R-complex C is a *left resolution* of an R-module M if there exists a quasi-isomorphism $C \xrightarrow{\simeq} M$ and $C^i = 0$ for $i > 0$; it is a *right resolution* if there is a quasi-isomorphism $M \xrightarrow{\simeq} C$ and $C^i = 0$ for $i < 0$. A *projective resolution* of a module is a left resolution P such that each P_i is projective; an *injective resolution* of a module is a right resolution I such that each I^i is injective.

2. Decomposing cotorsion flat modules

An R-module C is called *cotorsion* if $\operatorname{Ext}^1_R(F, C) = 0$ for every flat R-module F. All injective R-modules, as well as all R-modules of the form $\operatorname{Hom}_R(M, E)$ for any R-module M and injective R-module E, are cotorsion [7, Lemma 2.1]. The class of flat R-modules and the class of cotorsion R-modules form what is called a *cotorsion pair*; in particular, if F is any R-module such that $\operatorname{Ext}^1_R(F, C) = 0$ for every cotorsion R-module C, then F is flat [9, Lemma 7.1.4]. An R-module that is both cotorsion and flat will be called a *cotorsion flat* R-module.

Enochs showed [7, Theorem] that cotorsion flat R-modules have a unique decomposition indexed by Spec R: an R-module B is cotorsion flat if and only if there is an isomorphisms $B \cong \prod_{\mathfrak{p}\in \operatorname{Spec} R} \widehat{R_{\mathfrak{p}}^{(X_{\mathfrak{p}})}}^{\mathfrak{p}}$, for some sets $X_{\mathfrak{p}}$. Moreover, this decomposition is uniquely determined (up to isomorphism) by the ranks of the free $R_{\mathfrak{p}}$-modules $R_{\mathfrak{p}}^{(X_{\mathfrak{p}})}$. For any set X and $\mathfrak{p} \in \operatorname{Spec} R$, there is an isomorphism [23, Lemma 4.1.5]:

$$\operatorname{Hom}_R(E(R/\mathfrak{p}), E(R/\mathfrak{p})^{(X)}) \cong \widehat{R_{\mathfrak{p}}^{(X)}}^{\mathfrak{p}}. \tag{2.1}$$

The following lemma is one of our key tools in understanding the structure of complexes of cotorsion flat R-modules. The lemma captures the idea of recovering the $\mathfrak{p}$-component of a cotorsion flat R-module. Enochs employs this idea in [7, Proof of Theorem] but does not use completion and colocalization as is done below; the "in particular" of part (2) below can be found in [9, Proof of Lemma 8.5.25], for instance.

LEMMA 2.2. *Let B be a cotorsion flat R-module isomorphic to $\prod_{\mathfrak{q}\in \operatorname{Spec} R} T_{\mathfrak{q}}$, with $T_{\mathfrak{q}} = \widehat{R_{\mathfrak{q}}^{(X_{\mathfrak{q}})}}^{\mathfrak{q}}$ for some sets $X_{\mathfrak{q}}$. For an ideal $\mathfrak{a} \subseteq R$, prime ideal $\mathfrak{p} \in$ Spec R, and multiplicatively closed set S, there are isomorphisms*

(1) $\widehat{B}^{\mathfrak{a}} \cong \prod_{\mathfrak{a}\subseteq\mathfrak{q}} T_{\mathfrak{q}}$, *and*

(2) $\operatorname{Hom}_R(S^{-1}R, B) \cong \prod_{\mathfrak{q}\cap S=\varnothing} T_{\mathfrak{q}}$*; in particular,* $\operatorname{Hom}_R(R_{\mathfrak{p}}, B) \cong \prod_{\mathfrak{q}\subseteq\mathfrak{p}} T_{\mathfrak{q}}$.

Moreover, if B is a complex of cotorsion flat R-modules, the natural maps

$$\operatorname{Hom}_R(S^{-1}R, B) \hookrightarrow B \qquad \text{and} \qquad B \twoheadrightarrow \widehat{B}^{\mathfrak{a}}$$

are degreewise split morphisms. In particular, the complex $\operatorname{Hom}_R(R_{\mathfrak{p}}, \widehat{B}^{\mathfrak{p}})$ *can be identified with the subquotient complex*

$$\cdots \longrightarrow T_{\mathfrak{p}}^i \longrightarrow T_{\mathfrak{p}}^{i+1} \longrightarrow \cdots,$$

having differential $\partial_{\mathfrak{p}} = \operatorname{Hom}_R(R_{\mathfrak{p}}, \widehat{\partial}^{\mathfrak{p}})$ *induced from B.*

PROOF. For (1), first decompose B as

$$\prod_{\mathfrak{q}\in\operatorname{Spec} R} T_{\mathfrak{q}} \cong \Big(\prod_{\mathfrak{a}\subseteq\mathfrak{q}} T_{\mathfrak{q}}\Big) \oplus \Big(\prod_{\mathfrak{a}\not\subseteq\mathfrak{q}} T_{\mathfrak{q}}\Big).$$

For $\mathfrak{a} \not\subseteq \mathfrak{q}$ and $n \in \mathbb{N}$, $R/\mathfrak{a}^n \otimes_R T_{\mathfrak{q}} = 0$ since $T_{\mathfrak{q}}$ is $\mathfrak{q}$-local; thus

$$\varprojlim_n \Big(R/\mathfrak{a}^n \otimes_R \prod_{\mathfrak{a}\not\subseteq\mathfrak{q}} T_{\mathfrak{q}}\Big) \cong \varprojlim_n \prod_{\mathfrak{a}\not\subseteq\mathfrak{q}} (R/\mathfrak{a}^n \otimes_R T_{\mathfrak{q}}) = 0.$$

On the other hand, for $\mathfrak{a} \subseteq \mathfrak{q}$, the $\mathfrak{q}$-complete R-module $T_{\mathfrak{q}}$ is $\mathfrak{a}$-complete; see [16, Exercise 8.2]. As products commute with inverse limits, a product of $\mathfrak{a}$-complete modules is again $\mathfrak{a}$-complete; (1) follows.

For (2), setting $E := E(R/\mathfrak{q})$, we have:

$$\begin{aligned}
&\operatorname{Hom}_R\Big(S^{-1}R, \prod_{\mathfrak{q}\in\operatorname{Spec} R} T_{\mathfrak{q}}\Big)\\
&\cong \prod_{\mathfrak{q}\in\operatorname{Spec} R} \operatorname{Hom}_R\big(S^{-1}R, \operatorname{Hom}_R(E, E^{(X_{\mathfrak{q}})})\big), && \text{by (2.1)},\\
&\cong \prod_{\mathfrak{q}\in\operatorname{Spec} R} \operatorname{Hom}_R(E \otimes_R S^{-1}R, E^{(X_{\mathfrak{q}})}), && \text{by adjointness},\\
&\cong \prod_{\mathfrak{q}\cap S=\varnothing} \operatorname{Hom}_R(E, E^{(X_{\mathfrak{q}})}), && \text{as } E \text{ is } \mathfrak{q}\text{-local, } \mathfrak{q}\text{-torsion},\\
&\cong \prod_{\mathfrak{q}\cap S=\varnothing} T_{\mathfrak{q}}, && \text{again applying (2.1).}
\end{aligned}$$

The last remarks follow from the existence of natural maps $R \to S^{-1}R$ and $R \to \widehat{R}^{\mathfrak{a}}$ in conjunction with (1) and (2).

3. Minimality criteria for complexes of cotorsion flat modules

One of our main results is Theorem 3.5 below, where we present minimality criteria for complexes of cotorsion flat R-modules. As above, we use the notation

$$T_{\mathfrak{q}} = \widehat{R_{\mathfrak{q}}^{(X_{\mathfrak{q}})}}^{\mathfrak{q}}$$

for some prime $\mathfrak{q}$ and index set $X_{\mathfrak{q}}$. We start with two lemmas.

LEMMA 3.1. *For any homomorphism* $f\colon \prod_{\mathfrak{q}\in\operatorname{Spec} R} T_{\mathfrak{q}} \to \prod_{\mathfrak{q}\in\operatorname{Spec} R} T'_{\mathfrak{q}}$, *define* $f_{\mathfrak{p}}$ *to be the composite*

$$f_{\mathfrak{p}}\colon\ T_{\mathfrak{p}} \hookrightarrow \prod_{\mathfrak{q}} T_{\mathfrak{q}} \xrightarrow{\ f\ } \prod_{\mathfrak{q}} T'_{\mathfrak{q}} \twoheadrightarrow T'_{\mathfrak{p}},$$

where the outer maps are the canonical ones. If $f_{\mathfrak{p}}\colon T_{\mathfrak{p}} \to T'_{\mathfrak{p}}$ *is an isomorphism for all* $\mathfrak{p} \in \operatorname{Spec} R$*, then* f *is an isomorphism.*

PROOF. We define a well ordering on the set $Z = \operatorname{Spec} R$, for any noetherian ring R, as is done in [8]. This will allow us to avoid assuming finite Krull dimension. Let Z_0 be the set of maximal ideals of R. For any ordinal $\alpha > 0$, define Z_α to be the set of maximal elements of $Z \setminus \big(\bigcup_{\beta<\alpha} Z_\beta\big)$. If $Z \setminus \big(\bigcup_{\beta<\alpha} Z_\beta\big) \neq \varnothing$, then $Z_\alpha \neq \varnothing$, because R is noetherian. Moreover, there exists an ordinal κ such that $Z_\alpha = \varnothing$ for $\alpha \geq \kappa$ (else we would contradict the fact that Z is a set), hence Z is the disjoint union $Z = \bigcup_{\alpha<\kappa} Z_\alpha$. We may well order each Z_α (e.g., [9, Proposition 1.1.7]). By [9, Exercise 9a, page 7], we may use the well orderings of each Z_α to well order Z so that if $\mathfrak{p} \in Z_\alpha$ and $\mathfrak{q} \in Z_\beta$ with $\alpha < \beta$, then $\mathfrak{p} < \mathfrak{q}$. We may therefore index the primes in $Z = \operatorname{Spec} R$ by $\alpha < \lambda$ for some ordinal λ so that if $\beta < \alpha < \lambda$, then $\mathfrak{q}_\beta \not\subset \mathfrak{q}_\alpha$.

With this well ordering, the map above is $f\colon \prod_{\alpha<\lambda} T_{\mathfrak{q}_\alpha} \to \prod_{\alpha<\lambda} T'_{\mathfrak{q}_\alpha}$, with the assumption that f_{q_α} is an isomorphism for each $\alpha < \lambda$. For a fixed $\beta < \lambda$, we have [23, Lemma 4.1.8]:

$$\operatorname{Hom}_R\Big(\prod_{\beta<\alpha<\lambda} T_{\mathfrak{q}_\alpha}, T_{\mathfrak{q}_\beta}\Big) = 0. \tag{3.2}$$

For each $\beta < \lambda$, we may write

$$\prod_{\alpha<\lambda} T_{\mathfrak{q}_\alpha} = \Big(\prod_{\alpha\leq\beta} T_{\mathfrak{q}_\alpha}\Big) \oplus \Big(\prod_{\beta<\alpha<\lambda} T_{\mathfrak{q}_\alpha}\Big), \quad \text{and similarly for } \textstyle\prod_{\alpha<\lambda} T'_{\mathfrak{q}_\alpha}, \tag{3.3}$$

and so by (3.2), there exists a map $f_{\leq\beta}$ making the following diagram commute, where the vertical maps are the canonical split projections:

$$\begin{array}{ccc}
\prod_{\alpha<\lambda} T_{\mathfrak{q}_\alpha} & \xrightarrow{\ f\ } & \prod_{\alpha<\lambda} T'_{\mathfrak{q}_\alpha} \\
\downarrow & & \downarrow \\
\prod_{\alpha\leq\beta} T_{\mathfrak{q}_\alpha} & \xrightarrow{\ f_{\leq\beta}\ } & \prod_{\alpha\leq\beta} T'_{\mathfrak{q}_\alpha}.
\end{array}$$

Moreover, if $\beta' < \beta$ and we set $\pi_{\beta'\beta}\colon \prod_{\alpha\leq\beta} T_{\mathfrak{q}_\alpha} \twoheadrightarrow \prod_{\alpha\leq\beta'} T_{\mathfrak{q}_\alpha}$ (and similarly, $\pi'_{\beta'\beta}\colon \prod_{\alpha\leq\beta} T'_{\mathfrak{q}_\alpha} \twoheadrightarrow \prod_{\alpha\leq\beta'} T'_{\mathfrak{q}_\alpha}$) to be the canonical projections, a diagram chase shows that $\pi'_{\beta'\beta} \circ f_{\leq\beta} = f_{\leq\beta'}\pi_{\beta'\beta}$. Since $\prod_{\alpha<\lambda} T_{\mathfrak{q}_\alpha} = \varprojlim_\beta \prod_{\alpha\leq\beta<\lambda} T_{\mathfrak{q}_\alpha}$ (and likewise for $\prod_{\alpha<\lambda} T'_{\mathfrak{q}_\alpha}$) [8, Proof of Theorem 4.1], and $\{f_{\leq\beta}\}_{\beta<\lambda}$ is a morphism of inverse systems (alternatively, these inverse systems satisfy the Mittag-Leffler condition [21, Proposition 3.5.7]), to show f is an isomorphism, it is enough to show

$$f_{\le\beta}\colon \prod_{\alpha\le\beta<\lambda} T_{\mathfrak{q}_\alpha} \longrightarrow \prod_{\alpha\le\beta<\lambda} T'_{\mathfrak{q}_\alpha}$$

is an isomorphism for all $\beta < \lambda$. To do this, we apply transfinite induction (see e.g., [9, Proposition 1.1.18]).

When $\beta = 0$, the definition of $f_{\le 0}$ along with the decomposition in (3.3) shows $f_{\le 0} = f_{\mathfrak{q}_0}$, which is an isomorphism by hypothesis.

For an ordinal $\beta < \lambda$ such that $\beta = \epsilon + 1$ for an ordinal ϵ, there exists a map making the following diagram commute by (3.2):

$$\begin{array}{ccccc} T_{\mathfrak{q}_\beta} & \hookrightarrow & \prod_{\alpha\le\beta} T_{\mathfrak{q}_\alpha} & \twoheadrightarrow & \prod_{\alpha\le\epsilon} T_{\mathfrak{q}_\alpha} \\ \downarrow{\scriptstyle\exists} & & \downarrow{\scriptstyle f_{\le\beta}} & & \downarrow{\scriptstyle f_{\le\epsilon}} \\ T'_{q_\beta} & \hookrightarrow & \prod_{\alpha\le\beta} T'_{\mathfrak{q}_\alpha} & \twoheadrightarrow & \prod_{\alpha\le\epsilon} T'_{\mathfrak{q}_\alpha}. \end{array}$$

Appealing to the decomposition in (3.3), we see that the vertical map on the left agrees with $f_{\mathfrak{q}_\beta}$. The left and right vertical maps are isomorphisms, by hypothesis and assumption, respectively. Hence $f_{\le\beta}$ is an isomorphism in this case.

Finally, suppose β is a limit ordinal and that $f_{\le\beta'}$ is an isomorphism for all $\beta' < \beta$. Using the fact that $f_{\le\beta} = \varprojlim_{\beta'<\beta} f_{\le\beta'}$ and that $\{f_{\le\beta'}\}_{\beta'<\beta}$ is a morphism of inverse systems where each $f_{\le\beta'}$ is an isomorphism, we obtain that $f_{\le\beta}$ is an isomorphism in this case as well. It follows that f is an isomorphism.

The following observation seems to be known, but for lack of a reference we spell it out here. Compare this result also with [17, Lemma 3.5].

Lemma 3.4. *Let F be a semi-flat R-complex. If $R/\mathfrak{p} \otimes_R F$ is acyclic, then $\Lambda^{\mathfrak{p}}(F)$ is acyclic.*

Proof. The canonical short exact sequence $\mathfrak{p} \hookrightarrow R \twoheadrightarrow R/\mathfrak{p}$ induces a short exact sequence of R-complexes $\mathfrak{p} \otimes_R F \hookrightarrow R \otimes_R F \twoheadrightarrow R/\mathfrak{p} \otimes_R F$; hence the morphism $\mathfrak{p} \otimes_R F \to F$ is a quasi-isomorphism, by the long exact sequence in homology [21, Theorem 1.3.1]. Moreover, for each $n \ge 1$, there is a short exact sequence $\mathfrak{p}^{n+1} \otimes_R F \hookrightarrow \mathfrak{p}^n \otimes_R F \twoheadrightarrow \mathfrak{p}^n/\mathfrak{p}^{n+1} \otimes_R F$ (induced by the canonical maps in $\mathfrak{p}^{n+1} \hookrightarrow \mathfrak{p}^n \twoheadrightarrow \mathfrak{p}^n/\mathfrak{p}^{n+1}$). Further, [19, Proposition 5.7] yields that $\mathfrak{p}^n/\mathfrak{p}^{n+1} \otimes_R F \cong \mathfrak{p}^n/\mathfrak{p}^{n+1} \otimes_{R/\mathfrak{p}} R/\mathfrak{p} \otimes_R F$ is acyclic since $R/\mathfrak{p} \otimes_R F$ is acyclic and semi-flat over $R/\mathfrak{p}$. Thus the canonical injection $\mathfrak{p}^n \otimes_R F \hookrightarrow F$ is a quasi-isomorphism for all $n \ge 1$, hence $R/\mathfrak{p}^n \otimes_R F$ is acyclic for all $n \ge 1$. The inverse system

$$\cdots \twoheadrightarrow R/\mathfrak{p}^2 \otimes_R F \twoheadrightarrow R/\mathfrak{p} \otimes_R F$$

satisfies the Mittag-Leffler condition, and it follows by [21, Theorem 3.5.8] that $\varprojlim_n (R/\mathfrak{p}^n \otimes_R F) = \Lambda^{\mathfrak{p}}(F)$ is also acyclic.

To find appropriate minimality criteria for complexes of cotorsion flat R-modules, we turn to minimal complexes of injective R-modules for inspiration. Recall that an R-complex B is *minimal* if each homotopy equivalence $\gamma \colon B \to B$ is an isomorphism [3]; equivalently, if each map $\gamma \colon B \to B$ homotopic to 1_B is an isomorphism [3, Proposition 1.7]. This agrees with the classical notion of a minimal free resolution F of a finitely generated module in a local ring $(R, \mathfrak{m})$ (i.e., one that satisfies $\partial_F(F) \subseteq \mathfrak{m}F$), or that of a minimal injective resolution I (where the inclusions $\ker(\partial_I^i) \hookrightarrow I^i$ are essential). Furthermore, a complex I of injective R-modules is minimal if and only if $\operatorname{Hom}_R(R/\mathfrak{p}, I) \otimes_R R_{\mathfrak{p}}$ has zero differential for every $\mathfrak{p} \in \operatorname{Spec} R$ (see [14, Lemma B.1] and [13, Remark 3.15]). Compare this with condition (2) of the following result.

THEOREM 3.5. *Let R be a commutative noetherian ring and B be a complex of cotorsion flat R-modules. The following conditions are equivalent:*

(1) *the complex B is minimal;*
(2) *for every $\mathfrak{p} \in \operatorname{Spec} R$, the complex $R/\mathfrak{p} \otimes_R \operatorname{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}} B)$ has zero differential;*
(3) *there does not exist a subcomplex of B of the form*

$$\cdots \longrightarrow 0 \longrightarrow \widehat{R_{\mathfrak{p}}}^{\mathfrak{p}} \xrightarrow{\cong} \widehat{R_{\mathfrak{p}}}^{\mathfrak{p}} \longrightarrow 0 \longrightarrow \cdots$$

that is degreewise a direct summand of B, for any $\mathfrak{p} \in \operatorname{Spec} R$.

PROOF. (1) ⇒ (3): this implication follows from [3, Lemma 1.7].

(3) ⇒ (2): for each $\mathfrak{p} \in \operatorname{Spec} R$, the complex $\operatorname{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}} B)$ can be described as $\cdots \to T_{\mathfrak{p}}^i \xrightarrow{\partial_{\mathfrak{p}}^i} T_{\mathfrak{p}}^{i+1} \to \cdots$, with differential induced from B; in detail, Lemma 2.2 gives a commutative diagram for each $i \in \mathbb{Z}$:

$$\begin{array}{ccc}
\prod_{\mathfrak{q}} T_{\mathfrak{q}}^i & \xrightarrow{\partial^i} & \prod_{\mathfrak{q}} T_{\mathfrak{q}}^{i+1} \\
\downarrow & & \downarrow \\
\prod_{\mathfrak{p}\subseteq\mathfrak{q}} T_{\mathfrak{q}}^i & \xrightarrow{\Lambda^{\mathfrak{p}}\partial^i} & \prod_{\mathfrak{p}\subseteq\mathfrak{q}} T_{\mathfrak{q}}^{i+1} \\
\uparrow & & \uparrow \\
T_{\mathfrak{p}}^i & \xrightarrow{\partial_{\mathfrak{p}}^i := \operatorname{Hom}(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}}\partial^i)} & T_{\mathfrak{p}}^{i+1}
\end{array}$$

where all vertical maps are degreewise split by Lemma 2.2. The canonical degreewise split inclusion $\iota_{\mathfrak{p}}^i \colon T_{\mathfrak{p}}^i \hookrightarrow \prod_{\mathfrak{q}} T_{\mathfrak{q}}^i$ and degreewise split surjection $\pi_{\mathfrak{p}}^{i+1} \colon \prod_{\mathfrak{q}} T_{\mathfrak{q}}^{i+1} \twoheadrightarrow T_{\mathfrak{p}}^{i+1}$ agree with the splittings of the vertical maps in the diagram. Since $\operatorname{Hom}_R(T^i_{\mathfrak{p}}, T^{i+1}_{\mathfrak{q}}) = 0$ for $\mathfrak{p} \subsetneq \mathfrak{q}$ by [23, Lemma 4.1.8], a diagram chase shows $\pi^{i+1}_{\mathfrak{p}} \partial^i \iota^i_{\mathfrak{p}} = \partial^i_{\mathfrak{p}}$.

Now, aiming for a contradiction, suppose $\overline{\partial^i_{\mathfrak{p}}} = R/\mathfrak{p} \otimes_R \partial^i_{\mathfrak{p}} \neq 0$ for some $\mathfrak{p} \in \operatorname{Spec} R$ and $i \in \mathbb{Z}$. Thus there exists $u \in T^i_{\mathfrak{p}} \setminus (\mathfrak{p}R_{\mathfrak{p}})T^i_{\mathfrak{p}}$ such that $\pi^{i+1}_{\mathfrak{p}} \partial^i \iota^i_{\mathfrak{p}}(u) = v \in T^{i+1}_{\mathfrak{p}} \setminus (\mathfrak{p}R_{\mathfrak{p}})T^{i+1}_{\mathfrak{p}}$. As $T^i_{\mathfrak{p}}$ is an $\widehat{R_{\mathfrak{p}}}^{\mathfrak{p}}$-module, there is an $\widehat{R_{\mathfrak{p}}}^{\mathfrak{p}}$-linear map $\alpha\colon \widehat{R_{\mathfrak{p}}}^{\mathfrak{p}} \to T^i_{\mathfrak{p}}$ mapping $1 \mapsto u$. As $u \notin (\mathfrak{p}R_{\mathfrak{p}})T^i_{\mathfrak{p}}$, there exists a map $\rho\colon T^i_{\mathfrak{p}} \twoheadrightarrow \kappa(\mathfrak{p})$ such that $\rho(u) \neq 0$. As the composition $\rho\alpha$ is nonzero, it follows that $\rho\alpha$ is a flat cover by [23, Proposition 4.1.6], hence [23, Lemma 5.2.4] yields that α is a split injection. Similarly, $\partial^i \iota^i_{\mathfrak{p}}\alpha$ is a split injection, using instead that there exists a nonzero map $\rho'\colon \prod_{\mathfrak{q}} T^{i+1}_{\mathfrak{q}} \twoheadrightarrow \kappa(\mathfrak{p})$ such that $\rho' \partial^i \iota^i_{\mathfrak{p}}\alpha$ is nonzero, and then applying [23, Lemma 5.2.4].

Set $A = \cdots \to 0 \to \widehat{R_{\mathfrak{p}}}^{\mathfrak{p}} \xrightarrow{=} \widehat{R_{\mathfrak{p}}}^{\mathfrak{p}} \to 0 \to \cdots$, concentrated in degrees i and $i+1$, and define a map from $\phi\colon A \to B$ by setting $\phi^i = \iota^i_{\mathfrak{p}}\alpha$, $\phi^{i+1} = \partial^i \iota^i_{\mathfrak{p}}\alpha$, and $\phi^j = 0$ in all other degrees. Observe that ϕ is a degreewise split injective chain map: ϕ^i is a split injection because both α and $\iota^i_{\mathfrak{p}}$ are split injections; ϕ^{i+1} is a split injection by construction. This produces a subcomplex of B forbidden by (3), hence $R/\mathfrak{p} \otimes_R \partial^i_{\mathfrak{p}} = 0$ and (2) follows.

(2) ⇒ (1): let $\gamma\colon B \to B$ be a morphism that is homotopic to the identity 1_B. Recall from [3, Lemma 1.7] that B is minimal if and only if γ is an isomorphism. Set $\gamma_{\mathfrak{p}} = \operatorname{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}}(\gamma))$ and $\mathrm{id}_{\mathfrak{p}} = \operatorname{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}}(1_B))$. Since the functor defined as $R/\mathfrak{p} \otimes_R \operatorname{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}}(-))$ preserves homotopy equivalences, the morphism $R/\mathfrak{p} \otimes_R \gamma_{\mathfrak{p}}$ is homotopic to $R/\mathfrak{p} \otimes_R \mathrm{id}_{\mathfrak{p}}$. By hypothesis, the differential of $R/\mathfrak{p} \otimes_R \operatorname{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}}B)$ is trivial, and it follows that $R/\mathfrak{p} \otimes_R \gamma_{\mathfrak{p}} = R/\mathfrak{p} \otimes_R \mathrm{id}_{\mathfrak{p}}$ for each prime $\mathfrak{p}$. Fix i and $\mathfrak{p}$ and set $F = \cdots \to 0 \to T^i_{\mathfrak{p}} \xrightarrow{\gamma^i_{\mathfrak{p}}} T^i_{\mathfrak{p}} \to 0 \to \cdots$. Evidently, F is a semi-flat R-complex such that $R/\mathfrak{p} \otimes_R F$ is acyclic (since $R/\mathfrak{p} \otimes_R \gamma_{\mathfrak{p}} = R/\mathfrak{p} \otimes_R \mathrm{id}_{\mathfrak{p}}$). By Lemma 3.4, F ($\cong \Lambda^{\mathfrak{p}}F$) is acyclic. Hence $\gamma^i_{\mathfrak{p}}$ is an isomorphism for each $\mathfrak{p} \in \operatorname{Spec} R$. By Lemma 3.1, it follows that $\gamma^i\colon B^i \to B^i$ is an isomorphism for every $i \in \mathbb{Z}$, and so (1) follows.

We end this section by showing that, for a multiplicatively closed set S and an ideal $\mathfrak{a}$, the functors $\operatorname{Hom}_R(S^{-1}R, -)$ and $\Lambda^{\mathfrak{a}}(-)$ preserve minimal complexes of cotorsion flat R-modules.

PROPOSITION 3.6. *Let B be a minimal complex of cotorsion flat R-modules, $\mathfrak{a} \subseteq R$ an ideal, and S a multiplicatively closed set. The complexes $\Lambda^{\mathfrak{a}}(B)$ and $\operatorname{Hom}_R(S^{-1}R, B)$ are minimal complexes of cotorsion flat R-modules.*

PROOF. By Lemma 2.2, both $\Lambda^{\mathfrak{a}}(B)$ and $\operatorname{Hom}_R(S^{-1}R, B)$ are complexes of cotorsion flat R-modules and the maps

$$B \twoheadrightarrow \Lambda^{\mathfrak{a}}(B) \qquad \text{and} \qquad \operatorname{Hom}_R(S^{-1}R, B) \hookrightarrow B$$

are a degreewise split surjection and injection, respectively. Applying the functor

$$R/\mathfrak{p} \otimes_R \operatorname{Hom}_R(R_\mathfrak{p}, \Lambda^\mathfrak{p}(-))$$

to either of these shows the resulting complexes must have zero differential by Theorem 3.5, and hence the desired complexes are minimal.

4. Covers, envelopes, and minimal complexes

Herein we consider minimality of R-complexes more generally. For any class of R-modules $\mathscr{A}$ that is closed under isomorphisms, we will show that a complex built from $\mathscr{A}$-covers in every degree (or $\mathscr{A}$-envelopes in every degree) is minimal, as well as prove a partial converse. The results proved here are applied in the following section, where we study the case of cotorsion flat resolutions and replacements.

THEOREM 4.1. *Let $\mathscr{A}$ be a class of R-modules closed under isomorphisms and let A be an R-complex with $A_i \in \mathscr{A}$ for all $i \in \mathbb{Z}$. Suppose at least one of the following holds:*

(1) *the canonical surjection $A_i \twoheadrightarrow \operatorname{coker}(\partial_{i+1})$ is an $\mathscr{A}$-cover for all $i \in \mathbb{Z}$;*

(2) *the canonical injection $\ker(\partial_i) \hookrightarrow A_i$ is an $\mathscr{A}$-envelope for all $i \in \mathbb{Z}$.*

Then A is a minimal R-complex.

PROOF. For each $i \in \mathbb{Z}$, set $C_i = \operatorname{coker}(\partial_{i+1})$ and assume the canonical surjection $\pi_i \colon A_i \twoheadrightarrow C_i$ is an $\mathscr{A}$-cover. We first address the case where A is bounded on the right, so that for some $n \in \mathbb{Z}$, we have $A_i = 0$ for $i < n$. Let $\gamma \colon A \to A$ be a degree zero chain map such that $\gamma \sim 1^A$. Thus there is a homotopy σ such that $1^A_i - \gamma_i = \partial_{i+1}\sigma_i + \sigma_{i-1}\partial_i$ for each $i \in \mathbb{Z}$. We first claim the following diagram commutes:

$$\begin{array}{ccc} A_n & \xrightarrow{\pi_n} & C_n \\ \big\downarrow{\scriptstyle \gamma_n} & & \big\downarrow{\scriptstyle 1^C_n} \\ A_n & \xrightarrow{\pi_n} & C_n. \end{array}$$

This follows because $\partial_n = 0$ (by the assumption that $A_i = 0$ for $i < n$) and $\pi_n\partial_{n+1} = 0$, so

$$\pi_n(1^A_n - \gamma_n) = \pi_n\partial_{n+1}\sigma_n + \pi_n\sigma_{n-1}\partial_n = 0 \implies 1^C_n\pi_n - \pi_n\gamma_n = 0.$$

Because π_n is an $\mathscr{A}$-cover, we conclude that γ_n is an isomorphism.

Fix $i > n$ and proceed by induction, assuming γ_j is an isomorphism for all $j \leq i - 1$. As γ is a chain map, γ_{i-1} induces a map $\ker(\partial_{i-1}) \to$

$\ker(\partial_{i-1})$. Since γ_{i-1} and γ_{i-2} are isomorphisms, the five lemma yields that $\gamma_{i-1}\colon \ker(\partial_{i-1}) \to \ker(\partial_{i-1})$ is an isomorphism as well. Moreover, as $\gamma \sim 1^A$, we know that γ_{i-1} induces an isomorphism on homology. Consider the short exact sequence $B_{i-1} \hookrightarrow Z_{i-1} \twoheadrightarrow \mathrm{H}_{i-1}(A)$, where $B_{i-1} = \mathrm{im}(\partial_i)$ and $Z_{i-1} = \ker(\partial_{i-1})$. The diagram below, where the horizontal maps are the natural ones, commutes because γ is a chain map:

$$\begin{array}{ccccc} B_{i-1} & \hookrightarrow & Z_{i-1} & \twoheadrightarrow & H_{i-1}(A) \\ \downarrow{\scriptstyle \gamma_{i-1}} & & {\scriptstyle\cong}\downarrow{\scriptstyle \gamma_{i-1}} & & {\scriptstyle\cong}\downarrow \\ B_{i-1} & \hookrightarrow & Z_{i-1} & \twoheadrightarrow & H_{i-1}(A). \end{array}$$

The five lemma forces $\gamma_{i-1}\colon B_{i-1} \to B_{i-1}$ to also be an isomorphism. We also have a short exact sequence $\mathrm{H}_i(A) \hookrightarrow C_i \twoheadrightarrow A_i/Z_i \cong B_{i-1}$, which shows γ_i induces an isomorphism on C_i. Therefore we have the following commutative diagram:

$$\begin{array}{ccc} A_i & \xrightarrow{\pi_i} & C_i \\ \downarrow{\scriptstyle \gamma_i} & & {\scriptstyle\cong}\downarrow{\scriptstyle \overline{\gamma_i}} \\ A_i & \xrightarrow{\pi_i} & C_i. \end{array}$$

Since π_i is an $\mathscr{A}$-cover, we conclude that γ_i is an isomorphism as well. By induction, this shows γ is an isomorphism. Thus, when A is right bounded and we assume (1) holds, A is minimal.

Now, still assuming (1) holds, consider the case where A is no longer assumed to be bounded on the right. Let $\gamma\colon A \to A$ be such that $\gamma \sim 1^A$, so that $1_i^A - \gamma_i = \partial_{i+1}\sigma_i + \sigma_{i-1}\partial_i$ for each $i \in \mathbb{Z}$. Fix $j \in \mathbb{Z}$ and define a map $\widetilde{\gamma}\colon A_{\geq j-1} \to A_{\geq j-1}$, where $A_{\geq j-1}$ is a hard truncation, as:

$$\widetilde{\gamma} = \begin{cases} \gamma_i, & i \geq j, \\ \gamma_{j-1} + \sigma_{j-2}\partial_{j-1}, & i = j-1, \\ 0, & i < j-1. \end{cases}$$

Since $\sigma_{j-2}\partial_{j-1}\partial_j = 0$, the following diagram commutes and hence $\widetilde{\gamma}$ defines a chain map.

$$\begin{array}{ccccccc} A_{\geq j-1}\colon & \cdots \xrightarrow{\partial_{j+1}} & A_j & \xrightarrow{\partial_j} & A_{j-1} & \longrightarrow & 0 \\ \downarrow{\scriptstyle \widetilde{\gamma}} & & \downarrow{\scriptstyle \gamma_j} & & \downarrow{\scriptstyle \gamma_{j-1}+\sigma_{j-2}\partial_{j-1}} & & \downarrow \\ A_{\geq j-1}\colon & \cdots \xrightarrow{\partial_{j+1}} & A_j & \xrightarrow{\partial_j} & A_{j-1} & \longrightarrow & 0. \end{array}$$

Moreover, $\widetilde{\gamma} \sim 1^{A_{\geq j-1}}$ where the homotopy is given by

$$\sigma_{\geq j-1} = \begin{cases} \sigma_i, & i \geq j-1, \\ 0, & i < j-1. \end{cases}$$

Since $A_{\geq j-1}$ is bounded on the right, the work above shows that $\widetilde{\gamma}$ is an isomorphism, hence γ_j is an isomorphism. As $j \in \mathbb{Z}$ was arbitrary, this yields that γ is an isomorphism and hence A is minimal.

For case (2), the argument is dual, and is left to the reader. First consider the case where A is bounded on the left and proceed inductively; for an arbitrary R-complex A, hard truncate on the left and define $\widetilde{\gamma}$ by using the homotopy to modify the leftmost nonzero map as was done above.

We aim to prove a partial converse to this result, noting that the converse cannot hold in general since there are minimal R-complexes not built entirely from covers or entirely from envelopes:

EXAMPLE 4.2. Let R be a local ring, M a finitely generated R-module with $\mathrm{pd}_R\, M = 1$, and N an R-module with $\mathrm{id}_R\, N = 1$. Let $P = 0 \to P_1 \xrightarrow{\partial_1} P_0 \to 0$ be the minimal projective resolution of M and $I = 0 \to I_0 \xrightarrow{\partial_0} I_{-1} \to 0$ be the minimal injective resolution of N. Set $A = P \oplus \Sigma^{-1} I$, where $\Sigma^{-1} I$ is the complex I shifted one degree to the right.

Evidently, A is a minimal complex: any $\gamma\colon A \to A$ that is homotopic to 1^A restricts to maps on P and on I, both homotopic to their respective identity maps. As P and I are both minimal complexes, these maps are isomorphisms, and hence γ is an isomorphism.

However, A is not built entirely from covers or entirely from envelopes. Let $\mathscr{A}$ be any class of modules, containing 0 and closed under isomorphisms. Since $\partial_1\colon P_1 \hookrightarrow P_0$ is an injection, the canonical inclusion $0 = \ker(\partial_1^A) \hookrightarrow A_1 \neq 0$ cannot be an $\mathscr{A}$-envelope. Furthermore, since $\partial_0\colon I_0 \twoheadrightarrow I_{-1}$ is a surjection, the canonical map $0 \neq A_{-2} \twoheadrightarrow \mathrm{coker}(\partial_{-1}^A) = 0$ cannot be an $\mathscr{A}$-cover.

We can say more in the context of cotorsion pairs. Recall that, for any class of R-modules $\mathscr{A}$, one defines the orthogonal classes

$$^{\perp}\mathscr{A} = \{M \in \mathrm{Mod}(R) \mid \mathrm{Ext}_R^1(M, A) = 0 \text{ for all } A \in \mathscr{A}\};$$

$$\mathscr{A}^{\perp} = \{N \in \mathrm{Mod}(R) \mid \mathrm{Ext}_R^1(A, N) = 0 \text{ for all } A \in \mathscr{A}\}.$$

If $\mathscr{F}$ and $\mathscr{C}$ are classes of R-modules closed under isomorphisms, we say that $(\mathscr{F}, \mathscr{C})$ is a *cotorsion pair* if $\mathscr{F} = {}^{\perp}\mathscr{C}$ and $\mathscr{F}^{\perp} = \mathscr{C}$.

PROPOSITION 4.3. *Let $(\mathscr{F}, \mathscr{C})$ be a cotorsion pair of R-modules.*

(1) *If F is a minimal R-complex with $F_i \in \mathscr{F}$ and $\mathrm{im}(\partial_i^F) \in \mathscr{C}$ for all $i \in \mathbb{Z}$, then the canonical surjection $F_i \twoheadrightarrow \mathrm{coker}(\partial_{i+1}^F)$ is an $\mathscr{F}$-precover for all $i \in \mathbb{Z}$ and an $\mathscr{F}$-cover for $i \geq \sup\{j \mid \mathrm{H}_j(F) \neq 0\}$.*

(2) *If C is a minimal R-complex with $C^i \in \mathscr{C}$ and $\mathrm{im}(\partial_C^i) \in \mathscr{F}$ for all $i \in \mathbb{Z}$, then the canonical inclusion $\ker(\partial_C^i) \hookrightarrow C^i$ is a $\mathscr{C}$-preenvelope for all $i \in \mathbb{Z}$ and a $\mathscr{C}$-envelope for $i \geq \sup\{j \mid \mathrm{H}^j(C) \neq 0\}$.*

PROOF. Let F be a minimal R-complex as in (1) and for $i \in \mathbb{Z}$, set $C_i = \operatorname{coker}(\partial^F_{i+1})$.

Fix $n \in \mathbb{Z}$. To see that the canonical surjection $\pi_n\colon F_n \twoheadrightarrow C_n$ is an $\mathscr{F}$-precover, let $G \in \mathscr{F}$ and consider the short exact sequence $\operatorname{im}(\partial^F_{n+1}) \hookrightarrow F_n \twoheadrightarrow C_n$. Since $\operatorname{Ext}^1_R(G, \operatorname{im}(\partial^F_{n+1})) = 0$, the surjection π_n induces a surjection $\operatorname{Hom}_R(G, F_n) \twoheadrightarrow \operatorname{Hom}_R(G, C_n)$. It follows that any map $G \to C_n$ factors through F_n, hence $\pi_n\colon F_n \twoheadrightarrow C_n$ is an $\mathscr{F}$-precover.

Now assume $n \geq \sup\{j \mid \mathrm{H}_j(F) \neq 0\}$. Let $\gamma_n\colon F_n \to F_n$ be such that $\pi_n\gamma_n = \pi_n$. Since, for $i \geq n+1$, each $C_i \cong \operatorname{im}(\partial^F_i) \in \mathscr{C}$, the map γ_n extends to a map $\gamma_{\geq n}\colon F_{\geq n} \to F_{\geq n}$, thought of as a map of left resolutions of C_n which lifts 1^{C_n}. Moreover, $\gamma_{\geq n} \sim 1^{F_{\geq n}}$ by a standard argument, which we give in this case: because $C_{n+2} \cong \operatorname{im}(\partial_{n+2}) \in \mathscr{C}$, the natural map $\operatorname{Hom}_R(F_n, F_{n+1}) \twoheadrightarrow \operatorname{Hom}_R(F_n, C_{n+1})$ is surjective; furthermore, because $\pi_n 1^F_n - \pi_n\gamma_n = 0$, we have $1^F_n - \gamma_n \in \operatorname{Hom}_R(F_n, C_{n+1})$, and so there is a map $\sigma_n\colon F_n \to F_{n+1}$ such that $1^F_n - \gamma_n = \partial^F_{n+1}\sigma_n$. For $j > n$, we inductively assume that σ_{j-1} and σ_{j-2} have been constructed so that $1^F_{j-1} - \gamma_{j-1} = \sigma_{j-2}\partial^F_{j-1} + \partial^F_j\sigma_{j-1}$ (where we set $\sigma_{n-1} = 0$), and claim that $(1^F_j - \gamma_j) - \sigma_{j-1}\partial^F_j \in \operatorname{Hom}_R(F_j, C_{j+1})$. This follows because

$$\begin{aligned}
\partial^F_j\big((1^F_j - \gamma_j) - \sigma_{j-1}\partial^F_j\big) &= \partial^F_j 1^F_j - \partial^F_j\gamma_j - \partial^F_j\sigma_{j-1}\partial^F_j \\
&= \partial^F_j 1^F_j - \partial^F_j\gamma_j - (\sigma_{j-2}\partial^F_{j-1} + \partial^F_j\sigma_{j-1})\partial^F_j \\
&= \partial^F_j 1^F_j - \partial^F_j\gamma_j - (1^F_{j-1} - \gamma_{j-1})\partial^F_j \\
&= -\partial^F_j\gamma_j + \gamma_{j-1}\partial^F_j \\
&= 0,
\end{aligned}$$

since γ is a chain map. As before, there exists a map σ_j such that $(1^F_j - \gamma_j) - \sigma_{j-1}\partial^F_j = \partial^F_{j+1}\sigma_j$, hence $1^F_j - \gamma_j = \sigma_{j-1}\partial^F_j + \partial^F_{j+1}\sigma_j$. Thus $\gamma_{\geq n} \sim 1^{F_{\geq n}}$. Extend the map $\gamma_{\geq n}$ to a map $\gamma\colon F \to F$ by defining $\gamma_i = 1^F_i$ for $i < n$ and set $\sigma_i = 0$ for $i < n$. Inspection shows that $\gamma \sim 1^F$ via the homotopy σ. Minimality of F implies that γ, and therefore γ_n, is an isomorphism. Thus $\pi_n\colon F_n \twoheadrightarrow C_n$ is an $\mathscr{F}$-cover.

For an R-complex C as in (2), the argument is dual. One first remarks that there is an isomorphism $C^i/\ker(\partial^i_C) \cong \operatorname{im}(\partial^i_C) \in \mathscr{F}$, and so for any $E \in \mathscr{C}$, we have $\operatorname{Ext}^1_R(C^i/\ker(\partial^i_C), E) = 0$ and thus maps $\ker(\partial^i_C) \to E$ factor through $\ker(\partial^i_C) \hookrightarrow C^i$, implying that the natural inclusions are $\mathscr{C}$-preenvelopes. For $n \geq \sup\{j \mid H^j(C) \neq 0\}$, one argues that any map $\gamma^n\colon C^n \to C^n$ induces a map $C \to C$ that is homotopic to 1_C; minimality of C yields the desired result.

Combined with Theorem 4.1, one consequence of Proposition 4.3 is that an acyclic complex F of modules from $\mathscr{F}$ with syzygies from $\mathscr{C}$ is minimal if and only if $F_i \twoheadrightarrow \operatorname{coker}(\partial^F_{i+1})$ is an $\mathscr{F}$-cover for each $i \in \mathbb{Z}$; an acyclic complex C of modules from $\mathscr{C}$ with syzygies from $\mathscr{F}$ is minimal if and only if $\ker(\partial^i_C) \hookrightarrow C^i$ is a $\mathscr{C}$-envelope for each $i \in \mathbb{Z}$.

Another case of interest in the previous result is when F (or C) is a left (or right) resolution of a module. Let $\mathscr{P}$ be the class of projective modules, $\mathscr{I}$ be the class of injective modules, and $\mathscr{M}$ be the class of all R-modules, and recall [9, Example 7.1.3] that $(\mathscr{P}, \mathscr{M})$ and $(\mathscr{M}, \mathscr{I})$ are cotorsion pairs. We immediately obtain that a projective resolution P of a module is minimal if and only if $P_i \twoheadrightarrow \operatorname{coker}(\partial_{i+1})$ is a $\mathscr{P}$-cover for all $i \in \mathbb{Z}$; also, an injective resolution I of a module is minimal if and only if $\ker(\partial^i) \hookrightarrow I^i$ is an $\mathscr{I}$-envelope for all $i \in \mathbb{Z}$.

As a consequence, we can state:

Corollary 4.4. *Let R be a ring. Then the following are equivalent:*

(1) *every R-module has a $\mathscr{P}$-cover, i.e., R is perfect;*

(2) *every R module has a minimal projective resolution.*

Proof. The implication (1) $\Rightarrow$ (2) is by Theorem 4.1. For the other implication, since $(\mathscr{P}, \mathscr{M})$ is a cotorsion pair, Proposition 4.3 then implies the canonical surjections in a minimal projective resolution are all $\mathscr{P}$-covers, and (1) follows.

The conditions of Corollary 4.4 are also equivalent to every flat module being projective; additional equivalent conditions can be found in [23, Theorem 1.2.13].

5. Minimal cotorsion flat resolutions and replacements

As an application of the results in Sections 3 and 4, the goal of this final section is to show the existence of *minimal* left/right cotorsion flat resolutions and *minimal* semi-flat degreewise cotorsion flat replacements for modules.

A *left* (or *right*) *cotorsion flat resolution* of an R-module M is a complex B of cotorsion flat R-modules with a quasi-isomorphism $B \xrightarrow{\simeq} M$ (or $M \xrightarrow{\simeq} B$), such that $B_i = 0$ for $i < 0$ (or for $i > 0$). A *degreewise cotorsion flat replacement* of an R-module M is a complex of cotorsion flat R-modules which is isomorphic to M in $\mathrm{D}(R)$. We caution that although an R-module may not have a left (or right) cotorsion flat resolution, every R-module does have a degreewise cotorsion flat replacement (see Theorem 5.2).

Example 5.1. Let $(R, \mathfrak{m})$ be a complete local ring. Finitely generated projective R-modules are cotorsion flat; therefore, the minimal projective resolution of any finitely generated R-module is a minimal left cotorsion flat resolution.

Let $\mathscr{F}$, $\mathscr{C}$, and $\mathscr{PI}$ be the classes of flat, cotorsion, and pure-injective R-modules (see [9, Definition 5.3.6] for a definition of pure-injective module). Recall our convention from 1.5 that a covering $\mathscr{F}$-resolution is built from $\mathscr{F}$-covers and an enveloping $\mathscr{C}$- (or $\mathscr{PI}$-)resolution is built from $\mathscr{C}$- (or $\mathscr{PI}$-)envelopes.

THEOREM 5.2. *Let R be a commutative noetherian ring.*

(1) *Every cotorsion R-module has a minimal left cotorsion flat resolution; the covering $\mathscr{F}$-resolution is such a resolution.*

(2) *Every flat R-module has a minimal right cotorsion flat resolution; the enveloping $\mathscr{C}$- (or $\mathscr{PI}$-)resolution is such a resolution.*

(3) *Every R-module is isomorphic in* $\mathrm{D}(R)$ *to a minimal semi-flat complex of cotorsion flat R-modules; more precisely, for any R-module M there is a diagram of quasi-isomorphisms*

$$B \xleftarrow{\simeq} F \xrightarrow{\simeq} M,$$

where F is a minimal complex of flat R-modules and B is a minimal semi-flat complex of cotorsion flat R-modules.

PROOF. (1): let L be a cotorsion R-module and $F \xrightarrow{\simeq} L$ its covering $\mathscr{F}$-resolution; such a resolution exists and is a left resolution (that is, the augmented sequence $\cdots \to F_1 \to F_0 \to L \to 0$ is exact) by 1.5. As the $\mathscr{F}$-cover of a cotorsion R-module is cotorsion flat [7, Corollary] and the kernel of an $\mathscr{F}$-cover is cotorsion [7, Lemma 2.2], we note that F is a left cotorsion flat resolution; it is minimal by Theorem 4.1.

(2): let N be a flat R-module and $N \xrightarrow{\simeq} C$ its enveloping $\mathscr{C}$-resolution; such a resolution exists and is a right resolution (i.e., the augmented resolution is exact) by 1.5. The $\mathscr{C}$-envelope of a flat R-module is cotorsion flat and its cokernel is flat [23, Theorem 3.4.2]; see also [12] and [8, Lemma 1.1 and discussion following]. Thus C is a right cotorsion flat resolution, which is minimal by Theorem 4.1. As the $\mathscr{C}$-envelope and the $\mathscr{PI}$-envelope of a flat module are isomorphic [23, Remark 3.4.9], we see that C is isomorphic to the enveloping $\mathscr{PI}$-resolution.

(3): let $F \xrightarrow{\simeq} M$ be the covering $\mathscr{F}$-resolution of M and let $F_0 \xrightarrow{\simeq} C$ be the enveloping $\mathscr{C}$-resolution of F_0. Stitch these resolutions together as follows:

$$B_i = \begin{cases} F_i, & i > 0, \\ C^{-i}, & i \leq 0, \end{cases}$$

with differential

$$\partial_i^B = \begin{cases} \partial_i^F, & i \geq 2, \\ \iota \circ \partial_1^F, & i = 1, \\ \partial_C^{-i}, & i \leq 0, \end{cases}$$

where $\iota\colon F_0 \hookrightarrow C^0$ is the augmentation map. As F_0 is flat, C^{-i} is cotorsion flat for $i \leq 0$ by (2); since $\ker(F_0 \to M)$ is cotorsion by [7, Lemma 2.2], it follows that F_i is cotorsion flat for $i \geq 1$ by (1). Thus B is a complex of cotorsion flat modules such that $B \xleftarrow{\simeq} F \xrightarrow{\simeq} M$ is a diagram of quasi-isomorphisms.

To verify that B is minimal, Theorem 3.5 tells us it is enough to ensure that B has no subcomplex of the form $\cdots \to 0 \to \widehat{R_\mathfrak{p}}^\mathfrak{p} \xrightarrow{\cong} \widehat{R_\mathfrak{p}}^\mathfrak{p} \to 0 \to \cdots$ which is degreewise a summand of B. If such a forbidden subcomplex of B were to exist, it would follow that one would have to exist as a subcomplex of either F or C. This is clear in homological degrees at least 1 or at most 0, and for such a two term subcomplex of B concentrated in homological degrees 1 and 0, one uses the injection ι to show that it must also induce such a subcomplex of F. However, the complexes F and C are both minimal by Theorem 4.1, and so no such forbidden subcomplex can exist in B. Thus B is a minimal complex.

Finally, we show B is semi-flat: there is a short exact sequence of R-complexes $0 \to F \to B \to B/F \to 0$. The R-complex B/F is isomorphic to the complex $0 \to C^0/F_0 \to C^1 \to C^2 \to \cdots$, which is semi-flat because it is an acyclic complex of flat modules having flat syzygies by [23, Lemma 2.1.2]; the complex F is semi-flat because it is a bounded on the right complex of flat R-modules. Therefore, the short exact sequence implies that B is also semi-flat.

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